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Zanstra, H., 1289.
Zeist, W. van, 601.

SUR LA TECTONIQUE DE LA CORSE

PAR

H. A. BROUWER

(Communicated at the meeting of Dec. 17, 1949)

Il existe une grande diversité d'opinions sur la tectonique de la Corse. Il est très généralement admis que le mouvement d'ensemble des nappes de la Corse orientale est dirigé de l'Est à l'Ouest. D'après certains auteurs la plus puissante de ces nappes, la nappe des schistes lustrés, aurait subi un retour en arrière, qui a rejeté vers l'Est, par-dessus elle, des lambeaux arrachés au substratum. Cette hypothèse, dans une de ses formes, a été formulé par TERMIER et MAURY. [1]) Pour SALOMON-CALVI, [2]) qui s'en rapporte aussi à des opinions de TILMANN et STILLE, le mouvement des nappes est dirigé de l'Ouest à l'Est. Il croit que la zone d'écailles le long du bord oriental de la Corse autochtone, représente la zone de racines des nappes de la Corse et de l'Apennin. Il discute la distribution des granites déformés dans le pays de nappes de la Corse orientale, qu'il considère comme étroitement liés entre eux. Il lui semble impossible d'expliquer leur distribution par une avancée de l'Est à l'Ouest et un refoulement vers l'Est.

La discussion n'a qu'une importance historique parce que ces granites sont en grande partie des granites alpins, dont on trouve des affleurements depuis la côte orientale jusqu'à la bordure de la Corse autochtone. PILGER [3]) distingue une puissante nappe de schistes lustrés et une nappe supérieure, non métamorphique, dont il ne reste que des klippes; les racines des nappes se trouvent à l'Est. Sa conception est voisine de celle de STAUB [4]) qui avait déjà décrit une succession de nappes et écailles, appartenant d'après lui à plusieurs nappes pennines et à une nappe austro-alpine inférieure.

La superposition, assez généralement acceptée, d'une nappe non métamorphique sur une nappe de schistes lustrés a été révoqué en doute. MORET [5]) se demande pourquoi on invoquerait une origine austro-alpine

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[1]) P. TERMIER et E. MAURY. Nouvelles observations géologiques dans la Corse orientale. Essai de synthèse tectonique. C. R. Ac. Sc., t. 186, 1393—1396 (1928).

[2]) W. SALOMON-CALVI. Vorläufiger Bericht uber eine geologische Reise nach Korsika. Sitz. Ber. Heidelberger Ak. der Wiss., Math.-Naturw. Kl., Abh. 7 (1932).

[3]) A. PILGER, Der alpine Deckenbau Korsikas und seine Granit-Intrusionen. Abh. Ges. der Wiss. Göttingen, Math.-Phys. Kl., III Folge, 19, 1—43, (1939).

[4]) R. STAUB. Der Deckenbau Korsikas und sein Zusammenhang mit Alpen und Apennin. Viertelj. schr. Naturf. Ges. Zurich, 73, 298—348 (1928).

[5]) Réunion extraordinaire de la Société Géologique de France en Corse, du 1er au 8 octobre 1933. Bull. S. G. F., s. 5, t. III, 761 (1933).

pour des terrains qui rappellent ceux du Briançonnais et pour lui les terrains non métamorphiques, bien développés dans la région de San Colombano, viennent plus vraisemblablement de dessous les schistes lustrés. SCHNEEGANS [6]) est arrivé à la conclusion que l'ilôt de terrains non métamorphiques de la Cima di Pedani représente un copeau arraché au substratum, qui perce la nappe des schistes lustrés et les écailles non métamorphiques de Saint Florent représentent suivant GINDRAT [7]) sans aucun doute le substratum des schistes lustrés.

Cet aperçu montre suffisamment que les opinions varient sur l'origine et la superposition des unités tectoniques. Les récentes observations en Corse, faites par un groupe de mes élèves et moi-même, nous permettent de signaler quelques résultats préliminaires qui sont d'intérêt par rapport à l'extension des granites alpins et à la tectonique du pays de nappes.

La présence de granites alpins dans la chaîne du Tenda et près de la limite orientale de la Corse autochtone.

Il est très généralement admis que la „protogine" de la chaîne du Tenda fait partie de la Corse hercynienne, dont la partie principale forme le massif cristallin autochtone de la Corse occidentale. Dans sa coupe schématique qui accompagne la dernière en date des synthèses tectoniques de la Corse, DENIZOT [8]) s'est placé aussi dans l'hypothèse que le Tenda est hercynien. Dans une note antérieure en collaboration avec C. G. EGELER, [nous avons déjà mentionné [9]) que nos recherches ne nous permettent pas de confirmer la conclusion de PILGER (loc. cit. p. 7 et fig. 2), qu'à l'Ouest de St. Florent le granite syntectonique dans les schistes lustrés a été charrié sur des granites hercyniens autochtones de la chaîne du Tenda. Après nos études dans les environs de Santo Pietro di Tenda et du M. Asto, le plus haut sommet de la chaîne, nous sommes d'avis que ce charriage n'existe pas parce que tous ces granites, qui peuvent atteindre une grande épaisseur, alternent avec les roches de la nappe des schistes lustrés et englobent des morceaux de schistes plus ou moins injectés ou imbibés. La limite entre le granite et les schistes est souvent peu précise et on peut suivre les transitions des schistes normaux vers des schistes injectés et imbibés et des granites avec des zones minces où la roche est surchargée de minéraux colorés.

On pourrait supposer que les granites sont intrusifs et qu'ils ont

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[6]) D. SCHNEEGANS. Sur la position tectonique de l'écaille de la Cima Pedani a l'W. DE MOROSAGLIA (Corse). Bull. S. G. F., s. 5, t. III, 765 (1933).

[7]) H. GINDRAT. Les écailles de St. Florent (Corse). Thèse Genève, 47 (1942).

[8]) G. DENIZOT. Supplément à la Notice sur les recherches de Géologie, Août 1943, p. 120, Pl. XII.

[9]) H. A. BROUWER et C. G. EGELER, Sur les granites alpins de la Corse. Proc. Kon. Ned. Akad. v. Wet., 51, 302 (1948).

conservé une certaine liquidité jusqu'à une phase tardive des plissements alpins. Mais on peut se demander aussi si leur cristallisation ne se rapproche pas d'une cristallisation métamorphique. Nous reviendrons sur cette question après l'étude détaillée de nos collections de roches.

Près de la limite orientale de la Corse autochtone les granites gneissiques, accompagnés de schistes injectés et imbibés, forment une zone que Th. A. F. Netelbeek et S. B. Spijer (fig. 1 et 2) ont suivi depuis les environs de Venaco jusqu'au Sud du défilé des Strettes, à l'Est de Ghisoni. Vers Poggio di Naza, plus au Sud, nous n'avons plus trouvé des affleurements de cette zone de granites alpins.

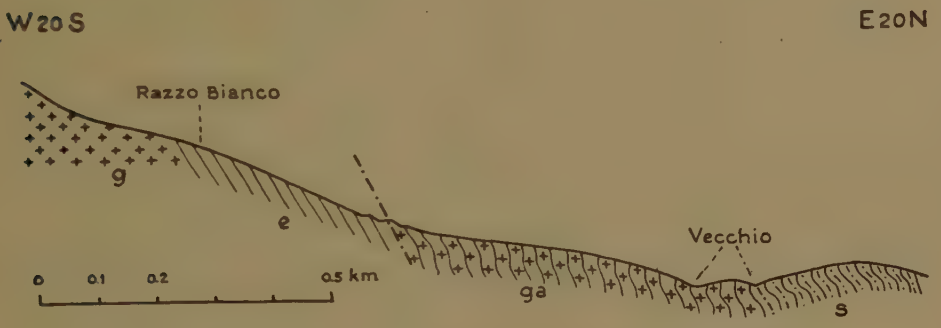

Fig. 1. *Coupe à 1.8 km au Sud de Venaco* (par S. B. Spijer).

g. granite hercynien

e. sédiments autochtones (calcaire du Razzo Bianco et poudingue de Venaco)

ga. granite alpin avec des traînées en partie reconnaissables comme appartenants à la nappe des schistes lustrés

s. schistes lustrés

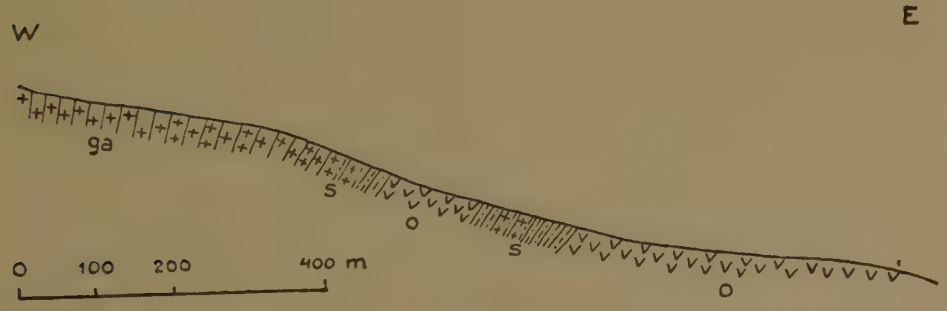

Fig. 2. *Coupe à 500 m au Sud de Vezzani* (par Th. A. F. Netelbeek).

ga. granite alpin avec des traînées en partie reconnaissables comme appartenants à la nappe des schistes lustrés

s. schistes lustrés, en partie alternants avec du granite alpin

o. gabbros et spilites des schistes lustrés

Des affleurements de granites alpins, dans les schistes lustrés, étaient déjà connus dans les environs de St. Florent et Bastia et dans la partie septentrionale du Cap Corse. [10]) Mais les masses les plus étendues affleurent dans une zone plus à l'Ouest. Les granites alpins de cette zone sont rares dans la partie aux environs des terrains non métamorphiques de Santa Lucia di Mercurio, à l'Est de la limite orientale de la Corse autochtone entre Corte et Ponte Leccia.

La grande extension de granites alpins dans les schistes lustrés de la Corse donne un caractère spécial à cette partie de la chaîne alpine.

La position tectonique des terrains non métamorphiques de Santa Lucia di Mercurio-Caporalino.

Un nombre d'auteurs a considéré des terrains non métamorphiques à l'Est de la Corse autochtone comme des klippes, qui reposent sur les schistes lustrés. Staub (loc. cit. p. 333) a distingué, à l'Est du géosynclinal des schistes lustrés, trois géanticlinaux et trois géosynclinaux, dont le géanticlinal le plus oriental aurait donné le faciès de la masse de Santa Lucia-Caporalino. D'après Pilger (loc. cit. p. 28, 40) l'ampleur des charriages est moins grande. Il pense que ce faciès provient d'un sueil ouest-ligure à la bordure orientale du géosynclinal des schistes lustrés. Le profil tectonique de Termier et Maury (loc. cit. p. 1394) entre Castirla et S. Nicolao montre les roches non métamorphiques de Santa Lucia-Caporalino comme la couverture sédimentaire de la Corse autochtone, laquelle a été disloquée et rejetée en arrière vers l'Est et repose dans un synclinal de nappe sur les schistes lustrés. Nous sommes d'accord qu'il ne faut pas chercher leur origine dans une région lointaine mais nos observations ne confirment pas que les terrains non métamorphiques sont rejetés en arrière, qu'ils reposent sur les schistes lustrés et qu'ils se trouvent dans un synclinal de nappe. Les pendages vers l'Ouest, que le flanc oriental de ce prétendu synclinal de nappe montre à divers endroits s'explique par un basculement, qui donne lieu à des structures en éventail.

Nous poursuivons l'étude de la partie méridionale de la prétendue klippe de Santa Lucia-Caporalino, qui n'a pas encore été étudié suffisamment. Dans la partie septentrionale, étudiée par T. de Booy, on voit plonger les couches éocènes en position anticlinale sous les roches basiques de la nappe des schistes lustrés, ce qui conduit à l'interprétation de Schneegans (loc. cit.) pour le Cima di Pedani. Les terrains non métamorphiques représentent le substratum des schistes lustrés (fig. 3), qui sont passés dessus dans une direction de l'Est à l'Ouest. Ils appartiennent à une zone anticlinale, qui apparaît en fenêtre.

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[10]) A. Pilger. loc. cit. p. 7. H. A. Brouwer et C. G. Egeler. loc. cit.

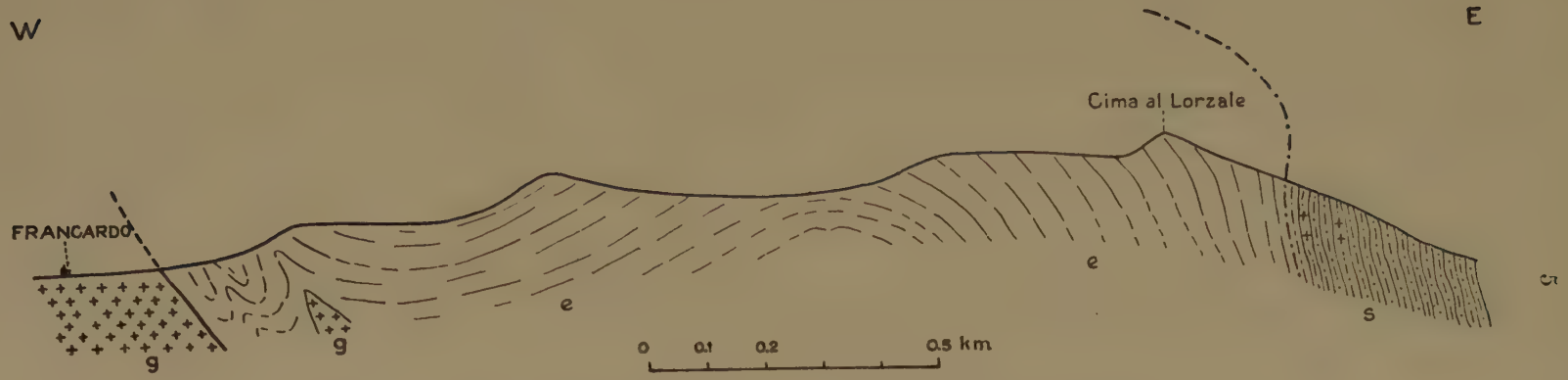

Fig. 3. *Coupe de la fenêtre de Santa Lucia — Caporalino* (par T. de Booy).

g. granite hercynien

e. sédiments(conglomérats de base, calcaires, schistes argilleuses et grèseuses) avec des zones fossilifères éocènes

s. nappe des schistes lustrés (avec de rares granites alpins)

Si les masses non métamorphiques et les granites [11]) qui les accompagnent étaient des lambeaux d'une nappe supérieure, reposant sur les schistes lustrés, on s'attendrait à trouver ces lambeaux en premier lieu dans les régions d'abaissement axial et dans les synclinaux de nappes. Dans une zone Est-Ouest, au Nord de Corte, on trouve les plus hauts sommets de la nappe des schistes lustrés, des terrains non métamorphiques et de la Corse autochtone. Si cette zone coincide approximativement avec une zone de culmination axiale dans le pays de nappes, les masses non métamorphiques de Santa Lucia-Caporalino sont situées au voisinage de cette culmination et il n'est pas étonnant d'y voir apparaître le substratum en fenêtre dans un anticlinal de nappe.

De ce qui a été dit, il résulte que la chaîne du Tenda, dont nous poursuivons l'étude pour établir la limite des granites alpins, ne peut pas être défini tout court comme un massif hercynien, contre lequel le charriage s'est arrêté. Les granites alpins de la chaîne appartiennent à la nappe des schistes lustrés, dont le chevauchement est dirigé de l'Est à l'Ouest.

La nappe des schistes lustrés est supérieure aux terrains non métamorphiques de Santa Lucia-Caporalino, qui apparaissent en fenêtre. A l'Ouest de ces terrains on trouve des schistes lustrés pincés dans les écailles parautochtones.

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[11]) Il faut attirer l'attention sur les différences des couleurs, qui indiquent la distribution des granites sur les cartes géologiques de PILGER (loc. cit.). Sur la carte géologique de la Corse (Tafel 1) des granites alpins sont indiqués au Nord-Est de Corte. Ces mêmes granites sont hercyniens sur la carte géologique de sa klippe de Santa Lucia (Tafel 4). Il résulte de sa description, qu'il considère ces granites, comme nous, d'âge hercynien.

Amsterdam, Institut géologique
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ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. VII [1])

BY

H. G. BUNGENBERG DE JONG AND H. J. VAN DEN BERG

(Communicated at the meeting of Dec. 17, 1949)

1. *Introduction.*

Besides giving a description of a contrivance for exciting rotational oscillations adapted for serial work (section 2) and some measurements of more theoretical importance with this apparatus (sections 3 and 4), this communication continues the investigation into the action of organic substances on the elastic behaviour started in Part VI of this series. The results of preliminarily investigating the action of hydrocarbons on the 1.2 % oleate system are given in section 6.

2. *Contrivance for exciting rotational oscillations adapted for serial work.*

For exciting the rotational oscillation we used an apparatus (fig. 1) which is convenient for investigations in which a number of spherical vessels of the same capacity have to be measured consecutively for a number of successive days. In performing the experiments in Part VI it was used already but so far we have not yet described it, but for the few words dedicated to it in Part III (section 4).

The principle is the same as that of the turning table described in Part I, but now the vessel is clamped round the neck in the centre of a ball bearing a, attached to a frame b by means of a brass plate c). During the measurements this frame is in a horizontal position (fig. I B) in such a way that the spherical part of the vessel is completely immersed into the water of the thermostate and the elastic oscillation can be observed (with the aid of the telescope of a kathetometer) through the glass window of the thermostate. After termination of the measurements the frame is turned upwards into an oblique position (dotted in fig. 1 B), so that the vessel can be replaced by the next of the series to be investigated. The thermostate is large enough for the surface left free by the frame, to hold simultaneously ten spherical vessels of nominally 500 ml immersed.

All the vessels used in such serial work have their necks wrapped up in a few layers of paper, which layers are covered with a piece of very thin rubber tubing (cylindrical fragments of cycle tubes), so as to give those necks the same external diameter.

[1]) Part I has appeared in these Proceedings **51**, 1197 (1948), Parts II—VI in these Proceedings **52**, 15, 99, 363, 377, 465 (1949).

For fastening a vessel into the frame, two brass clamps of the form depicted in fig. 1 C are placed around its neck and the cylinder they constitute is slided into the hollow brass cylinder, fastened in the central

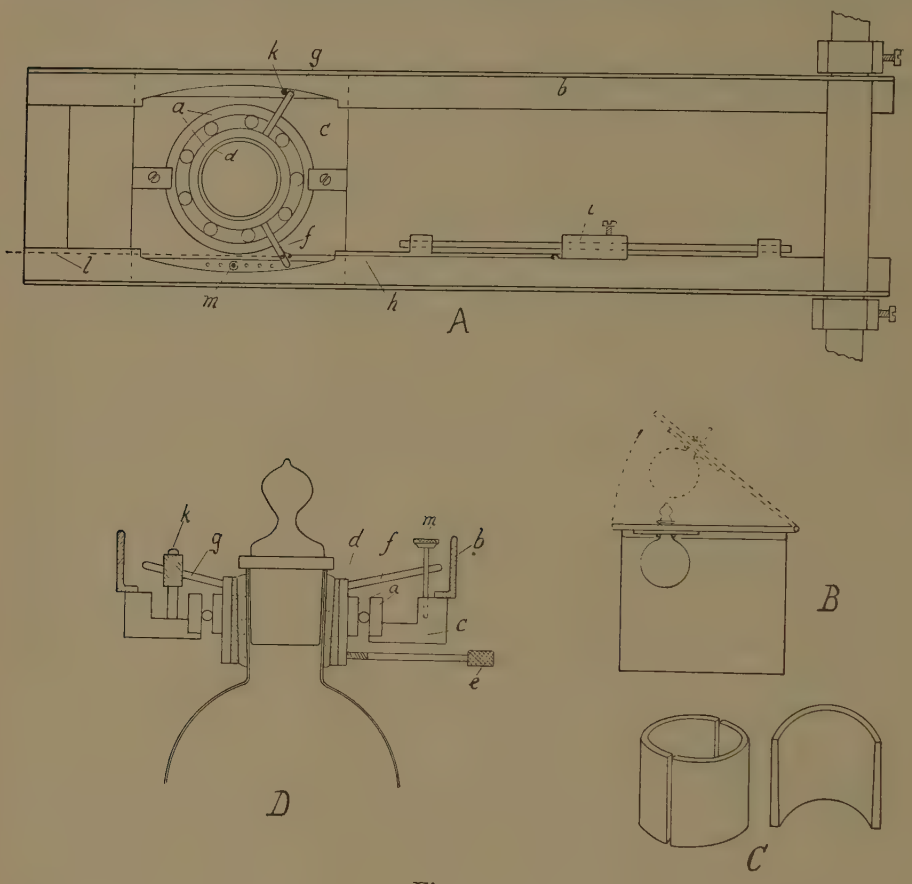

Fig. 1

hole of the ball bearing (d in fig. 1 A and 1 D), and is fixed by turning on screw e. The hollow brass cylinder carries two spikes f and g which limit the possibility of rotation of the ball bearing and consequently that of the vessel too (see below). A rubber string h stretched between the hooks on spike f and brass piece i always tends to make the vessel rotate anticlockwise, but this is prevented by the spike g pressing against the rod k attached to the ground plate. See fig. 1 A and 1 D.

For starting the rotational oscillation we pull very gently at string l (dotted), till spike f touches pin m (which may be placed at will into one of a series of holes), then we let the string l go. String h now sets the vessel rotating anticlockwise till spike g hits against rod k. In order to render the impact between g and k non-elastic, k is covered with a piece of rubber tubing. The vessel comes to a sudden standstill and the elastic oscillations of the oleate system in the vessel set in.

The impulse given may be regulated in two ways: *a*) by shifting brass piece i, which will modify the tension in string h, when this is in the position of rest, *b*) by placing pin m into one of the other holes, which will impart more or less extra extension to the string h when we pull at l.

3. *Influence of the intensity of the impulse used for exciting the damped rotational oscillation on the events during the first and following periods.*

The following survey gives a number of effects in consequence of increasing the impulse (imparted by the string) on a 1.2 % oleate system (15°; 1.43 N KCl + 0.05 N KOH, i.e. nearly the salt concentration of minimum damping) contained in a half filled spherical vessel of nominally 500 ml capacity.

pin in hole	work done g—cm	$10 \times T/2$ between turning points 5—15 sec.	$10 \times T/2$ between turning points 20—30 sec.	b_1/b_3 (5, 6, 7, 8)	b_1/b_3 (15, 16, 17, 18)	n	amplitude between turning points 0—1 at 0.5 R (scale divisions ocular micrometer)	depth of dimple top beneath surface (scale divisions ocular micrometer)
I	33	5.44 ±0.02	5.39 ±0.02	1.250 ±0.004	—	34.3 ±0.3	between 5 and 15	12
II	68	5.54 ±0.02	5.41 ±0.02	1.249 ±0.006	—	39.5 ±0.2	between 25 and 30	20
III	114	5.66 ±0.02	5.39 ±0.02	1.249 ±0.006	—	43.9 ±0.2	±50	29
IV	162	5.71 ±0.02	5.40 ±0.02	1.251 ±0.006	—	44.7 ±0.1	±60	38
V	213	5.71 ±0.02	5.41 ±0.02	—	—	44.8 ±0.1	±60	45
VI	321	—	—	—	1.247 ±0.006	—	±60	52
urning able	—	5.60 ±0.02	5.39 ±0.02	1.252 ±0.006	—	44.7 ±0.2	—	—

Some of the effects were already known qualitatively and are now illustrated by numbers, some conform suppositions already made, others again furnish entirdly new data.

1. We were aware already (cf Part I, section 6) that the quandrantal oscillation, which is excited simultaneously with the rotational oscillation gains relatively in importance by increasing the impulse. This is elucidated when one compares the columns 8 and 9, which show that the vertical dimension of the dimple — to be regarded as a measure of the initial amplitude of the quadrantal oscillation (cf Part I, section 12) is ever increasing, whereas the amplitude of the rotational oscillation soon reaches its maximum value.

2. It was also known that the number of observable oscillations n is independent of the intensity of the impulse, except in the case of very weak impulses, where n becomes smaller (cf Part I, section 9). This phenomenon is illustrated in column 7.

3. Fresh information is given in the columns 5 and 6, which show

that b_1/b_3, hence Λ (because $\Lambda = 2.303$ log. b_1/b_3) has the same value if measured from four consecutive turning points shortly after exciting the oscillation (no 5—8) or much later (no 15—18), when the oscillation has veritably grown synchronous. It was already known that Λ is independent of the intensity of the impulse in the latter case (Part I). Column 5 shows this to be true too shortly after starting the oscillations.

4. The findings in columns 5, 6, 7 and 8 combined, confirm the explanation we gave provisionally (Part I, section 9) of the constancy of n (provided very weak impulses are not used). We do see indeed that in column 7 n grows constant at the same impulse, as where in column 8 the amplitude between the turning points 0—1 becomes constant. Then it is comprehensible that if Λ is always constant (columns 5 and 6) n too must be constant (the number of oscillations required to reduce the same initial amplitude to the same very small amplitude, which has just grown imperceptible physiologically).

The usefulness in emergency cases of the change in n as an approximate measure of the change in $1/\Lambda$ in a certain types of comparative investigations (with vessels of the same radius) becomes then understandable too (compare Part I, section 10; Part VI, section 5 and in the present communication section 7) as $n \times \Lambda$ should be constant.

5. Finally, the well known fact that the first oscillations have too large a period and may not be used for the calculation of G (compare Part I, section 6a) is illustrated by comparing the columns 3 and 4. We notice, that the difference between the values of column 3 and 4 increases, but finally becomes constant. This seems to occur at the same intensity of the impulse where n (column 7) and the amplitude (column 8) grow constant, for which phenomenon we have no explanation. When the turning points 20—30 are used, the period proves to be independent of the intensity of the initial impulse, and, as further experiments showed, the period has the same value as when we use the turning points 15—25, 25—35 or 30—40. The disturbances which were present initially have died out and the damped oscillation is now truly synchronous.

4. *The half-periods between consecutive turning points before the damped oscillation has become synchronous.*

Using an analogous soap system, though in a 1.5 l vessel and with an unknown, but constant, impulse (turning table, see Part I) we have made measurements of the term between the turning points 0—20, 1—20, 2—20, etc. which when subtracted, gave the time between the turning points 0—1, 1—2, 2—3, etc. They are given in figure 2, in which the horizontal dotted line represents the value of $T/2$ (0.801 $\pm$0,002) from measurements of the term between the turning points 25—35. Though the values for $T/2$ between two consecutive turning points

have a relatively great mean error (0.03—0.04: every time indicated in the figure) as a consequence of the subtraction (e.g. 0—20 = 17.02 ± 0.02 sec.; 1—20 = 16,11 ± 0.02 sec.; therefore 0—1 = 0.91 ± 0.04 sec.), one gets the impression, that the wavy nature of the curve before the level, characteristic of damped isochronous oscillations, has been reached, is nevertheless real. An explanation cannot be given, but we think that it stands in relation with the phenomena we observed in Part I, section 12, where the spherical wave travelled towards the centre and back, and went a few times to and fro.

When we now turn to the practical side of determining the period, which we need for the calculation of G and which consists in measuring the time for ten consecutive turning points, it will be clear that we must not use the first ten for this aim. In fig. 2 we have also inserted

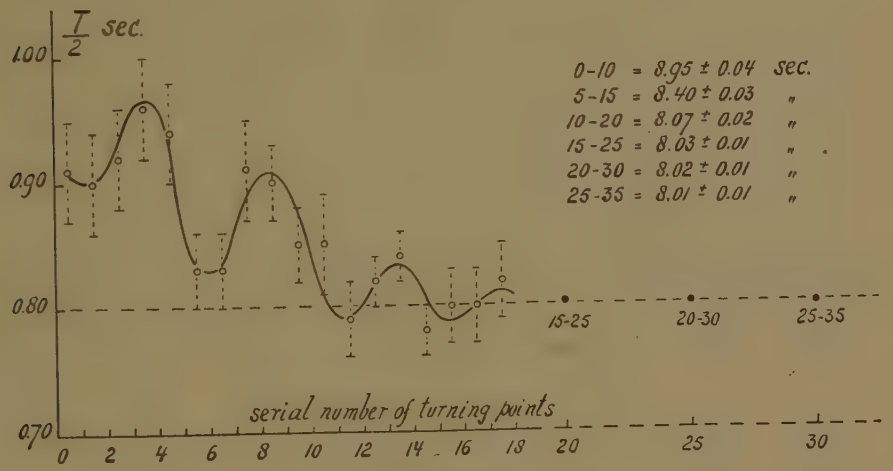

Fig. 2

$10 \times T/2$ each time starting from a later turning point. The values for 0—10 and 5—15 follow directly by subtraction from the above determinations. The others have been determined separately. We see that $10 \times T/2$ is within the experimental error equal for the determinations in which we use 20—30, and 25—35. This applies even for 15—25, though we may assume from fig. 2 that at the beginning of this tract there are still small undulations of the individual half periods. As these undulations occur practically around the isochronous level, now they cancel out each other.

A systematic error exceeding the experimental error is introduced when the turning points 10—20 are used, the mean value of T being already 0.6 % too high, that is the calculated value of G would be 1.2 % too low. The systematic error is much greater if we use the turning points 5—15 (T is 4.7% too high). If we use 0—10 the systematic error is very great (T is 11.6 % too high).

In the given example, where we worked with an oleate system of relatively small decrement, we must therefore allow some 15 turning points to pass, so about one third of the total number of observable oscillations, before beginning with the measurement of the period.

In studying the influence of various factors (such as of temperature, KCl concentration, oleate concentration, of an added organic substance) we often had to measure oscillations which were far more damped. It is our general impression that the number of turning points we had to leave unused is proportionally smaller then. If for instance n is only 12, we may safely use the turning points 5—10 for measuring T.

5. *A simplified method of investigating the actions of organic substances on the elastic behaviour giving a maximum of information.*

In part VI of this series the actions of n hexylalcohol and of ethanol have been studied on the positions of the curves which represent G (the elastic shear modulus), $1/\Lambda$ (Λ = logarithmic decrement), λ (relaxation time) and n (the maximum number of observable oscillations) as functions of the KCl concentration. The results, combined with those obtained concerning the actions of other n. primary alcohols and of the n. fatty acid anions C_8—C_{12} at only one particular KCl concentration (minimum damping of the blank oleate system) lead to a scheme, which may play a useful rôle in the study of the actions of organic substances on the elastic behaviour of the oleate system. In this scheme, which we reproduce here once more as fig. 3, the fully drawn curves relate to the

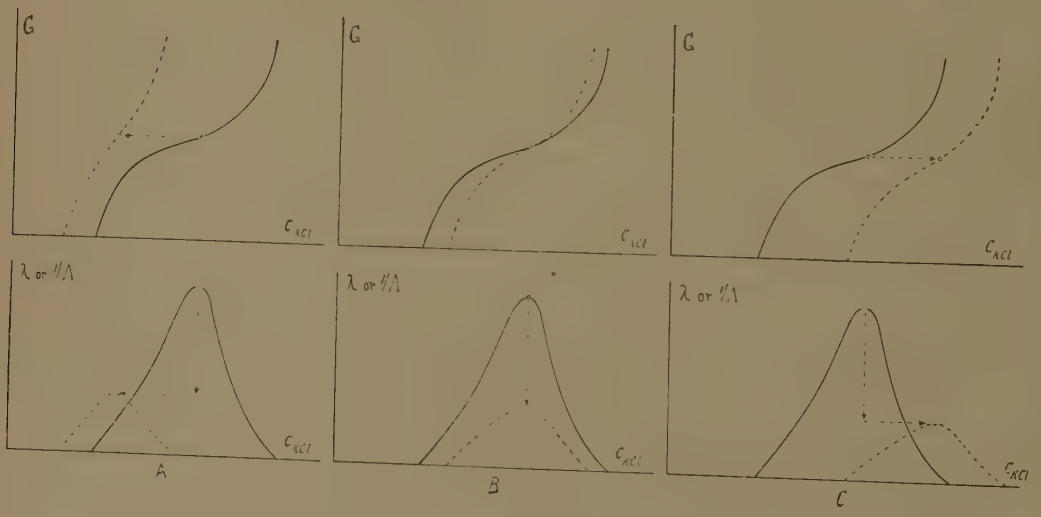

Fig. 3

blank oleate system, the dotted ones to oleate systems containing a certain amount of an organic substance. We see now that the added substance may shift the G, $1/\Lambda$ (or λ or n) curves either in the direction

of smaller KCl concentrations (Type A, e.g. n. hexylalcohol) or in the opposite direction (type C, e.g. the undecylate anion). In between these alternatives a type B may be conceived showing not any shift in horizontal direction. In Part VI we found that ethanol approached this type closely already, as the horizontal shift, though still of type A, is relatively small here. Further particulars about these shifts, which we found experimentally for hexylalcohol, we have provisionally assumed to be valid not only for other substances belonging to case A, but for those belonging to case C as well. These particulars are the following:

a. Both in the absence and presence of an organic substance the inflexion point of the G curve always lies at the same KCl concentration as the maximum of the $1/\Lambda$, λ or n curves.

b. The ordinate value of the inflexion point of the G curve does not alter when the G curve shifts towards either lower or higher KCl concentrations.

c. The displacement of the maximum on the $1/\Lambda$, λ or n curves consists of two components: I) a horizontal component towards either lower or higher KCl concentrations, and II) a vertical component which has always a downward direction.

The above induces one to attribute a twofold action (possibly at two different points of attack on the elastic structure) to an added organic substance:

1. The shifts in horizontal direction (viz. those of the inflexion point of the G curve, horizontal component of the shift of the maximum on the $1/\Lambda$, λ or n curves), indicate that an organic substance may aid KCl in building up the typical elastic viscous system (case A; a smaller KCl concentration is needed than for the blank) or it may counteract KCl (case C: a greater KCl concentration being needed now) or it may be indifferent in this respect (case B).

2. For all three types A, B and C the shift in vertical direction however is downwards (the maximum of the shifted $1/\Lambda$, λ or n curves always being lower than that of the blank). This strongly suggests that the organic substance has a second point of attack on the elastic structure, which in general leads to more damping. For a further study of the actions of organic substances it would be ideal if for each substance we had got time and enough material to investigate it so thoroughly as we did it in Part VI on hexylalcohol and ethanol. In actual practice it is hardly possible to follow this way, and it has besides the great disadvantage that it is not possible then to compare a number of substances in the same series (so with the same oleate sol).

A simpler form of experimentation, but in which such comparison (of e.g. some 8 substances) is possible consists in working at one constant KCl concentration. But then we have to consider how to choose this

KCl concentration that the results should give a maximum of information. In principle we can choose KCl concentrations which are lower than, equal to, or higher than the KCl concentration of minimum damping (of the blank).

From the scheme fig. 3 we can now read for these three KCl concentrations the sign of the change of G and of $1/\Lambda$, λ or n, if an organic substance of the types A, B or C is added.

The nine cases constructed in this way have been given schematically in fig. 4. We perceive from this scheme *that maximum information is*

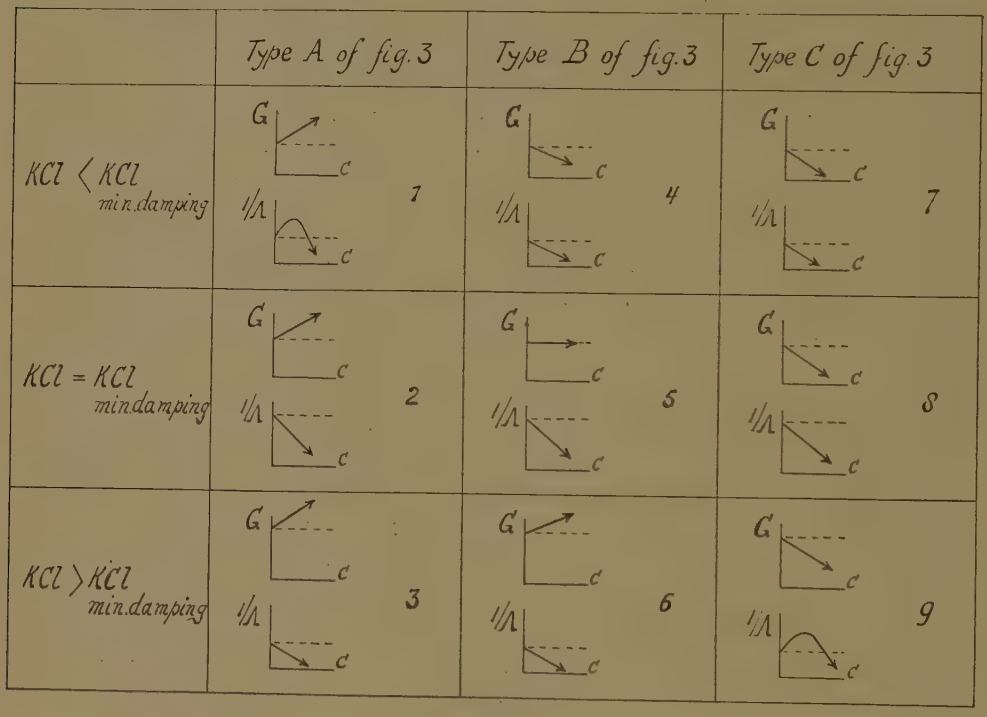

Fig. 4

obtained only, if we choose the KCl concentration of minimum damping for our experiments, as it is possible then to distinguish between the types A, B and C (cf. fig. 4 cases 2, 5 and 8). At other KCl concentrations we get less information, while any discrimination between each two types is no longer possible. At a lower concentration than that of minimum damping the types B and C cannot be distinguished (In fig. 4 case 4 = case 7), and at a higher concentration than that of minimum damping the same applies to the types A and B (In fig. 4 case 3 = case 6).

We have already availed ourselves of the above considerations in Part VI, though we had not yet given them explicitly there. For the fatty acid anions C_8—C_{12} we obtained results as those in case 9 of fig. 4, and as we know that the constant KCl concentration used in this

experiment (1.52 N) was slightly greater than that of minimum damping (1.43 N), it was concluded that these fatty acid anions belong to type C.

6. *Influence of some hydrocarbons on the elastic behaviour of the* 1.2 % *oleate system at the* KCl *concentration of minimum damping.*

As our stock of Na oleinicum medicinale pur. pulv. MERCK was exhausted and could not possibly be replenished, we had (just as with the alcohols C_1-C_6 and the fatty acid anions C_8-C_{14} in Part VI of this series), to make use of the large volume of oleate system, which had already served in previous experiments (viz. those on the dependence of T and Λ on R: cf. Part II of this series). It had the composition 1.2 g oleate per 100 ml 1.52 N KCl + 0.08 N KOH, which electrolyte composition lies near to the minimum damping of the elastic oscillations. This system was used as such for the following experiments.

TABLE I

Influence of some hydrocarbons on the elastic behaviour of the 1.2 % oleate system at approximately the KCl concentration of minimum damping (15°)

Substance	concentration millimoles/l	G corr.	Λ	λ sec.	n	100 Λ_0/Λ	100 λ/λ_0	100 n/n_0
tetrahydronaphtalene	blank	41.8	0.219	2.52	47.5	100	100	100
	0.6	43.8	0.264	2.04	42.7	83.0	81.0	89.9
	1.8	50.9	0.395	1.27	26.0	55.4	50.4	54.7
	3.5	—	—		0—1	—		0—2
toluene	blank	41.9	0.221	2.49	47.8	100	100	100
	2.1	49.9	0.425	1.19	32.9	52.0	47.8	68.8
	4.2	56.7	1.398	0.34	14.1	15.8	13.7	29.5
benzene	blank	42.0	0.222	2.48	47.7	100	100	100
	2.1	46.8	0.292	1.79	39.5	76.0	72.2	82.8
	7.7	58.3	1.586	0.29	10.8	14.0	11.7	22.6
cyclohexane	blank	41.1	0.225	2.53	47.4	100	100	100
	3.2	47.1	0.291	1.83	39.5	77.3	72.3	83.3
	6.4	49.3	0.375	1.39	25.9	60.0	54.9	54.6
	9.7	47.1	0.759	0.70	13.0	29.6	27.7	27.4
heptane	blank	41.8	0.220	2.51	47.6	100	100	100
	2.0	42.3	0.226	2.43	45.3	97.3	96.8	95.2
	4.0	42.6	0.229	2.39	44.9	96.1	95.2	94.3
	8.1	38.2	0.230	2.51?	44.1	95.7	100.0?	92.6
	13.1	31.9	0.278	2.27	43.0	79.1	90.4	90.3
	18.1	22.5	1.193	0.63	12.8	18.4	25.1	26.9
	20.6	16.6	2.286	0.38	6.0	9.6	15.1	12.6
decahydronaphtalene	blank	42.0	0.221	2.49	47.7	100	100	100
	1.5	41.8	0.230	2.39	45.8	96.1	96.0	96.0
	7.4	37.3	0.287	2.04	40.6	77.0	81.9	95.1
	12.3	26.2	0.593	1.18	20.0	37.2	47.4	41.9
	16.3	12.8	2.381	0.42	5.0	9.3	16.9	10.5

Mean value of the blanks: $G = 41{,}8$; $\Lambda = 0.221$; $\lambda = 2.50$; $n = 47.6$

For the methods (rotational oscillation in spherical vessels of known capacity, 15°) we refer to Parts I, II and III of this series, with the only exception that instead of using completely filled spherical vessels we used exactly half filled vessels (as already described in details in Part VI of this series; cf. there sections 1 and 3).

The results are given in Table I and fig. 5.

In this figure the values of G (the shear modulus in dyne/cm²) are given as such. For the purpose of comparing the influence of the hydrocarbons on $1/\Lambda$ (Λ = the logarithmic decrement), λ (relaxation time in seconds) and n (the maximum number of oscillations observable through the telescope of the kathetometer), the corresponding graphs have been drawn on the same scale. This is reached by using as ordinates the values $100\ \Lambda_0/\Lambda$, $100\ \lambda/\lambda_0$ and $100\ n/n_0$, in which the index zero indicates the value of Λ, λ and n for the blanks.

A glance at fig. 5 reveals that the hydrocarbons investigated excert

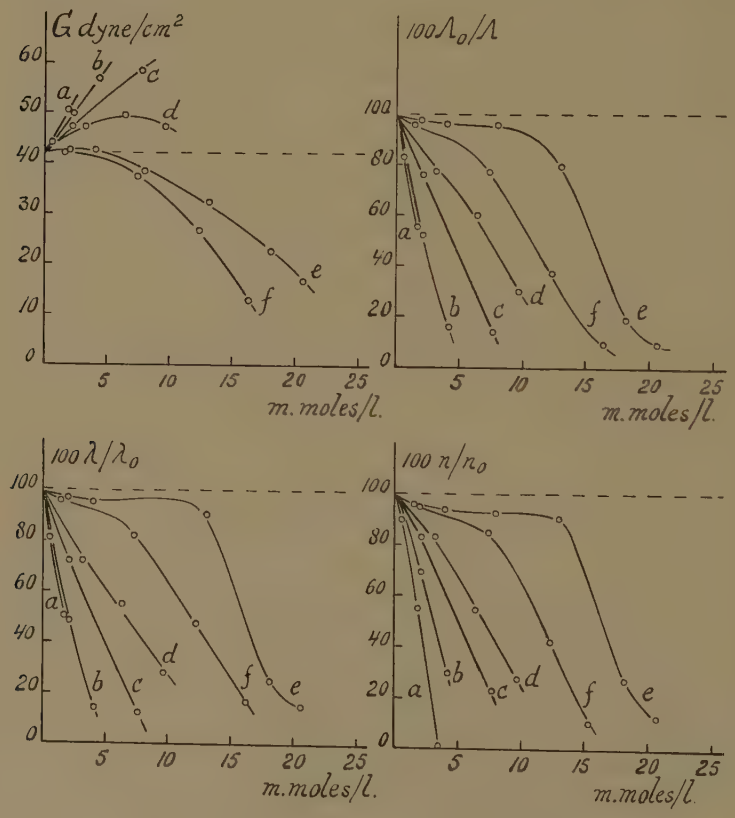

Fig. 5

their influence at very different concentrations, and what is above all interesting, we can distinguish two groups. The hydrocarbons of the first group, comprising tetrahydronaphtalene, toluene, benzene and cyclohexane increase G, the hydrocarbons of the second group comprising heptane and decahydronaphtalene decrease G. Both groups, however, do not differ as regards the direction in which the remaining quantities $1/\Lambda$, λ and n are changed. All hydrocarbons diminish these quantities.

Since we are actually working in the above experiments with hydrocarbons very near to the KCl concentration of minimum damping, we must conclude from the discussion in section 5 that tetrahydronaphtalene, toluene, benzene and cyclohexane shift the G, $1/\Lambda$, λ and n curves in the direction of smaller KCl concentrations (in a certain sense they aid the KCl in building up the elastic system), that heptane and decahydronaphtalene on the other hand shift these curves in the reverse direction (they counteract KCl in setting up the elastic system).

It is noteworthy that in the first group we find hydrocarbons (tetrahydronaphtalene, toluene, benzene) which contain one aromatic ring, and that in the second group we find saturated hydrocarbons (heptane, decahydronaphtalene). Yet this criterion is not sufficient, since we perceive that cyclohexane, though saturated belongs to the first group nevertheless. It will be clear that much more research on the action of hydrocarbons will be necessary in order to come to a better understanding of the connection between structure and action. Systematic work in this direction is now going on, the results of which will be published in this series later on.

7. *Correlation between n and Λ.*

On comparing in fig. 5 the $100\ n/n_0$ graph with the $100\ \Lambda_0/\Lambda$ graph we find a striking similarity. Other examples of this correlation were already given in Part VI, section 5, where the experimental conditions for its occurrence and also its significance and practical importance have already been discussed.

Summary.

1. A contrivance for exciting rotational oscillations adapted for serial work has been described. It makes it possible to impart impulses of known, different intensities.

2. The influence of the intensity of the impulse on the events during the first and following periods has been investigated. The results confirm inter alia a supposition made in Part I of this series how to explain the constancy of n (the maximum number of observable oscillations).

3. Particulars have been given concerning the half periods between consecutive turning points before the damped oscillation has become isochronous.

4. A simpler form of experimentation (in which the KCl concentration is kept constant) has been discussed for the study of the action of organic substances on the elastic behaviour of oleate systems. It has been shown by diagrams that maximum information can be obtained only if the KCl concentration corresponds to that of the minimum damping of the blank oleate system.

5. A preliminary investigation has been made on the influence of hydrocarbons on the elastic behaviour of the 1.2 % oleate system at the KCl concentration of minimum damping. From its results it was concluded that tetrahydronaphtalene, toluene, benzene and cyclohexane aid KCl in building up the elastic system, and that heptane and decahydronaphtalene counteract KCl in setting up the elastic system.

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THE OZONOLYSIS OF QUINOLINE AND SOME OF ITS HOMOLOGUES, IN CONNECTION WITH THE FINE STRUCTURE OF THE QUINOLINE RING *)

BY

J. P. WIBAUT AND H. BOER [1])

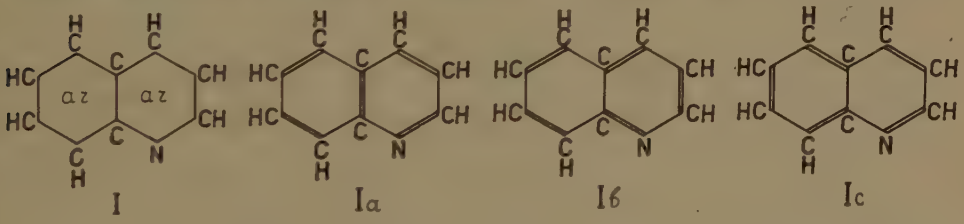
(Communicated at the meeting of December 17, 1949)

§ 1. Structural formula I of quinoline, drawn up by KÖRNER and DEWAR, accounts for the ways in which this compound is formed and for many chemical conversions it may undergo.

In the classical theory of organic chemistry efforts have been made to define the structures to be assigned to bicyclic aromatic ring systems by assuming a certain distribution of single and double bonds in these systems.

In formula I the distribution of the single and double bonds has been left undecided; the letters "ar" in a six-carbon ring indicate that this six-carbon ring has an aromatic character.

Formulae Ia, Ib and Ic show three valency structures, which are distinguished by a different arrangement of the single and double bonds.



I Ia Ib Ic

In the literature there are only few papers on the problem of the distribution of the single and double bonds in quinoline. These investigations will not be discussed in the present paper. Quinoline, just like pyridine, shows a typical aromatic character in that this compound behaves in various chemical reactions as a saturated rather than as an unsaturated compound. On the strength of calculations and on the analogy with a purely aromatic, bicyclic compound such as naphthalene, it is assumed that the quinoline ring system contains neither single nor double bonds; the *C-C* linkages in the ring are considered as aromatic bonds. According to the modern theory of the chemical bond the quinoline ring system is considered as a mesomeric form resonating between

*) Complete experimental data will be published in the Recueil des Travaux Chimiques des Pays-Bas.

various valency structures. According to PAULING [2]) the resonance energy of quinoline is 69 cal. This value approaches that of the resonance energy of naphthalene (75 cal.). The valency structures Ia, Ib and Ic are assumed to be the main contributors to the mesomeric form. These structures, which are similar to those of the Kekulé benzene structures, will be indicated as KÖRNER-DEWAR structures.

In some cases, such as in catalytic reduction or in the reaction with ozone, quinoline reacts as a compound with double bonds. The question is therefore justified according to which valency structure the quinoline molecule reacts in such cases. Investigations carried out by WIBAUT [3]) and collaborators showed that the study of ozonolysis of aromatic compounds may give an answer to these questions. In this paper we shall discuss the application of ozonolysis to the study of the fine structure of the quinoline ring system, i.e. of the valency structures involved.

§ 2. We ozonized quinoline and some of its homologues at — 25° or — 35°, using chloroform as solvent. Probably, ozonides will be primarily formed, though we have not been able to isolate them. After ozonization the reaction product was treated with an aqueous hydroxylamine solution or with a solution of *p*-nitrophenylhydrazine in dilute hydrochloric acid. This operation decomposes the primary oxidation products (ozonides?), thus forming compounds with one or more aldehyde or ketone groups. These carbonyl compounds react either with hydroxylamine or with *p*-nitrophenylhydrazine, thus forming oximes and nitrophenylhydrazones respectively. These derivatievs were isolated and identified.

From the data thus obtained we can infer which primary scission products have been formed on ozonolysis.

It was found that first the benzene ring of the quinoline ring system is attacked by ozone, the pyridine ring being attacked in a subsequent reaction.

In co-operation with F. L. J. SIXMA we carried out some measurements on the velocity of the ozonization of quinoline and some of its homologues. A mixture of ozone and oxygen was passed through a quinoline (or homologue)-chloroform solution; the flow rate of this gas mixture as well as the ozone content were kept constant during the experiment. By determining the ozone content of the exit gases at certain intervals the quantities of ozone reacting with the quinoline base during successive periods could be calculated.

In this way the ozonization rate of quinoline and of some monomethyl and dimethylquinolines were measured. These measurements were carried out at — 40°.

It was found that the ozonization proceeds from the outset at a nearly constant rate until two mols of ozone have reacted with one mol of

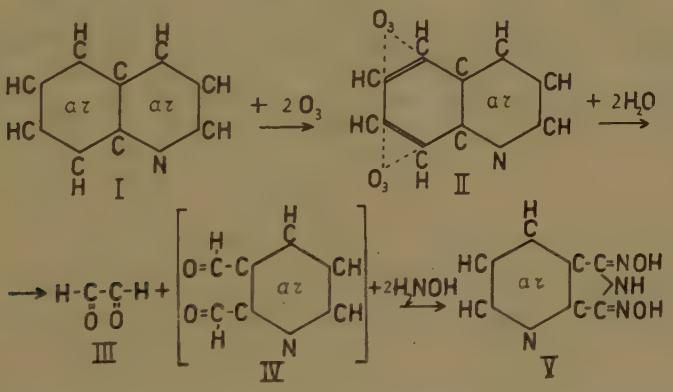
quinoline (or homologue). At this moment the reaction velocity suddenly drops to a far lower value. The quinoline derivative therefore reacts fairly easily with two molecules of ozone, a diozonide being probably formed. *) The further reaction of ozone with this diozonide proceeds far more slowly.

In other experiments the ozonization was interrupted at a certain moment. The reaction product was treated with water, as a result of which the primarily formed products (ozonides) decomposed. The liquid was then alkalized and heated to split off ammonia.

The ammonia has been formed from the scission products of the pyridine-ring. The primary formed diozonide probably reacts with two molecules of ozone, which attack the pyridine ring. In this way a tetrozonide is formed. From the scission products of this tetrozonide ammonia is splitt off by heating in alkaline solution.

Our measurements also show that the attack on the benzene ring is the main reaction. In the following section we shall discuss the scission products formed in this main reaction.

§ 3. In the ozonolysis of quinoline (I) itself the characteristic primary scission products are glyoxal (III) and a pyridine dialdehyde (IV). These compounds can be formed by scission of a diozonide (II). The reactions may be represented by the following formulae:



The pyridine dialdehyde, an unknown substance, has not been isolated. However, we obtained several derivatives of this hypothetical compound. On treating the mixture of scission products with *p*-nitrophenylhydrazine we obtained an orange substance, which gave a purple colour with an alcoholic solution of sodium hydroxide. According to BAMBERGER this colour reaction is characteristic for bis-*p*-nitrophenyl-hydrazones. Although the figures obtained in the analysis of this substance showed some deviations from the calculated figures, there seems to be no doubt that

*) This investigation will be published in detail in the Recueil des Travaux chimiques des Pays Bas.

this compound — melting point 250° — represents the bis-*p*-nitrophenylhydrazone of pyridine dialdehyde.

By treating the reaction mixture obtained by ozonolysis of quinoline with hydroxylamine, we obtained a white substance, melting at 260°, which according to its empirical formula $C_7H_6O_2N_4$, and its chemical behaviour must be considered as the dioxime of *quinolinic acid imide* (V).

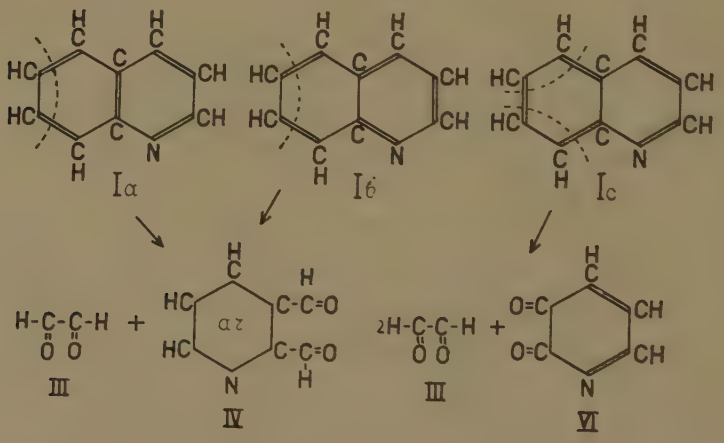
The formation of this compound by the action of hydroxylamine on pyridine dialdehyde is quite analogous to the formation of the dioxime of phthalimide by the action of hydroxylamine on orthophthalaldehyde.

On treating the mixture of scission products formed in the ozonolysis of quinoline with hydrogen peroxide, we obtained quinolinic acid, which is formed by oxidation of IV.

Therefore it seems very probably that pyridine dialdehyde is a primary scission product of quinoline.

In the ozonolysis of quinoline glyoxal (in the form of the dioxime) was obtained in a yield of about 12 mol %. The yield of the dioxime V was only $2\frac{1}{2}$ mol %; however, the isolation of this compound is difficult, so that the amount actually formed may be much greater.

It is clear that pyridine dialdehyde may be formed from structure Ia as well as from Ib, but not from structure Ic.

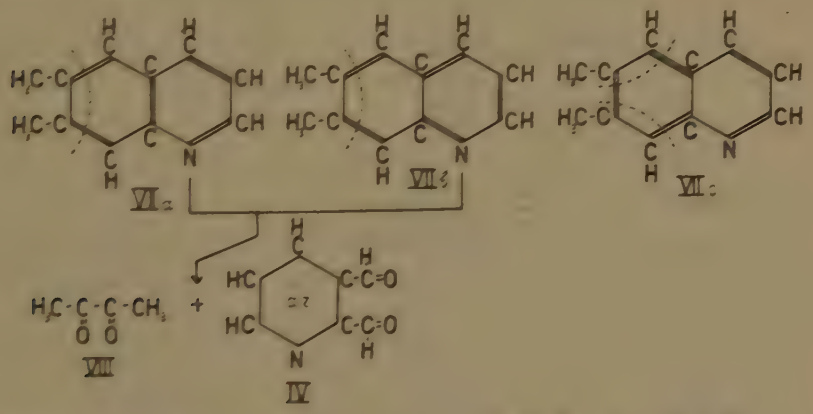


If quinoline reacted according to structure Ic, pyridione-2,3 (VI) should be formed. This compound has been described by KUDERNATSCH; [4]) its structure has been elucidated by v. SCHICK [5]) and collaborators. Following the directions of these authors we prepared this pyridione, which forms a dark brown powder, almost insoluble in water and in the common organic solvents. This substance shows a red colour reaction with concentrated sulphuric acid and a blue colour reaction with a solution of caustic soda. Moreover, we found that pyridione-2,3 suspended in chloroform, is not attacked by ozone at — 20°.

On account of these properties even small amounts of this substance should be detected if present in a mixture of scission products. However, we did not detect this pyridione among the products of the ozonolysis of quinoline. Therefore our conclusion is that in the ozonization reaction quinoline reacts according to structures Ia and (or) Ib, but not according to Ic.

§ 4. In the ozonolysis of 6,7-dimethylquinoline (VII) dimethylglyoxal (VIII) is formed in a yield of 13 mol % and pyridine dialdehyde (IV) in a yield of about 2 mol %. Here again the pyridine dialdehyde was isolated in the form of its bis-*p*-nitrophenylhydrazone. The sample of this compound isolated from the mixture of scission products of VII was found to be identical with a sample isolated from the mixture of scission products of quinoline.

The dimethylglyoxal has been isolated in the form of its bis-*p*-nitrophenylhydrazone as well as in the form of its dioxime. Pyridine dialdehyde and dimethylglyoxal can be formed either from structure VIIa or from structure VIIb.



We could detect neither methylglyoxal *) $H_3C-C(=O)-C(=O)-H$ nor pyridione in the mixture of scission products. These compounds should have been formed if the 6,7-dimethylquinoline had reacted according to structure VIIc.

In the ozonization of 6,7-dimethylquinoline this compound therefore reacts according to structure VIIa and or VIIb, but not according to structure VIIc.

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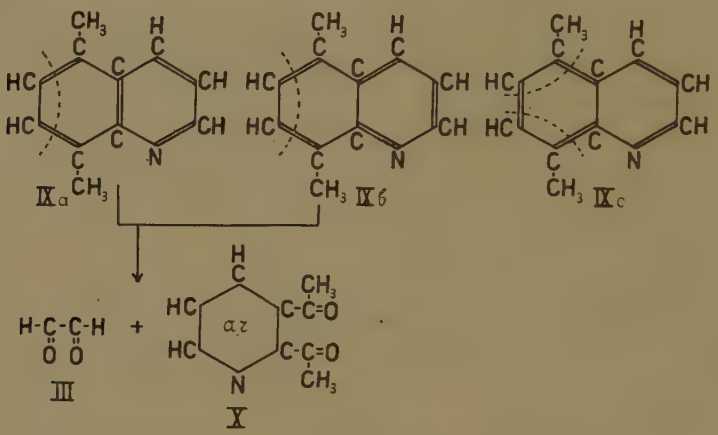
*) To determine the presence of a small quantity of methylglyoxal we worked out a method of analysis based on infra-red spectra. The infra-red spectra of dimethylglyoxime, methylglyoxime and glyoxime show characteristic differences.

In a mixture of dimethylglyoxime and 4—5 % of methylglyoxime the presence of the latter compound can be detected by infra-red analysis.

§ 5. Analogous results were obtained in the ozonolysis of 5,8-dimethylquinoline (IX). This compound was ozonized at — 30° in chloroform solution. The reaction mixture was treated either with an aqueous solution of hydroxylamine containing sodium carbonate, or with a solution of *p*-nitrophenylhydrazone in dilute hydrochloric acid.

The carbonyl compounds formed in the ozonolysis thus react either with hydroxylamine or with *p*-nitrophenylhydrazine; the resulting derivatives are isolated. The characteristic scission products formed in the ozonolysis of 5,8-dimethylquinoline are glyoxal (III) and 2,3-diacetylpyridine (X).



The dioxime of glyoxal was isolated in a yield of about 8 mol %. A sample of this oxime (melting point 178°) was subjected to infra-red analysis; methylglyoxime, which might have been formed from structure IXc, could not be detected. In another experiment the bis-*p*-nitrophenylhydrazone of glyoxal was isolated in a yield of 12 mol %.

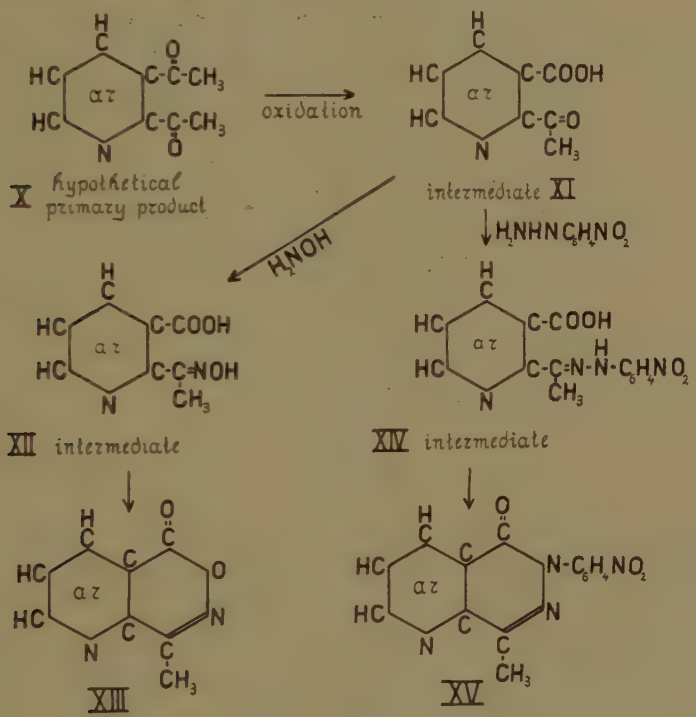
In the ozonization of 5,8-dimethylquinoline this compound therefore reacts according to the structures IXa and/or IXb, but not according to structure IXc.

The 2,3-diacetylpyridine (X), an unknown compound, was not isolated, but the following may serve as a proof that this compound is formed as a primary scission product. After treating the reaction product of the ozonolysis with an alcoholic solution of hydroxylamine we isolated not only glyoxime, but also a compound of structure XIII, i.e. the cyclic anhydride of the oxime of 2-acetylpyridine-3-carboxylic acid. This compound has been described long ago by Rosenheim and Tafel, [6]) who prepared it by treating 2-acetylpyridine-3-carboxylic acid XI with hydroxylamine hydrochloride. The sample of XIII, which we obtained from the reaction product of the ozonolysis, was identical with a sample prepared by the method of Rosenheim and Tafel.

We assume that the primary scission product of the ozonolysis, 2,3-diacetylpyridine (X), is oxidized under the prevailing experimental

conditions, yielding XI, which is transformed via XII into XIII by the action of hydroxylamine.

When the mixture of scission products of the ozonolysis of 5,8-dimethylquinoline was treated with a solution of *p*-nitrophenylhydrazine in dilute hydrochloric acid we obtained XV, i.e. the anhydride of the *p*-nitrophenylhydrazone of 2-acetylpyridine-3-carboxylic acid. For comparison we prepared this compound by treating XI with *p*-nitrophenylhydrazine. Both samples proved to be identical.



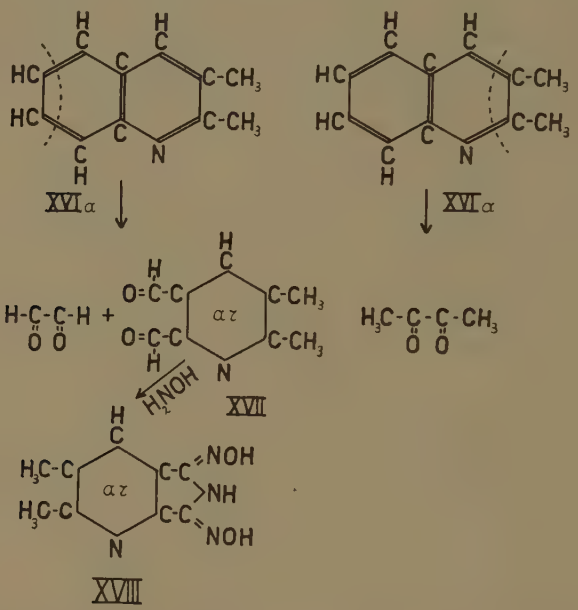
§ 6. In the foregoing we discussed the ozonolysis of dimethylquinolines having their methyl groups in the benzene nucleus. We shall now review the ozonolysis of 2,3-dimethylquinoline (XVI), in which the pyridine ring contains the methyl groups.

Ozone was passed for two hours through a solution of 22 millimols of 2,3-dimethylquinoline in chloroform at — 40°. The quinoline base had then reacted with 32 millimols of ozone, i.e. with less than the quantity required for the formation of a diozonide. This quantity of ozone had been chosen to prevent a subsequent attack on primarily formed diozonide.

After the reaction the chloroform solution was treated with an aqueous hydroxylamine solution. On working-up we obtained:

a. a mixture of aliphatic oximes which, judging from the infra-red analysis, probably consisted of glyoxime and dimethylglyoxime.

b. a compound with the empirical formula $C_9H_{10}O_2N_4$, melting at 293° with decomposition. This compound (XVIII) must be considered as the dioxime of 2,3-dimethylpyridine-5,6-dicarboxylic acid imide. We assume that XVIII has been formed by the action of the primarily formed scission product XVII.



XVIα XVIα

XVII

XVIII

The scission products I and XVII were formed by the attack of ozone on the benzene ring, the pyridine ring being retained. This is in agreement with what was found with the above quinoline derivatives. The formation of dimethylglyoxal as scission product shows that in a second reaction developing simultaneously, the pyridine ring is attacked by ozone. This is in accordance with the experiments described in § 2. However, the ozonolysis of 2,3-dimethylquinoline requires further investigation.

Conclusion.

In the reaction with ozone the ring system of quinoline reacts according to structure Ia and/or Ib, but not according to Ic.

Amsterdam, December 1949

Laboratory for Organic Chemistry of the University of Amsterdam

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EARTH'S CRUST DEFORMATIONS IN GEOSYNCLINES

BY

F. A. VENING MEINESZ

(Communicated at the meeting of Dec. 17, 1949)

The belts of strong negative gravity anomalies found in the Indonesian, Caribbean and Japanese areas have led to the hypothesis [1]) that in these areas the Earth's crust has in recent geological periods buckled downwards thus forming a large bulge of crustal matter at the lower boundary of the crust. This crustal bulge has pushed away the plastic subcrustal matter and, being of smaller density, has caused the deficiency of mass revealed by the gravity anomalies. This explanation of the anomalies has been more and more widely accepted although other views have also been advanced; it will form the base of this paper.

For the problems dealt with here we need not go into the question that the rigid crust probably consists of more than one layer of increasing density when we go downwards, and that thus the down-buckling gives rise to more areas in the crustal cross-section where lighter rocks have been substituted to denser ones, each area contributing its share to the anomalies.

In this paper the writer wants to attack the problem how this crustal deformation got into being and this involves the question how far this deformation had a plastic and how far an elastic character. For this purpose he shall often make use of the investigations of BYLAARD about this matter and his conclusions will for a great part coincide with BYLAARD'S results. He shall especially base his discussions on the Indonesian Archipelago because so much is known about this area and among BYLAARD'S papers he may refer particularly to those of 1935 and 1936 which likewise deal with this area.

In the first place we have to choose between the supposition of a purely elastic beginning of the buckling of the crust and, at least partly, a plastic start of the deformation. The second assumption, defended by BYLAARD, has several advantages with regard to the first but it offers also difficulties; when the lateral compression in the crust has increased to a value that plastic deformation sets in, the problem arises e.g. whether the crust will not bulge to both sides and continue to do so in such a way that isostatic balance will remain valid (fig. 1b) instead of the development of a geosyncline leading to a bulging downwards as fig. 1a shows. As the gravity anomalies clearly indicate that in the critical

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1) F. A. VENING MEINESZ, 1930, 1934, etc.

belt isostasy is strongly disturbed and that, therefore, the actual crustal deformation must be much nearer to 1a than to 1b, it is only possible to assume a plastic behaviour of the crust if we can prove that the deformation in that case will not lead to 1b.

Fig. 1a.

Fig. 1b.

A second problem we have to consider is whether in case we may assume deformation 1a the plastic behaviour of the crust can lead to a second crustal wave as shown by the left part of fig. 2 which apparently

Fig. 2.

is present in the western part of Indonesia viz. in the oilgeosynclines of Java and Sumatra. [2])

Before starting an investigation of these problems we shall first examine the difficulties involved in the hypothesis of a purely elastic beginning of the phenomenon.

In the first place we find in that case that the stress needed to bring about buckling of a rigid crust of a thickness of 30 km must be of the order of 40.000 to 50.000 kg/cm^2 [3]) and it seems hardly likely that the crust could stand stresses of such an amount. It has, however, already been pointed out by SMOLUCHOWSKI, [4]) who arrived at the same conclusion, that much smaller stresses would be sufficient if we could assume that the crust consists of several layers. If we may admit that the friction between these layers is negligible, we find that the stress required for buckling would be $1/n$ th part of the above figure, n being the number of layers, if we assume the layers to have equal thickness. For bringing down the buckling-stress to a more acceptable value of 3000 kg/cm^2 we would, therefore, have to assume the crust to consist of some 15 layers and this seems a rather large number. It seems, moreover, questionable whether the friction between the layers would not prevent such a development.

There is still another way out of this difficulty. GRIGGS [5]) has suggested

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[2]) Called by UMBGROVE idiogeosynclines, p. 43 e.s., Umbgrove 1942, or 2nd Ed. 1947, p. 49 e.s.

[3]) F. A. VENING MEINESZ, 1934, p. 48 or 1948, p. 70.

[4]) M. SMOLUCHOWSKI, 1909.

[5]) D. A. GRIGGS, 1939.

that the downbulging of the crust might be caused by the sinking part of a convection system descending below the belt. The drag exerted by the horizontal parts of this system on the lower boundary of the crust to both sides of the belt is on each side directed towards this belt. This may thus explain the strong compression of the crust in the belt. At the same time the effect of the sinking current on the crust may lead to the crustal down-bulging at stresses below the buckling-limit.

This hypothesis is no doubt attractive and it would solve our problem. The writer, however, sees difficulties. In the first place it appears likely that such a convection-current could not have great dimensions; as the sinking current can not have much larger cross-section than a few hundred kilometers and as the distance of the two belts of the Banda arc is not more than 600 km, the total horizontal dimension of the system could hardly exceed 300—500 km. As from other sources we must suppose the speed to be at least a few cm per year, it is unlikely that such a current could continue for the whole period of an active geosyncline which persists probably over some fifty million years or more; in that time it would make more than one complete revolution while the study of convection in the Earth leads to the conclusion that such currents never make more than a half turn. This is also GRIGG's hypothesis as given in the paper mentioned.

It would of course be possible to suppose that a current was only present during the periods of actual folding which occur from time to time during the history of the geosyncline and each of which cover a much shorter time. But in that case it would be surprising that the current would repeat itself at the same place after a first one had occurred. This first current-system must already have been causing a certain concentration of crustal matter in the belt above the sinking current and the higher percentage of radio-active minerals in this matter must thus bring about a gradual rising of temperature in the belt which makes it unlikely that a sinking current below it would again come into being.

A second and still more serious difficulty with regard to GRIGG's hypothesis of local convection-currents causing the down-bulging of the crust is based on the impression that the belt in Indonesia as well as that in the Caribbean area each form a great unit extending over many thousands of kilometers and showing a systematic course and a systematic occurring of the anomalies: this makes it unlikely that local current-systems could have brought them about.

Concluding the writer thinks that this mechanism of bringing forth the down-bulging of the crust is difficult to admit and that thus the problem remains how such a phenomenon could take place if we maintain the supposition of elastic deformation of the crust at the start of it.

A second difficulty is brought about by the hypothesis, in recent times arrived at by the author, that the substratum itself is no viscous fluid but that it has a small strength which must be overcome before flow

can set in. [6]) It has likewise been advanced by GRIGGS in his paper of 1939 and since long by JEFFREYS. The writer shall not go here into the arguments leading to this hypothesis but he may point out that it obviously makes it difficult to adhere to an elastical down-buckling of the crust. For maintaining this idea we should need an initial downward deformation of the crust which would make it possible that the momentum thus caused by the resultant of the horizontal stress in the crust would not only overcome the resistance against bending of the crust but also the elastic resistance offered by the substratum. It is, besides, likely that this last resistance will involve a considerably larger value of the buckling-stress needed in the crust and that thus the number of separate layers in the crust which we had to assume for bringing this stress down to an admissible value would have to be still greater.

A third problem offered by the hypothesis of a purely elastic beginning of the crustal deformation in the tectonic areas is to find an explanation for the curved and often tortuous course the deformed belts mostly show. It is hardly likely that this is caused by an intricate pattern of forces bringing about these deformations. If we examine e.g. the main anomaly belt in Indonesia we come to the conclusion that it is probably caused by a regular field of forces showing an unique or nearly unique direction [7]) given by the arrows on the map and so it seems out of the question to attribute the curved shape of the belt to the forces. It is, however, possible to account for part of the irregular courses of the mobile belts by the hypothesis of pre-existant zones of weakness of the Earth's crust as advanced by the writer in 1943. [8]) The presence of such zones must make it likely that the crustal deformation will in part follow them. The pattern of shearzones derived in the papers referred to has been based on the assumption of a polar shift in a very early stage of the crust's history; the directions derived for Indonesia have been indicated in the map.

It is unlikely, however, that the presence of such zones of weakness can explain all the features of the complicated belt-pattern of the orogenic areas; in Indonesia e.g. the part parallel to Java and several other parts have still to be accounted for. This is difficult in case we assume the pattern to develop as an elastic deformation of the crust. As BYLAARD has remarked such a deformation must be expected to occur in belts at about right angles to the force and although the curvature of the Earth might somewhat affect it, it could not deviate as much as the existing belts, as e.g. that south of Java, actually do.

As we shall afterwards examine more in detail a plastic deformation could give a much better explanation of these directions. It is well known that in a plate subjected to a single stress-direction the plastic deformation

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[6]) F. A. VENING MEINESZ, 1948, p. 39.

[7]) VENING MEINESZ, Gravity Expeditions at Sea, Vol. II, p. 122, Vol. IV, p. 30 etc.

[8]) VENING MEINESZ, 1943 and 1947 (see e.g. 1947, p. 56).

after the elastic limit is exceeded will occur under an angle of about 55° with this direction and not at right angles to it; this direction is e.g. shown by the lines of HARTMANN developing at this moment. BYLAARD and afterwards VAN ITERSON have given an explanation of this angle and have also derived the formula for the angle occurring in the general case of a more complicated stress condition in the plate. BYLAARD has already shown that these directions of the plastic deformation may give a more satisfactory explanation of the pattern of the tectonic belts. We shall come back to this point.

Concluding we may say that the assumption of an elastic deformation at the beginning of the phenomenon involves many difficulties which are avoided by the supposition that from the start it has a plastic character. We shall now examine the two problems raised by the plastic hypothesis, the question whether a downbuckling of the crust can be thus explained and whether it is possible that an adjoining wave develops.

We shall begin by attacking the first problem. Let us assume that in a belt over a breadth b the compression exceeds the elastic limit of the crust and that a plastic deformation has set in, leading to a thickening of the crust, which towards the edges of the belt disappears as represented by fig. 3. We may probably assume that the rate of thickening is extremely

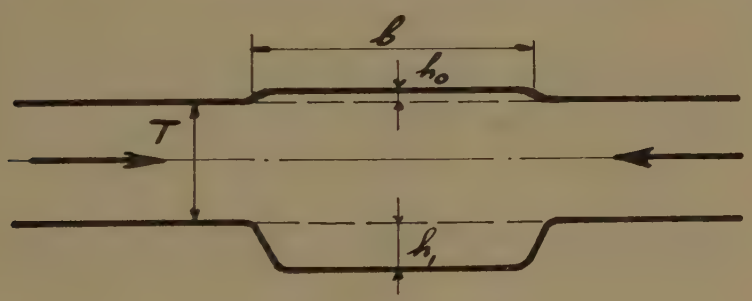

Fig. 3.

slow, e.g. of the order of a few mm per year or less, and in that case we may practically neglect the viscous resistance of the substratum against the forming of the bulge at the lower boundary of the crust. [9]) As a consequence of this we may expect the heights h_0 and h_1 of the upper and lower bulges to have a ratio corresponding to isostatic balance and

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[9]) This may be derived from the speed of isostatic readjustment which the writer found to be given by the formula of p. 662 of his paper of 1937

$$w = 5\,\zeta\,L \text{ cm/year}$$

in which w = vertical speed of the crust in cm/year
ζ = vertical deviation of crust's surface from the equilibrium position in km
L = diameter in 1000 km of equilibrium deviation
The formula has been derived from the rate of post-glacial uplift of Scandinavia.

if we adopt the usual values for the densities of the crust and the substratum of 2.67 and 3.27, this gives

(1) $$h_1 = 4.45\, h_0.$$

This asymmetry is important for our problem because it must involve a deviation from the regular distribution of stress in a vertical cross-section of the crust. As h_0 and h_1 are no doubt small with regard to the breadth b of the bulge we shall probably not make a great error in neglecting the complications in the transition zones at the edges of the belt and in assuming that the normal stress-component in each cross-section is a linear function of the vertical coordinate z. As we also know that the resultant of these components must coincide with the original axis of the crust, our assumption allows us to derive the distribution.

If we denote the values of the normal stress σ at the upper and lower boundaries by σ_0 and σ_1, their difference by $\sigma_\triangle$ and their mean value

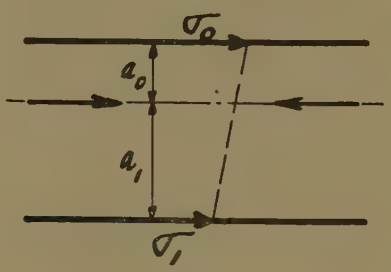

Fig. 4.

by σ_m which is also the mean value of the stresses σ over the whole cross-section, we can derive

(2a) $$\frac{\sigma_\triangle}{\sigma_m} = 6\,\frac{a_1 - a_0}{a_1 + a_0} = 6\,\frac{h_1 - h_0}{T + h_1 + h_0}$$

and if we use (1) and neglect $h_1 + h_0$ in the denominator

(2b) $$\frac{\sigma_\triangle}{\sigma_m} = 20.70\,\frac{h_0}{T}.$$

The difference $\sigma_\triangle$ of the normal stresses at the two crustal boundaries must lead to a difference in the shortening of the horizontal dimensions at both boundaries and, therefore, to a down-bending of the crust. We see that thus a downward movement of the crust may be explained. We must, however, derive its value to find out whether it can compensate and even exceed the rising h_0 of the upper surface so that a geosyncline is formed.

The downward movement affects the formules (2a) and (2b). If we denote the sinking of the crustal axis in an arbitrary cross-section by

z and in the middle of the belt by z_c, the value of $a_1 - a_0$ (see fig. 4) is increased by 2 z, resp. 2 z_c, and so instead of (2a) and (2b) we get

$$\frac{\sigma_\triangle}{\sigma_m} = 6\,\frac{h_1 - h_0 + 2z}{T + h_1 + h_0} \tag{3a}$$

and using again (1) and neglecting $h_1 + h_0$ in the denominator

$$\frac{\sigma_\triangle}{\sigma_m} = 12\,\frac{(1.725\,h_0 + z)}{T}. \tag{3b}$$

For the middle cross-section we have to substitute z_c to z.

As the dimensions of the phenomenon in the sense of the belt are large we may assume the rate of shortening of each particle of the crust in the sense of the compression by the stress σ (x direction) to be equal to the rate of thickening in the vertical sense (z direction). As we suppose no stress to be working in vertical direction on the crust the usual stress-strain relation of viscous fluids gives here for the rate of both deformations

$$-\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right) = \frac{\sigma}{2\eta} \tag{4}$$

where u and w are x and z comp. of displacement and η the coefficient of viscosity.

If we denote the length in x direction after deformation of the whole deformed area at the surface by b_0 and at the lower boundary by b_1 (undeformed both $= b$) we get

$$\frac{\partial}{\partial t}(b_1 - b_0) = \frac{1}{2\eta}\int_0^b \sigma_\triangle\,dx = 6\,\frac{\sigma_m}{\eta T}\int_0^b (1.725\,h_0 + z)\,dx.$$

Contenting us with an approximate solution we may — if z is zero at both ends of the deformed part — assume that the mean value of z over this area is $\frac{2}{3}\,z_c$. It would, however, be possible that the adjoining non-plastic parts of the crust also take part in the bending and in that case we should have to add a figure for the sinking of the ends of the plastic part. Supposing this to be proportional to z_c we get a mean value of z over the plastic part of

$$z_m = (\tfrac{2}{3} + \mu)\,z_c.$$

Introducing this we obtain:

$$\frac{\partial}{\partial t}(b_1 - b_0) = 6\,\frac{\sigma_m\,b}{\eta T}\,[1.725\,h_0 + (\tfrac{2}{3} + \mu)\,z_c]. \tag{5}$$

We shall now express $b_1 - b_0$ in z_c and h_0 in the time t.

From fig. 5 we derive:

$$\varrho : T = b : (b_1 - b_0)$$

and so

$$\varrho = \frac{bT}{b_1 - b_0}$$

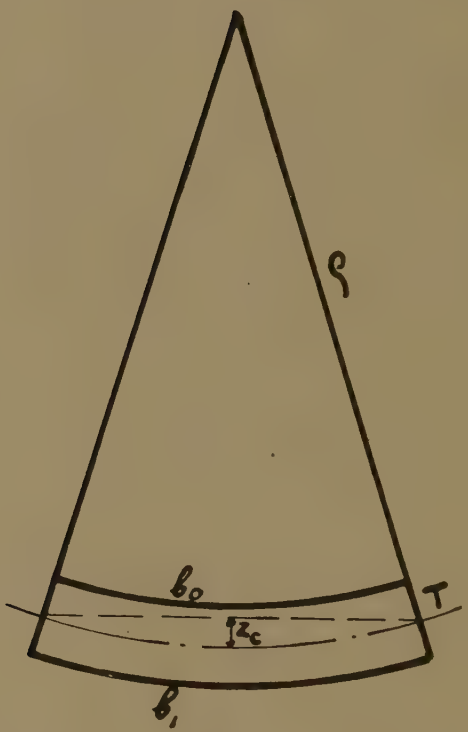

Fig. 5.

and

$$z_c = \frac{1}{8}\frac{b^2}{\varrho} = \frac{1}{8}\frac{b}{T}(b_1 - b_0). \tag{6}$$

Multiplying (5) by $b/8\,T$ we get

$$\frac{\partial z_c}{\partial t} = \frac{3}{4}\frac{\sigma_m b^2}{\eta T^2}\left[1.725\,h_0 + \left(\tfrac{2}{3} + \mu\right) z_c\right]. \tag{7}$$

The value of h_0 and of the total increase of the crust's thickness which we shall denote by h ($h = 5.45\,h_0$) can be found because we know that the compressive stress σ_m and, therefore, the rate of thickening of the crust since the beginning of the plastic deformation, i.e. since the time $t = o$, has been constant and that this rate is given by formula (4).

We thus obtain:

$$\frac{\partial h}{\partial t} = \frac{\sigma_m T}{2\eta} \tag{8a}$$

and integrating:

$$h = \frac{\sigma_m}{2\eta} T t. \tag{8b}$$

Introducing the value of $h_0 = 0.1835\ h$ in (7) we get the following first order differential equation in z_c:

$$\frac{\partial z_c}{\partial t} - (1 + \tfrac{3}{2}\mu) \frac{\sigma_m}{2\eta} \frac{b^2}{T^2} z_c - 0.475 \left(\frac{\sigma_m}{2\eta}\right)^2 \frac{b^2}{T} t = 0. \tag{9}$$

We can simplify this equation by choosing a new variable instead of the time t, viz. the increase of the crust's thickness h which is proportional to t. By means of (8) we find

$$\frac{\partial z_c}{\partial h} - (1 + \tfrac{3}{2}\mu) \frac{b^2}{T^3} z_c - 0.475 \frac{b^2}{T^3} h = 0. \tag{10}$$

This no longer contains explicitly the quantities σ_m and η. The solution of the equation is simple. Taking into account that $z_c = 0$ for $t = h = 0$ we obtain

$$z_c = \frac{0.475}{1 + \frac{3}{2}\mu} \left[\frac{1}{1 + \frac{3}{2}\mu} \frac{T^3}{b^2} \left(e^{(1+\frac{3}{2}\mu)\frac{b^2}{T^3} h} - 1\right) - h\right]. \tag{11}$$

For finding the total amount of the sinking of the crustal axis in the middle of the plastic belt we have to add the sinking at the edges of this belt which we assumed to be μz_c and so we have to multiply our formula by $1 + \mu$. Denoting the result by z' we get

$$z' = 0.475 \frac{1+\mu}{1 + \frac{3}{2}\mu} \left[\frac{1}{1 + \frac{3}{2}\mu} \frac{T^3}{b^2} \left(e^{(1+\frac{3}{2}\mu)\frac{b^2}{T^3} h} - 1\right) - h\right]. \tag{12}$$

For $\mu = 0$, i.e. when the adjoining non-plastic parts of the crust do not take part in the bending, the formula becomes

$$z' = 0.475 \left[\frac{T^3}{b^2} \left(e^{\frac{b^2}{T^3} h} - 1\right) - h\right]. \tag{13}$$

We obtain the formula for the sinking z_0' of the crust's surface in the middle of the belt by subtracting the rising $h_0 = 0.1835\ h$ and so we get

$$z_0' = 0.475 \frac{(1+\mu)}{(1 + \frac{3}{2}\mu)^2} \frac{T^3}{b^2} \left(e^{(1+\frac{3}{2}\mu)\frac{b^2}{T^3} h} - 1\right) - 0.658 \frac{1 + 1.139\,\mu}{1 + \frac{3}{2}\mu} h, \tag{14}$$

which for $\mu = 0$ simplifies to

$$z_0' = 0.475 \frac{T^3}{b^2} \left(e^{\frac{b^2}{T^3} h} - 1\right) - 0.658\, h. \tag{15}$$

By multiplying these equations by $\frac{b^2}{T^3}$ we see that $\frac{b^2}{T^3} z_0'$ is only function of $\frac{b^2}{T^3} h$ and μ, a conclusion which we could already have drawn

from the differential equation (10). So we find that z_0' and h are proportional to the third power of T and inversely proportional to the square of b. We may add that by computing numerical values for these quantities for different values of μ we find that as a rough approximation both quantities are inversely proportional to $1 + \mu$, at least inside the limits $0 < \mu < 1$, which will probably not be exceeded by the cases occurring in practice.

Formulas (14) and (15) give the following values (Table I) for three values of μ, i.e. $\mu = 0$, $\mu = \frac{1}{2}$ and $\mu = 1$. The curve derived from these values for $\frac{b^2 z_0'}{T^3}$ as a function of $\frac{b^2 h}{T^3}$ is given by fig. 6; it is practically the same for the three cases but on different scales. For all three the vertical dimensions are four times the horizontal ones. The z_0' axis is positive downwards.

TABLE I

$\mu = 0$		$\mu = 0.5$		$\mu = 1.0$	
$\frac{b^2 h}{T^3}$	$\frac{b^2 z_0'}{T^3}$	$\frac{b^2 h}{T^3}$	$\frac{b^2 z_0'}{T^3}$	$\frac{b^2 h}{T^3}$	$\frac{b^2 z_0'}{T^3}$
0.327	− 0.031	0.212	− 0.0207	0.1572	− 0.0154
0.618	0	0.401	0	0.2971	0
0.654	+ 0.009	0.424	+ 0.0056	0.3144	+ 0.0047
0.981	+ 0.149	0.636	+ 0.0997	0.4716	+ 0.0765

The horizontal axis of fig. 6 gives h which is proportional to t; it may, therefore, be considered as a time-axis. The first line of Table I shows the maximum negative values of z_0', i.e. the maximum rise of the Earth's surface in the belt. We see that after this first rise, the effect of the sinking of the crust's axis begins more and more to dominate the effect of the rise caused by the plastic thickening of the crust. At the values of h given by the second line of Table I the rise in the centre of the belt is already neutralized and, as fig. 6 shows, the sinking becomes afterwards more and more marked. So we see that our attempt is successful and that a down-buckling of the crust as a result of plastic deformation can not only well be explained but appears to be an unavoidable consequence of it.

We shall now assume values for T and b in order to derive the values of z_0' for the maximum rise and for other parts of the curve. For T we shall assume the value of 30 km which follows from the great majority of the observed gravity values. It is more difficult to make an estimate of the breadth of the plastic belt b. It is in fact likely that it will be variable, even for parts of the same tectonic belt. We shall choose three values for our computations viz. 50 km, 100 km and 150 km; the last is probably too large. Table II gives a list of the maximum rise in the

centre of the belt for the different cases and of the sinking after a three times longer time-interval.

TABLE II

b	maximum rise at time t_m			sinking at time 3 t_m		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1.0$	$\mu = 0$	$\mu = 0.5$	$\mu = 1.0$
50 km	335 m	223 m	166 m	1610 m	1080 m	826 m
100 „	84 „	56 „	42 „	401 „	269 „	207 „
150 „	37 „	25 „	18 „	178 „	120 „	92 „

We shall now make an attempt to find the time-intervals involved in the phenomenon. We can begin by deriving h by means of the same

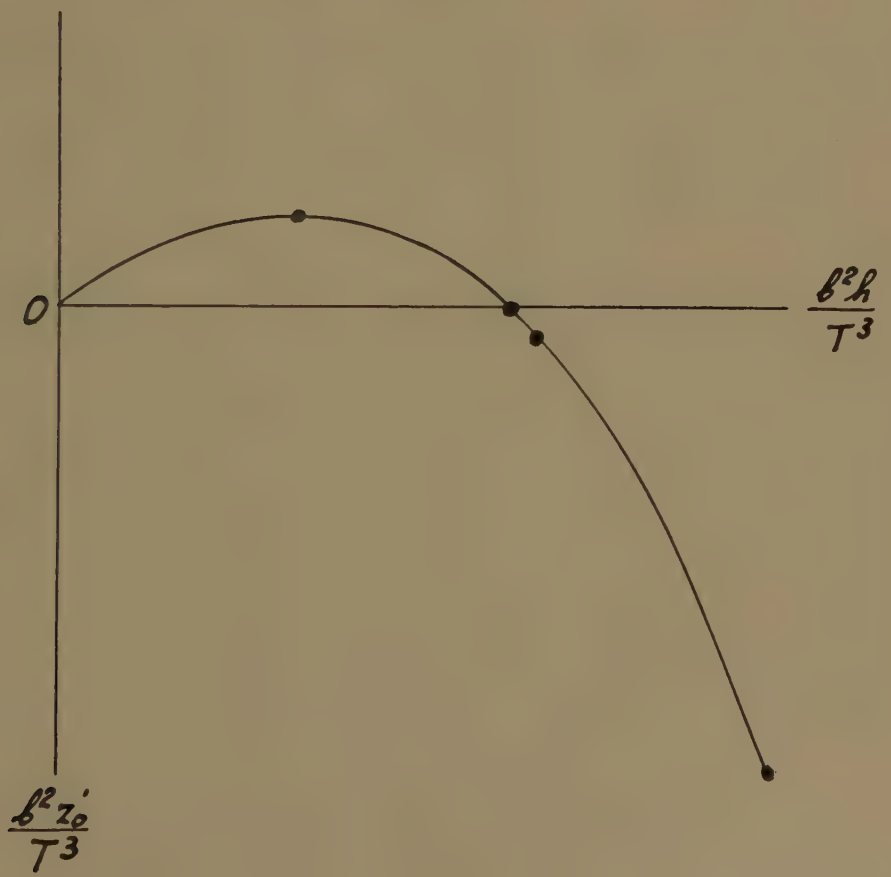

Fig. 6.

assumptions about T and b as used above. Next we have to apply formula (8b) for obtaining the corresponding times t. Here, however, a great deal of uncertainty sets in because of our lack of knowledge

about the stress σ_m and the viscosity η. The stress σ_m must slightly exceed the strength of the crust and so the writer thinks that

$$\sigma_m = 3000 \text{ kg/cm}^2 \tag{16}$$

may be a reasonable value.

Of η we know practically nothing. It must no doubt considerably exceed the viscosity of the substratum which has been derived from the post-glacial uplift in Scandinavia; the writer found in this way a dynamic viscosity η of 3×10^{22} poises. [10]) From the fact, moreover that never any evidence has been found of flowage of rocks in mountains in case of strong shearing-stress, an indication of an even higher lower limit of η may be derived; we thus find that it is likely to exceed at least 10^{23} poises. For our purpose we shall put

$$\eta = 10^{24} \text{ poises.} \tag{17}$$

This value is of course subject to considerable uncertainty and the same is, therefore, true for the following estimates.

Before applying these assumptions to our main problem we shall use them for obtaining an idea of the rate of thickening of the crust. By means of formula (8a) we find

$$\frac{\partial h}{\partial t} = 0.15 \text{ cm per year.} \tag{18}$$

As far as the order of magnitude is concerned this result appears acceptable. As we have already mentioned on p. 31, a rate of a few mm per year allows us practically to neglect the viscous resistance of the substratum and this is necessary, for otherwise this resistance would play a part in preventing the free development of the down-bending and down-bulging of the crust.

The above value of $\partial h/\partial t$ implies a rate $\partial h_0/\partial t$ of the rise at the surface of 0.03 cm per year.

Returning to our main problem we shall apply formula 8b and use the values mentioned before of T and b for changing from the quantity b^2h/T^3 to the corresponding time. We shall do so for computing the time-interval t_m in which the maximum rise at the centre of the belt comes into being. We thus find the values of Table III.

TABLE III

Time t_m of maximum rise in years

b	$\mu = 0$	$\mu = 0.5$	$\mu = 1.0$
50 km	2.490.000	1.610.000	1.200.000
100 „	620.000	403.000	298.000
150 „	280.000	179.000	133.000

[10]) Vening Meinesz, 1937.

These results for the time elapsed since the beginning of the plastic deformation of the crust till the maximum rise occurs, must be doubled to obtain the time-interval between this maximum rise and the sinking at the time $3\,t_m$ given in Table II. Most of that sinking is occurring in the second half of this last interval, that is to say in an interval equalling that of the table. The order of magnitude of these results seems acceptable. It is hardly necessary to add that the sinking will continue with increasing speed after the time $3\,t_m$ till the whole phenomenon will change when it leads to the entire breaking together of the crust.

Considering the long periods we have found [11]) it appears that the value we assumed for the viscosity η of the crust could not be many times larger; a value of 10^{25} for instance would give ten times greater time-intervals and such intervals seem highly improbable.

Before leaving our subject we may point out that in the developing geosyncline sedimentation must occur and, because of its load and the isostatic readjustment, the rate of sinking must be considerably increased. In general we may estimate the sediments at four to five times greater thickness than the depth of the geosyncline if no sedimentation or filling up by water had occurred.

When during the further development the crust breaks together and is pushed downwards, the buoyancy of the whole structure must become larger and larger and this must lead to the lifting up of the adjoining crustal belts. In the gravity field this shows up by belts of positive anomalies accompanying on both sides the belt of strong negative anomalies caused by the downbulge of the crust. This proves again the extremely high viscosity of the crust; if it were less the central belt would have been able to rise for readjusting its isostatic equilibrium and no negative anomalies could have been left. The time elapsed since the downbulging of the crust is some 15—20 million years and so during this whole period no large isostatic readjustment has apparently taken place.

The second problem to be attacked is whether for a crust in a plastic state we can explain that a second syncline develops on the outside of one of the raised belts adjoining the main syncline; we have already mentioned that this seems to have occurred in the idio-geosynclines in Java and Sumatra, i.e. on the inside of the curved arc. As these last synclines do not appear to lead to an ordinary tectonic belt with great folding and overthrusting of the surface layers as we have discussed up to now (although there is evidence of some folding of these layers) it seems indicated to surmise another phenomenon. In particular it appears unlikely that the whole crust has become plastic here and has started by a plastic thickening.

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[11]) The last line of Table III will probably not occur as it corresponds to a larger value of b than is likely.

In view of these facts it seems reasonable to suppose that the crust was near the limit of plastic flow before the wave developed and that during this development the part of the cross-section where the bending caused a decrease of the compressive stress remained in the elastic state while the part where the compression increased became in its entirety subject to plastic flow (fig. 7). We denote the limit of plastic flow by σ_v and the stresses at surface and lower boundary by σ_0 and σ_1.

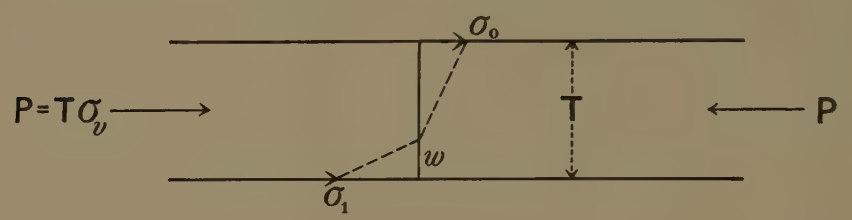

Fig. 7.

The lengthening per unit of length at the lower boundary has an elastic character and can, therefore, be written

$$\frac{\sigma_v - \sigma_1}{E}$$

while the speed of the shortening at the surface is given by

$$\frac{\sigma_0}{2\eta}$$

and, in case σ_0 remains constant, the total shortening by

$$\frac{\sigma_0 t}{2\eta}.$$

We have two conditions for σ_0, σ_1 and w. In the first place the cross-section must remain plane, and, therefore, if we neglect the thickening of the cross-section because of the plastic flow in the upper half

$$\frac{w}{T-w} = \frac{\sigma_v - \sigma_1}{\sigma_0} \cdot \frac{2\eta}{Et}. \tag{19}$$

In the second place the total integral of the stresses over the cross-section added by the bending must be zero. Denoting the increase of the thickness T of the crust because of the flow by s, this condition gives

$$\tfrac{1}{2}(\sigma_v - \sigma_1)\, w = \tfrac{1}{2}(\sigma_0 - \sigma_v)(T-w) + s\,\sigma_0$$

and as

$$s = \frac{\sigma_0 t}{2\eta} \cdot \tfrac{1}{2}(T-w)$$

we find

$$\sigma_v - \sigma_1 = \frac{T-w}{w}\left(\sigma_0 - \sigma_v + \frac{\sigma_0^2 t}{2\eta}\right). \tag{20}$$

Eliminating $\sigma_v - \sigma_1$ from (19) and (20) we obtain

$$\left(\frac{w}{T-w}\right)^2 = \frac{\sigma_0 - \sigma_v}{\sigma_0} \cdot \frac{2\eta}{Et} + \frac{\sigma_0}{E} \tag{21}$$

in which the second term of the right member is brought along by the plastic thickening of the crust.

Taking the X axis horizontal and the Z axis vertical and denoting the deviation of the crustal axis from its original position as it is caused by the bending by z we find for the momentum $\sigma_v Tz$ of the bending-stresses of the crustal cross-section (neglecting s)

$$M = \sigma_v Tz = \tfrac{1}{2} w (\sigma_v - \sigma_1) \cdot \tfrac{2}{3} T = \tfrac{1}{3} wT (\sigma_v - \sigma_1). \tag{22}$$

For the curvature of the crust's axis caused by the bending we obtain

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{w} \frac{\sigma_v - \sigma_1}{E}. \tag{23}$$

Combining (22) and (23) we get the differential equation for the curve of the crust's axis

$$\frac{\partial^2 z}{\partial x^2} = \frac{3\,\sigma_v}{w^2 E} z \tag{24}$$

which has the same shape as the equation for buckling of the crust in the elastic state provided $w = \frac{1}{2} T$, which in the elastic state is of course true. The half wave-length L of the curve is given by

$$L^2 = \pi^2 \frac{z}{\frac{\partial^2 z}{\partial x^2}} = \frac{\pi^2}{3} w^2 \frac{E}{\sigma_v}. \tag{25}$$

The distance w is provided by (21); we see that the wave-length L is proportional to it.

Examining (21) we find that for small values of t we obtain large values of w and L. These values become smaller for increasing time till for t infinite we should get the following limits

$$w = \frac{T}{1 + \sqrt{\frac{E}{\sigma_v}}} \tag{26a}$$

$$L = \frac{\pi}{\sqrt{3}} \frac{T}{1 + \sqrt{\frac{\sigma_v}{E}}} \tag{26b}$$

which for

$T = 30$ km
$E = 1.000.000$ kg/cm^2
$\sigma_v = 3000$ kg/cm^2

give

$w = 1.58$ km
$L = 52$ km

We may conclude that there must be a tendency towards a shortening of the wavelength with time but we can not assume a gradual shortening as mentioned here because such a shortening would imply a continual

change of σ_0 and so our formula of p. 40 for the shortening at the upper boundary of the crust of $\sigma_0 t/2\,\eta$ would no longer be valid; this would invalidate all the deductions.

Formulas (21) and (25) lead to the further important conclusion that waves of this type can not originate without being started from the outside by an effect enforcing a crustal deformation. This must continue for such a time-interval that the first term of the right member of (21) gets down to an acceptable value. In order to obtain an idea about this question we may point out that for

$$\eta = 10^{24} \text{ poises}$$
$$E = 1.000.000 \text{ kg/cm}^2$$
$$\text{and } t = 100.000 \text{ years}$$

we get

$$\frac{2\,\eta}{E\,t} = 0.64.$$

This would lead to reasonable values of w and L. Putting e.g.

$$\sigma_0 - \sigma_v = 120 \text{ kg/cm}^2$$

we obtain

$$w = 4.3 \text{ km}$$
$$\text{and } L = 140 \text{ km}$$

For smaller values of $\sigma_0 - \sigma_v$ the value of t could of course be smaller.

Resuming we may conclude that this type of wave can only come into being by an outside disturbance working for a considerable time. This is of course the case for our problem. The lifting by the main belt's buoyancy of the adjoining crustal belts must have a disturbing effect on the crust outside these belts and if this crust would fulfill the condition of being near the limit of plastic flow it seems probable that a downward wave of the type here described would develop. As the crust in this case is not entirely plastic it appears unlikely that such a wave would lead to a complete giving way and to a down-bulging of the crust as occurs in the main tectonic belt and this is in good harmony with what is known about the idio-geosynclines of Java and Sumatra. As this type of wave can only come into being in the special condition mentioned, it is likewise clear that it does not occur everywhere near a tectonic belt.

We may point out here the possibility that such waves will also originate when the compression σ of the crust is not exactly equal to σ_v but slightly below it. Our formulas then become somewhat more complicated but the principle remains the same. We may safely assume that such waves will be more difficult to bring about for σ differing more from σ_v.

We may lastly remark that the tendency towards a shortening of the wave-length in a later stage is perhaps discernible in the idio-geosyncline

in Java, where in the area south of Surabaya it shows clearly a division in two synclines by a secondary ridge in the middle. A similar breaking up in smaller synclines appears to occur in the Sumatra idio-geosyncline.

Resuming our investigation we may conclude that a plastic state of the crust may provide us with a good explanation of a crustal down-bulging in the main tectonic belt and that it does not make it impossible to explain the originating of the idio-geosynclines of Java and Sumatra. In view also of the other advantages there appears, therefore, good reason to accept this hypothesis.

As we have already mentioned it enables us to come to a better understanding of the complicated pattern of the tectonic belts. BIJLAARD has already pointed this out in 1935[12]) and shown this to be the case[13]) for the whole tectonic area east and southeast of Asia including Indonesia. The writer shall deal here with the latter area for which his views slightly differ from BIJLAARD'S treatment although in many regards he shall follow the same lines.

Examining the map we see that the direction of the belt of negative anomalies west of Sumatra follows one of the two directions of the world's shear-pattern already mentioned on p. 30 and supposed to have been brought about by a shift of the poles in an early stage of the crust's history. The same is approximately true for the belt following the eastcoast of the Philippine Islands. In both areas the gravity anomalies, the seismic movements and the topography are in harmony with the view that the main component of the relative movement of the crustal blocks of Indonesia with regard to the outside crust is parallel to these parts of the belt and that these blocks only show a slight overriding over the outside crust combined with a small compression. It is not difficult to understand that the tectonic belt in part follows pre-existant zones of crustal weakness. The character of the relative movement of the crustal blocks points to a direction of this movement in the sense of the arrows, one for the western and one for the eastern half of the archipelago; the two directions nearly coincide and perhaps the small divergence is not more than an error of interpretation.

The part of the anomaly-belt south of Java differs from the crustal shear-pattern. Here the hypothesis of BIJLAARD, dealt with in our paper can give a good explanation. As it has already been mentioned the direction of the belt of plastic deformation for our case of a single stress-direction may be expected to make an angle α of 35° with the cross-section on which this stress works, i.e. an angle of 55° with the stress-direction itself.[14]) If, besides this main principal stress ϱ_1, there is a second smaller principal stress ϱ_2 working in a crustal cross-section at

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[12]) BIJLAARD 1935.

[13]) BIJLAARD 1936, see e.g. map on p. 20.

[14]) BIJLAARD 1931, 1935 and 1936 and many other papers, ITERSON 1943 and other papers.

right angles to that of the stress ϱ_1, the angle α has a different value for which BIJLAARD derived the formula

$$\cos 2\alpha = \frac{1}{3}\frac{\varrho_1 + \varrho_2}{\varrho_1 - \varrho_2} \tag{27}$$

which for $\varrho_2 = 0$ gives $\alpha = 35°$ (i.e. 55° with the stress ϱ_1 itself).

As the map shows this value of α agrees well with the belt south of Java at least with the western part of it. Towards the east the belt curves slightly and α diminishes towards a value of about 20°. This may perhaps be explained by the direction of flow which we may expect in the belt in the beginning of the phenomenon when no great down-bulging has yet occurred; according to BIJLAARD the flow in the belt, if free to do so, would make an angle of $\alpha = 35°$ with the stress; this direction is indicated in the map by the dotted arrow. This flow-tendency must probably give rise to a secondary stress in the sense of the belt gradually increasing towards the east. According to formula (27) a value of $\varrho_2 = 0.4\ \varrho_1$ would bring down the value of α to 20°.

The eastern half of the archipelago shows a much more complicated pattern of the tectonic phenomena. From north to south we have two belts of strong negative anomalies of which the southern one is the prolongation of that south of Java. The northern one ends over east Celebes; it shows still stronger negative values of the gravity anomaly, viz — 204 mgal against about — 130 mgal in the southern belt. We must, therefore, assume that it means a greater shortening of the crust; it is, however, more irregular in the sense of its axis. Its disappearance in Celebes must lead us to the supposition that at least in the northern part of the archipelago a fault-zone must exist dividing the eastern from the western Indonesian block; along this fault-zone a strong relative movement must have taken place.

We shall not further elaborate this problem and only mention a study by the writer published in Gravity Expeditions at Sea, Vol. IV, p. 31 e.s., where he is led to suppose a major fault-zone in one of the two shear directions already mentioned, running over S.E. Celebes towards the NNW and ending at the eastern end of the Himalayas. The crustal shortening of the second Indonesian belt here discussed would thus represent the continuation of that of the Himalayan zone. The SSE movement of the eastern Indonesian block would have about the same direction as this fault-zone; it is given by the arrow of the map.

Returning now to our subject we see on the map that the northern and southern belts show similar directions making an angle in the middle and that in both cases these directions from both sides are enclosing angles of 55° with the sense of the force as given by the arrow. So here again the hypothesis of a plastic beginning of the crustal deformation may give a satisfactory explanation of the course of the two belts.

If we can accept this also for the southern belt or, in other words,

if we have no accidental coincidence here, we can draw the important conclusion that the course of this belt would not primarily be determined by the shape of the Australian continent as has often been assumed. This view would be in harmony with the opinion given by many geologists that the whole area of the Banda Sea has continental character and that this sea has fairly recently been formed by the sinking down of the crust.

The further course of the southern tectonic belt to the north of the part here considered, i.e. from the Key Is towards the eastpoint of Ceram is given by the NNW direction of the crustal shear pattern and adjoining this part we get the Ceram-Buru stretch which may perhaps again be considered to make an angle of 55° with the arrow. Near Buru, where the negative anomalies become weak, it begins to deviate from this direction.

The northern belt of negative anomalies practically stops towards the north where the part considered above comes to an end.

Resuming we may say that the intricate pattern of the tectonic belts east of the supposed major shear-zone through S.-E. Celebes and the Mangkalihat Peninsula can for the greatest part be well explained by a combination of the crust's shear-zone pattern and the direction with regard to the force in which plastic deformation will be likely to occur.

There is still one part of the southern belt not yet discussed, viz. that between W. Sumba and E. Timor. In the same way as it is the case for the part further to the east which we have already examined, the inner Banda arc is more regular here than the outer arc where we find the belt of negative anomalies and the great crustal deformation. More than one supposition can be made to explain the curve of the latter belt in the Sumba-Timor area but the writer thinks that the data are not yet sufficient for coming to a satisfactory conclusion.

Resuming our investigations we may say that BIJLAARD'S hypothesis about the crust's deformation in the tectonic belts having from the beginning a plastic character can explain many questions otherwise difficult to account for.

Summary.

In comparing the hypothesis that the great deformations of the Earth's crust in the tectonic belts begin by having an elastic or a plastic character the writer comes to the same conclusion as formerly BIJLAARD already arrived at that the latter is more likely. By assuming a deformation of the plastic kind throughout the whole phenomenon a satisfactory explanation can be given of the downbulging of the crust which the belts of negative gravity anomalies lead us to suppose in these belts. At the same time we can thus explain the intricate pattern of the belts in Indonesia and it appears likely that this is also possible in other orogenic areas.

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Indonesian Archipelago. Gravity anomaly map.

CRYSTAL STRUCTURE OF δ-BENZENE HEXACHLORIDE

BY

A. J. VAN BOMMEL, BEELDJE STRIJK AND J. M. BIJVOET

(Communicated at the meeting of December 17, 1949)

In continuation of our structure determination of GAMMEXANE, [1]) the γ-isomer of benzene hexachloride, the crystal structure of the δ-isomer has been determined by *X*-ray analysis.

Rotation diagrams of the δ-isomer around the three axes and zero-layer WEISSENBERG diagrams around [010] and [100].

Cell dimensions: $a = 9{,}64\ A$; $b = 8{,}73\ A$; $c = 14{,}09\ A$; $\beta = 118°$; space group $P\,2_1\,c$ from absences, four molecules pro cell.

By systemetical combination of the 10 highest independent PATTERSON maxima of the [100] PATTERSON projection it was possible to determine the rough y and z coordinates of 5 of the six chlorine atoms.

Repeated FOURIER syntheses gave the [100]-FOURIER projection of fig. 1 and satisfactory agreement between calculated and observed intensities.

The model with chair-form of the carbon ring and the $\varepsilon\varkappa\varkappa\varkappa\varkappa\varkappa$ configuration of the chlorines appears to fit this projection, ε chlorine atom marked *A* in the figure. [2])

By this projection and the supposed model (fig. 2) there remains as unknown parameter the x of the molecule centre; after the latter had been roughly determined by trial and error, repeated [100] FOURIER syntheses gave fig. 3.

Further refinement of the coordinates is in progress.

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[1]) G. W. v. VLOTEN, CH. A. KRUISSINK, B. STRIJK and J. M. BIJVOET. Nature **162**, 771 (1948). Acta Cryst. in press.

[2]) We owed HASSEL the private communication that by electron diffraction gaseous benzene hexachloride was found by him to be of the $\varepsilon\varkappa\varkappa\varkappa\varkappa\varkappa$ type. In the meantime all benzene hexachloride isomers have been determined in the gaseous state by HASSEL and coworkers. Research **2**, 248 (1949).

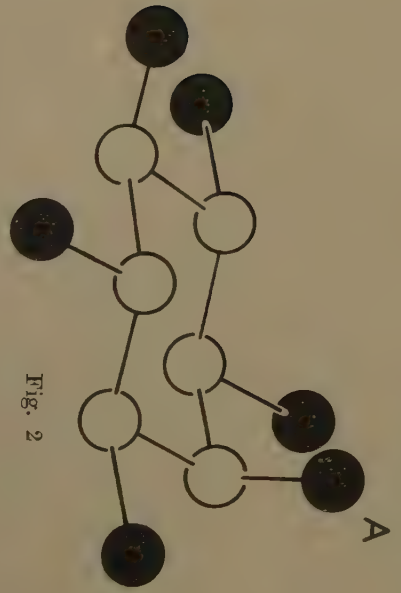

Fig. 2

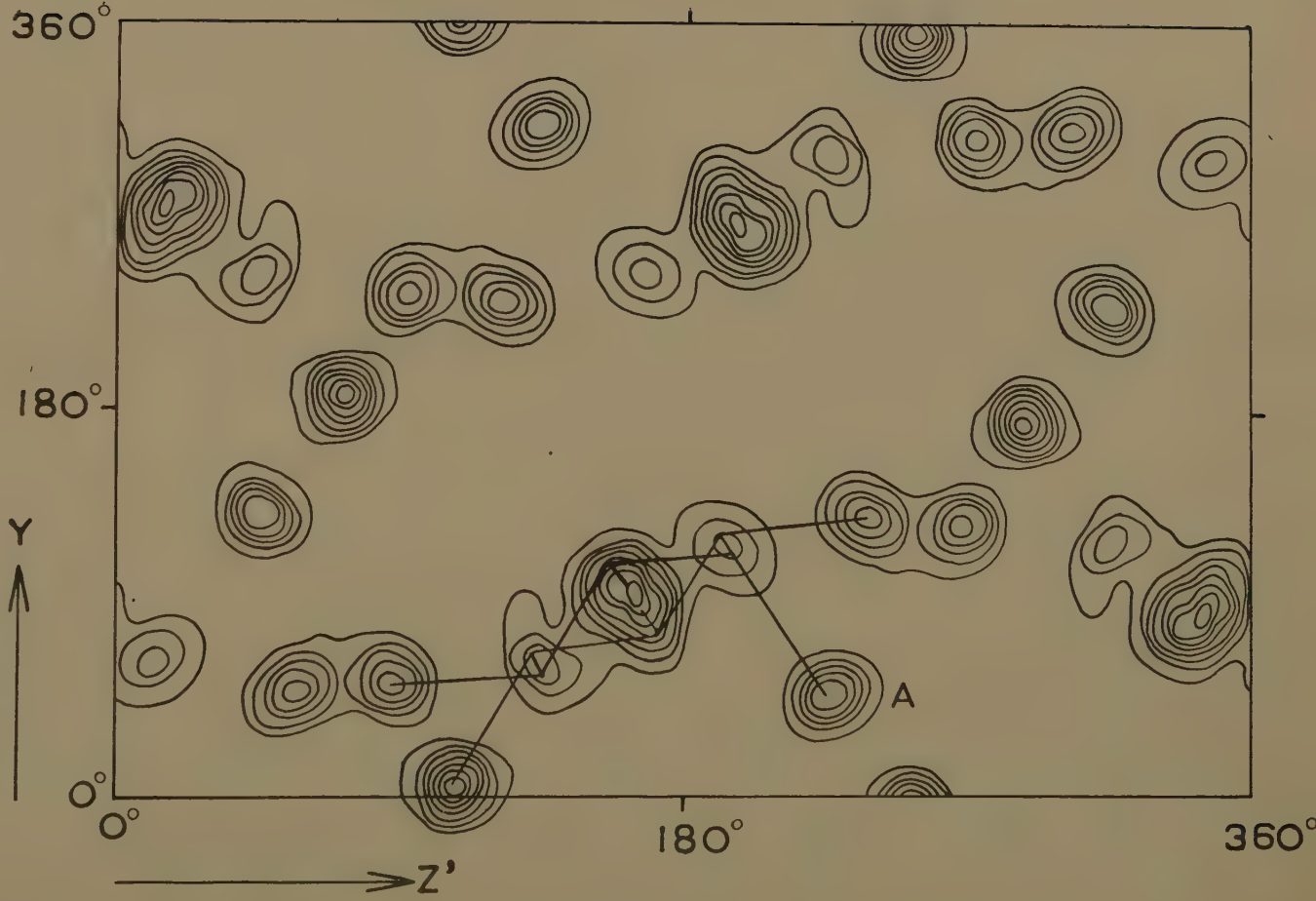

Fig. 1

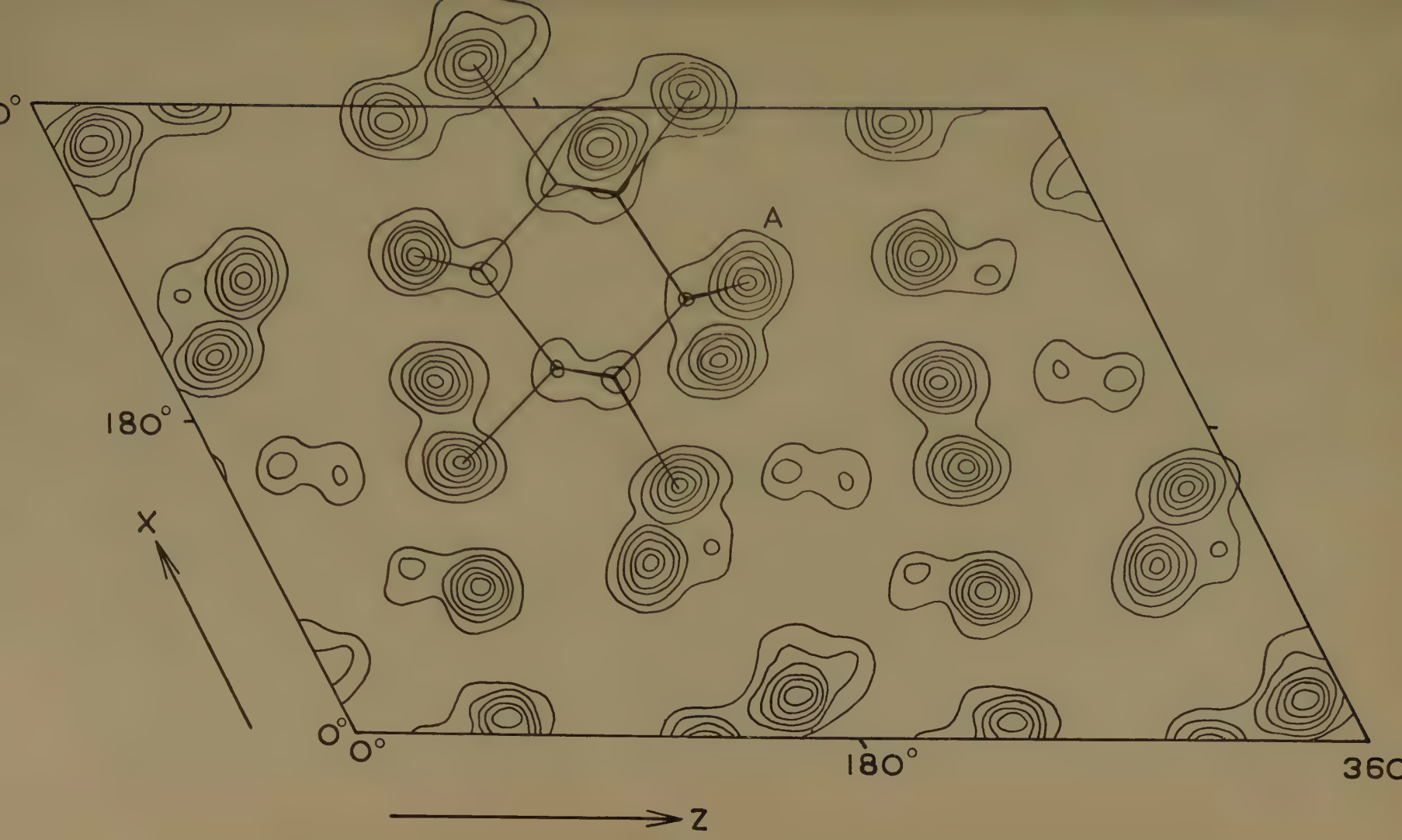

Fig. 3

RELATIVITÉ ET STATISTIQUE

PAR

ANDRÉ MERCIER

(Communicated by Prof. J. CLAY at the meeting of Dec. 17, 1949)

Resumé.

Le présent travail, de nature épistémologique et méthodologique, a pour but de montrer, par une analyse des théories existantes, que l'effort de généralisation de la mécanique classique basé sur la relativité (dans le but d'obtenir une véritable dynamique et non seulement un cadre cinématique relativiste pour une dynamique), tout comme l'effort de généralisation de la même mécanique basé sur la canonicité, fournissent deux procédés de quantification. Dans le titre du travail, il faut comprendre plus spécialement ce qu'il y a de canonique dans la statistique. Or, du fait généralement ignoré que le procédé canonique est lié à une exigence d'ordre statistique à l'endroit des cellules necessairement finies de l'expace de phase (BOLTZMANN-EHRENFEST), on peut soupçonner une double origine, relativiste ou statistique, du quantique. Les deux principes correspondants auraient alors quelque chose de commun (que l'on trouve au niveau très élevé du principe philosophique d'absence de raison suffisante), et leur superposition dans le calcul d'une prévision physique ne va pas sans danger.

1. *Relativité et canonicité, caractères des deux adaptations de la mécanique à la description des phénomènes d'électricité atomiques.*

La mécanique ne se prêtant pas à la description correcte des phénomènes électriques, on a cherché à l'y adapter. De cette recherche sont nées deux théories; l'une, théorie de la relativité restreinte, s'appuie sur le caractère de relativité de la mécanique en en modifiant la description formelle (abandon des transformations de GALILÉE au profit de celles de LORENTZ: l'autre, théorie des quanta, s'appuie sur son caractère de canonicité en en modifiant l'applicabilité (substitution de l'algèbre opératorielle ou matricielle à l'algèbre ordinaire).

Cependant la théorie de la relativité restreinte n'est pas en soi une mécanique complète, parce qu'elle n'emploie la notion correspondant à celle de force que d'une manière boiteuse, et pour la compléter, il fallait introduire l'essence dynamique manquante. Suivant EINSTEIN, ce qui avait été perdu était l'action particulière pour la description unique de laquelle NEWTON avait créé la mécanique, à savoir la gravitation. On sait comment EINSTEIN rétablit celle-ci, créant une nouvelle cosmologie. La théorie de relativité générale est souvent présentée comme un

prolongement nécessaire de la relativité restreinte, la raison se trouvant dans la tensorialité riemannienne. Cependant la mécanique non relativiste est déjà maîtrisée par le tenseur, mais par le tenseur cartésien seulement; personne n'a estimé devoir généraliser cette mécanique qui opère avec un seul temps; en réalité, le désir de rétablir la gravitation a joué un rôle décisif dans l'établissement d'une relativité généralisée.

Dans les tentatives d'unification des théories quantique et relativiste, seul le désir de réaliser la variance lorentzienne correcte a joué un rôle sérieux. Il est pour le moment tout à fait justifié de faire abstraction d'une intégration de la gravitation dans la théorie des particules et atomes, ne serait-ce que pour la raison pratique de l'infinie petitesse de l'interaction gravifique comparée aux interactions électrique ou nucléaire.

Le problème est entièrement celui d'une théorie quantique et relativiste restreinte. Quelles sont, à part la quantification elle-même avec la dualité et l'incertitude ou la transformation de LORENTZ elle-même, les innovations spécifiques qui ont été apportées par la théorie des quanta et par la relativité restreinte? (Et par spécifique, entendons: qui détermine l'intensité des phénomènes plus que leur forme). Du côté de la relativité, c'est l'équivalence de masse et énergie ainsi que le dédoublement de principe en énergies positives et négatives. De celui des quanta c'est l'interaction d'échange. Le spin, lui, nous apparaît actuellement comme le résultat d'une composition du quantique et du relativiste. Il est vrai qu'à l'époque de la découverte d'UHLENBECK et GOUDSMIT on avait trouvé le spin en vertu du désir de respecter la conservation du moment de rotation, mais c'est là déjà une certaine combinaison du quantique avec une relativité provisoire, et à ce niveau, on ne pouvait pas démontrer la conservation, il fallait l'admettre. Par contre, au niveau des transformations de LORENTZ, le spin a surgi de lui-même de la composition du quantique et du relativiste. Il n'est cependant pas démontré que seule cette composition puisse le faire apparaître.

Bien que théorie de relativité et théorie des quanta soient, séparément, bonnes et satisfaisantes, leur composition nous met en présence d'antinomies qu'on cherche à soulever par des artifices plus ou moins cachés (que les plus astucieux se plaisent à découvrir), si bien qu'en fin de compte les antinomies persistent toujours. Dans cette composition néanmoins, le spin qui en surgit n'est pas un artifice, c'est une propriété des corpuscules appelée probablement à jouer un rôle décisif dans toutes les théories à venir. C'est lui qui décide des statistiques particulières et peut-être aussi de la distinction des particules élémentaires de celles qui ne le sont pas. De plus, les spineurs ont montré que tout ne s'arrête pas à la notion de tenseur d'ordre entier, si bien que le problème de la généralisation de la relativité a été repoussé à un autre horizon.

Ce qui importe tout spécialement dans l'électrodynamique, c'est le champ électromagnétique. Or ce dernier n'est pas véritablement incorporé

à la mécanique par le seul fait qu'on a relativisé la mécanique. Comme la partie statique du champ n'a pas d'intérêt théorique ici, on peut en vertu des équations de MAXWELL (et de la condition de LORENTZ) se borner à considérer le champ comme une propagation d'ondes, à propos de laquelle l'introduction du vecteur d'onde d'espace-temps $(\vec{\varkappa}, \nu)$ s'impose. Il fallait en faire des systèmes mécaniques, qu'EINSTEIN a découverts et décrits, les photons. La cinématique relativiste ayant fait également de $(\vec{p}, E)$ un vecteur (dont les composantes portent les propriétés de nature dynamique au sens de la mécanique employée jusqu'alors) L. DE BROGLIE a compris qu'il faut également identifier ces deux vecteurs par l'intermédiaire de la constante de PLANCK à propos des particules possédant une masse au repos non nulle, donc dans tous les cas. Donc le photon comme l'onde-corpuscule sont du ressort direct de la relativité. Tel est, sur la base relativiste des exigences posées par l'électricité, l'acte de création d'une mécanique au plein sens du mot. La mécanique classique ne requiert rien de pareil. Elle n'empêche pas qu'on essaie de le faire, et on sait que NEWTON avait déjà tenté de construire des corpuscules lumineux, sans succès d'ailleurs parce qu'il ne parvenait pas à bien décire la réfraction. Mais *ce* qu'il faisait (par opposition à la *manière* dont il le faisait) ne lui était pas dicté par sa mécanique, en particulier pas par la relativité galiléenne à laquelle il se référait, tandis que du moment où l'on se pose le problème de réaliser une dynamique de la lumière dans l'acceptation de la relativité lorentzienne, le photon et l'onde-corpuscule s'imposent.

Au moment où ces découvertes ont été faites et au milieu du développement de la physique proprement dite, les choses pouvaient ne pas paraître si simples; maintenant que nous y revenons avec la curiosité de l'épistémologiste, leur nécessité nous apparaît saisissante. Et nous pouvons résumer en disant qu'il est de ce point de vue nécessaire qu'une dynamique relativiste prenne la forme d'une mécanique ondulatoire. Plus d'un savant s'est étonné que L. DE BROGLIE ait établi les ondes de matière par une méthode relativiste, alors que la mécanique ondulatoire qui a suivi n'est pas relativiste. Les explications données ici fourniront la réponse.

Quant à l'equation de SCHRÖDINGER, établie apparemment dans l'oubli de la base relativiste que devrait avoir une mécanique ondulatoire logiquement correcte, elle fournit simplement une description non relativiste parce que le formalisme de la mécanique dont elle est issue constitute lui-même une théorie non-relativiste. Mais la question n'est pas ici d'obtenir une équation (du type SCHRÖDINGER) de variance relativiste par un procédé en dehors de la relativité. Nous pensons qu'on ne devrait pas chercher à la fois dans la relativité et dans la canonicité le germe d'une nouvelle mécanique. Apparemment, SCHRÖDINGER l'a cherché dans la canonicité parce que, comme il l'a fait lui-même, son

équation semble établie à partir de celle d'HAMILTON-JACOBI. Mais le procédé iconal invoqué dans les raisonnements est en realité issu de l'optique ondulatoire, c'est à dire d'une théorie relevant entièrement de la relativité. D'ailleurs on y pose la fonction ψ proportionnelle à l'exponentielle

$$e^{\frac{i}{h}(\int \Sigma p_x dx - \int H dt)},$$

ce qui constitue aussi une soumission à la relativité.

Ce qu'il y a de canonique dans la dynamique analytique influe bien entendu d'une manière précise sur la mécanique ondulatoire, et parce que cette dernière se borne à être une approximation non-relativiste, c'est la canonicité qui lui décerne le pouvoir de la quantification. Ce déplacement d'influence du relativiste au canonique est une chose singulière qui donne à la mécanique ondulatoire une souplesse d'analyse que ne possède pas la mécanique des matrices qui a dû en chercher une compensation dans la seconde quantification et la théorie quantique des champs. Il est vrai qu'en l'absence de cette souplesse, la mécanique des matrices s'est tracé un chemin admirable pour réaliser le passage du classique au quantique grâce à un énoncé tout à fait précis du principe de correspondance qui n'existe pas, — au moins au sens de BOHR, — dans l'établissement de la mécanique ondulatoire de SCHRÖDINGER. Il serait désirable de pouvoir établir une dynamique relativiste des ondes-corpuscules sans s'appuyer essentiellement sur un procédé canonique. Même dans la théorie quantique des champs où l'on introduit, si possible, une densité lagrangienne relativiste, le procédé qui mène à la quantification est d'origine purement canonique. L. DE BROGLIE par contre ne procède précisément pas ainsi dans la théorie qu'il élabore depuis plus de dix ans: au contraire il procède selon l'esprit même d'une théorie ondulatoire qui s'oppose à une quantification du type canonique menant à la théorie des champs. [1]) Nous tenons pour correct aussi bien le procédé qui se fonde sur la relativité que celui qui part de la canonicité, mais leur confusion pour l'origine d'antinomies ou tout au moins de désaccords fâcheux.

En résumé, malgré la substitution d'un procédé fondé sur la canonicité, il reste que la mécanique ondulatoire est le résultat d'un effort d'adapter la mécanique à la description des phénomènes atomiques par l'application du postulat de relativité, du fait que la fonction ψ de SCHRÖDINGER est le symbole d'ondes-corpuscules qui sont nécessairement du ressort de la relativité.

L'autre modification apportée à la mécanique, celle qui a amené à la mécanique quantique des matrices, ne se fonde en aucune manière

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[1]) Ces lignes ont été écrites avant qu'ait paru l'article de L. DE BROGLIE dans le volume jubilaire d'ALBERT EINSTEIN (Review of Modern Physics, 21, 345, (1949)).

sur un postulat de relativité, par contre elle suit un chemin clairement tracé par la correspondance.

L'exemple des relations d'incertitude illustrera la distinction d'appartenance des deux théories. L'incertitude $\triangle p \triangle q \approx \hbar$ est une conséquence de la canonicité; si la mécanique modifiée à partir de la canonicité était en même temps le fait d'une exigence relativiste, on devrait trouver une incertitude $\triangle E \triangle t \approx \hbar$. Or il n'en est rien, on ne peut pas établir $\triangle E \triangle t \approx \hbar$ au moyen du formalisme canonique, car il faudrait pour cela que l'énergie et le temps aient un commutateur non nul, et alors, en vertu d'un théorème de DIRAC [2]), l'énergie et le temps auraient des spectres continus, ce qui serait en contradiction avec l'expérience. D'ailleurs, déjà en mécanique non quantique, le formalisme canonique ne peut pas donner lieu à la relation de réciprocité relativiste entre le temps et l'énergie, comme le montre une analyse rigoureuse. [3])

2. *Les postulats de relativité et de statistique.*

L'isomorphie de la mécanique ondulatoire et de la mécanique quantique est connue. Comment se fait-il qu'on arrive à deux théories isomorphes en se fondant sur deux caractères de la mécanique qui semblent, au premier abord du moins, n'avoir rien de commun? Il est peut-être important de rappeler que les deux théories, si elles sont isomorphes, ne sont pas identiques: on ne peut les unifier sans l'aide d'une opération singulière (fonction delta, J. VON NEUMANN).

Aux issues qui ouvrent le passage des conceptions classiques vers ces deux théories, on trouve soit la dualité onde-corpuscule comme idée maîtresse, soit le quantum d'action en tant que mesure minimum des cellules de l'espace de phase. Mais d'un côté la dualité n'est rien sans l'exigence relativiste qui est à sa base: NEWTON avait déjà tenté la construction d'ondes-corpuscules sans parvenir cependant à décrire correctement la réfraction: ce n'est qu'en postulant les transformations de LORENTZ au lieu de celles de GALILÉE qu'on fait suivre aux photons le chemin réel des rayons lumineux. Dans ce fait, il faut voir d'ailleurs une justification aussi remarquable des transformations de LORENTZ que celle qui a donné raison à la formule relativiste de variation de la masse sur celle du modèle d'ABRAHAM. Ici, comme dans l'application des équations de MAXWELL, les transformations de LORENTZ ont l'avantage décisif sur celles de GALILÉE; cet avantage ressort d'expériences directes. Par contre, dans le fait que l'hypothèse des transformations de LORENTZ (ou si l'on veut la soumission des équations de MAXWELL à la covariance [4]))

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[2]) P. A. M. DIRAC, Les principes de la mécanique quantique, (trad. francaise, Paris, 63, (1931)).

[3]) A. MERCIER et E. KEBERLE, Archives des Sciences, 2, 186, (1949).

[4]) Nous nous proposons de publier un mémoire sur les fondements de l'électrodynamique, dans lequel la théorie de la relativité restreinte n'apparaîtra pas, au contraire de l'habitude, comme fondée sur le postulat de l'équivalence des systèmes

conduit nécessairement à la dualité quand elle est imposée à la mécanique, [5]) il n'y a plus rien qui permette de justifier le choix de ces transformations parmi d'autres: admettre leur justesse est un a priori qui donne le ton à la théorie consécutive. Si donc on parle de ce acte comme d'un postulat de relativité, ce postulat devient fondamental par son caractère de priorité.

D'un autre côté, on dit facilement que c'est la théorie des quanta qui a attribué une extension finie aux cellules de l'espace de phase. [6]) Cependant, il est possible d'invoquer à cette fin la statistique elle-même. [7]) Cela est d'une grande importance, puisqu'il en découle que la nécessité d'une quantification dans le cadre du formalisme canonique est liée à l'exigence d'une interprétation mécanique (statistique) de la thermodynamique. [8]) Là de nouveau, l'idée, maîtresse de la statistique n'est pas soumissible à un contrôle expérimental. Cette idée réside dans la dénombrabilité, notion a priori qui, précisée par BOLTZMANN et les grands statisticiens, est peut-être l'expression seule actuellement correcte de l'atomisme. [9]) La proposition de la statistique qu'on adopte alors sur la base de cette dénombrabilité étant qu'à des cellules égales correspondent des probabilités égales, elle a le même caractère d'a priori que celle qui, ailleurs, faisait adopter les transformations de LORENTZ. Elle constitue un postulat de statistique ayant même priorité que celui de relativité.

Il est alors concevable qu'on en vienne à admettre, comme le fait KEBERLE (loc. cit.) une appartenance initiale ou d'ordre supérieur commune à ces deux postulats.

Avant d'en venir là, préparons les explications finales par quelques remarques.

3. *Remarques.*

Tout d'abord, selon qu'on considère la mécanique des matrices ou la mécanique ondulatoire, on a le cas des relations d'incertitude ou celui de la dualité. Mais la complémentarité n'a lieu qu'en vertu de l'effort

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galiléens combinée à l'universalité de la vitesse de la lumiere, mais sur la covariance des équations de MAXWELL et sur l'hypothèse presque évidente que la constante d'influence électrique (ou constante diélectrique du vide) ε_0 est une constante universelle.

[5]) Cette nécessité que nous avons expliquée plus haut, ne semble pas avoir été reconnue pleinement jusqu'ici.

[6]) Au hasard des ouvrages, citions Saha et Srivastava, A textbook of Heat (Allahabad, 649, (1931)).

[7]) Voir P. et T. EHRENFEST, Enzyklop. der math. Wissenschaften, IV, 4, note 110 au bas de la page 36.

[8]) Pour la démonstration, voir E. KEBERLE, Thèse de doctorat (Berne 1948).

[9]) On pourrait objecter que la seconde quantification fournit de fait la notion de particule, et que c'est elle qui, par là, donne l'expression de l'atomisme. Cependant, il n'y a pas de seconde quantification sans la première, qui elle-même vient d'être ramenée à l'exigence de dénombrabilité de la statistique.

d'adapter les interprétations historiques (classiques, non-quantiques....) aux exigences d'une théorie quantique nouvelle: d'une part la description spatio-temporelle qui est l'essence d'une relativité, d'autre part la description canonique qui est celle d'une évolution où le temps est seule variable indépendante.

Ensuite, nous avons des raisons pour dire que la mécanique ondulatoire, bien qu'issue historiquement de l'équation d'HAMILTON-JACOBI, n'a pas été créée à partir de la mécanique analytique en tant que formalisme canonique uniquement sans qu'un certain effort de relativisation [10]) soit de la partie. On ne le voit pas dans les procédés ordinairement exposés qui partent de l'équation d'HAMILTON-JACOBI ou autre employant l'action, parce que c'est justement un procédé de ce genre qui laisse à désirer au point de vue formel (formel: par opposition au procédé iconal du passage de l'ondulatoire au géométrique). Nous avons montré récemment [11]) que l'équation de SCHRÖDINGER est une traduction de ce qu'on appelle la condition accessoire du formalisme canonique homogénéisé et non pas celle de l'équation d'HAMILTON. Or l'homogénisation est une tentative du genre relativiste, ce qui ne veut pas dire qu'elle satisfasse aux exigences formulées d'une manière précise dans la transformation de LORENTZ. [12]) Elle n'y satisfait pas puisque même pour un seul corpuscule, énergie et temps ne sont pas canoniquement conjugués (compte tenu du signe —). Aussi ne peut-on s'empêcher de voir là une certaine incompatibilité entre relativité et canonicité. Il faut entendre par là qu'on ne peut sans danger, peut-être même sans contradiction, imposer à la fois des conditions relevant d'une part de la canonicité, de l'autre de la relativité; les premières conduisent à énoncer un postulat d'ordre statistique par l'invariance adiabatique qu'elles impliquent, les autres relèvent de la géométrie d'un espace-temps ou coordonnées et temps sont unifiés, elles doivent mener à un postulat qui, s'il implique une invariance, ne peut le faire que dans ledit espace-temps.

Enfin, on remarque dans le cas de la relativité que si tous les systèmes de référence se trouvent également justifiés, on ne peut faire autrement que de s'en référer à l'un d'eux, quel qu'il soit d'ailleurs. Dans le cas de la statistique, on remarque quelque chose de comparable, mais qui s'exprime de façon opposée: Ce qui, dans la mécanique, prépare la statistique, c'est en connexion avec la canonicité, l'existence de la propriété très générale d'invariance adiabatique; là-dessus repose la possibilité de postuler l'égalité des probabilités des cellules de phases

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[10]) Effort toutefois distinct de celui de la relativité lorentzienne.

[11]) Pour paraître dans les Archives des Sciences.

[12]) Cela n'est d'ailleurs pas en contradiction avec le fait qu'une transformation de Lorentz peut être considérée comme une transformation canonique.

H. KRAMERS a déjà eu l'idée d'un passage de la condition accessoire à l'équation de SCHRÖDINGER, sans en tirer cependant toutes les conséquences (Grundlagen der Quantentheorie, Leipzig 1938, § 17).

égales. Pour pouvoir parler des cellules, il faut qu'elles soient finies, et la quantification en découle en principe. Mais, on doit toujours considérer l'ensemble des cellules (par opposition à l'un des systèmes de référence cités plus haut), et les propriétés sont attribuées à chacune d'entre elles.

Autrement dit: C'est aux idées de relativité et de statistique qu'on peut en fin de compte attribuer l'apparition des nouvelles théories, et la mécanique classique contenait déjà les germes des deux tendances dans la description géométrique et dans la description canonique. En poussant à l'extrême les exigences du type géométrique, on courre le risque d'y perdre la canonicité, et en donnant à l'espace de phase qui repose sur la canonicité la signification si puissante de la statistique, on risque de devoir renoncer à toute implication spatio-temporelle.

Nous pouvons alors résumer, préciser et conclure comme suit.

4. *La double origine de la quantification.*

a) Le postulat de relativité offre la possibilité d'une attribution mutuelle onde-corpuscule, et lorsqu'on l'applique à une description dynamique des phénomènes électriques atomiques, ce postulat a pour conséquence la dualité de L. DE BROGLIE.

Cette attribution mutuelle entraîne la quantification aussitôt qu'il s'agit non pas d'ondes-corpuscules totalement libres mais d'ondes-corpuscules avec couplage (par exemple dans une enceinte de paroi intérieure réfléchissante). Et comme il n'y a pas d'onde-corpuscule totalement libre, on peut dire avoir trouvé un premier procédé, celui qui mène de la relativité à la quantification.

b) Le postulat de statistique qui attribue la même probabilité à des cellules d'égale extension dans l'espace de phase, n'a de sens que si ces cellules sont d'extension finie, si petites soient-elles. Cette dernière phrase („si petites soient-elles") ne peut pas être comprise au sens du passage à la limite où le discontinu se confond en un continu: En appliquant le postulat de statistique au sens d'un dénombrement continu, on ferait un non-sens (comme si on voulait construire une roue dentée ayant une infinité non-dénombrable de dents, ce qui exclurait la transmission même par frottement). L'idée-même de statistique et en particulier la possibilité des cellules d'égale extension est liée au dénombrement (malgré l'impossibilité de fixer exactement les propriétés des éléments à dénombrer). En d'autres termes, le formalisme statistique entraîne de son côté la quantification de principe. Et puisque, grâce aux invariants adiabatiques, la mécanique fournit les éléments d'une base de la thermodynamique (existence de l'entropie), puisque le formalisme statistique est dès lors non seulement suggéré par le formalisme canonique mais en est comme la continuation naturelle, il y a lieu de dire que la quantification est une conséquence (médiate) du formalisme canonique.

Tel est le second procédé, celui qui mène de la canonicité à la quantification.

c) Pour expliquer qu'on puisse, par deux procédés, atteindre à la quantification, rien n'est plus naturel d'admettre que les deux points de départ ont quelque chose de commun qui les rend apparentés. Jusqu'à ce jour, le postulat de relativité et celui de statistique étaient (en dehors du domaine des considérations d'ordre cosmologique dont nous avons fait abstraction), ce qu'il y avait de plus profond dans la construction des théories physiques.

Postulons à un ordre supérieur une parenté des deux postulats. En ce faisant, on explique qu'on puisse en particulier obtenir une théorie des quanta par deux procédés. Mais on ne justifie pas l'application simultanée du postulat de relativité et de celui de statistique; au contraire, il apparaît douteux qu'on puisse avec fruit en faire, à un seul et même objet, l'application simultanée; nous avons de fortes raisons pour le croire; ainsi s'expliquerait l'espèce d'incompatibilité que nous avons relevée entre relativité et canonicité. Il y a plus, on s'explique l'apparition de graves difficultés dans les théories physiques modernes.

C'est une alternative avec exclusion que nous présentent les deux postulats. Ils sont apparentés mais distincts; chacun livre, à sa manière, une théorie.

La question se pose de savoir en quoi réside leur parenté. La réponse se trouve dans le manque de raison suffisante d'en admettre d'autres. [13]) En effet: d'une part (relativité) tous les systèmes de référence sont équivalents dans la référence qu'ils permettent de faire, d'autre part (statistique) toutes les cellules de phase d'égale extension sont équivalentes dans les probabilités qu'on leur attache, — cela, parce qu'il n'y a aucune raison qu'il n'en soit pas ainsi. La différence réside dans le fait que de tous les systèmes de référence, un seul importe dans la description (donc un seul d'entre eux prend à sa charge ce que tous sont censés faire), tandis que c'est toujours l'ensemble des cellules de phase qui domine (chaque cellule abandonne son individualité aux exigences de l'ensemble).

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[13]) Voir E. Keberle, loc. cit. Il est extrêmement intéressant de voir quel rôle peut jouer un principe philosophique tel que celui de raison suffisante dans le domaine si réel des théories physiques.

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP AND PHOSPHATIDE COACERVATES — X [1])

THE INFLUENCE OF NORMAL ALCOHOLS ON OLEATE COACERVATES

BY

H. L. BOOIJ, C. J. VOGELSANG *) AND J. C. LYCKLAMA *), [2]) [3])

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of Dec. 17, 1949)

1. *Introduction.*

It has been demonstrated (BOOIJ and BUNGENBERG DE JONG) that normal fatty anions show a peculiar relation between the number of carbon atoms and their action on an oleate coacervate. An "opening" of the coacervate has been observed, beginning with an anion with 8 C-atoms, having a maximum at 11 C-atoms, a minimum at 15 C-atoms, after which the action increases once more. This phenomenon was explained by assuming a disturbing effect of the fatty anion added, on the oleate micelle, which effect is at a minimum when the length of the fatty anion closely resembles the length of the "substratum" (the oleate within the micelles).

Starting from this investigation it seemed worth while to study the influence of a homologous series of "condensing" substances on an oleate coacervate and we chose for this study the series of normal alcohols. It was already known (BUNGENBERG DE JONG, SAUBERT and BOOIJ) that the first terms of this series (methyl and ethyl alcohol) exhibit an opening action, that propyl alcohol first condenses the coacervate but that a reverse action occurs at higher concentration, while the higher alcohols (as far as they have been investigated) showed only a condensing action. The question we asked ourselves after the study of the fatty anions was, whether the alcohols too would show some extraordinary activity when the chain length was in the neighbourhood of 15 C-atoms.

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*) Aided by grants from the "Netherlands Organisation for Basic Research (Z.W.O.)".

[1]) Publication no. IX of this series will be found in Proc. Kon. Ned. Akad. Wetensch. Amst. **52**, 1100 (1949).

[2]) Our stock of Na-oleate was practically exhausted at the end of the war. A generous gift of oleate from The Rockefeller Foundation provided the means for the experiments described in this paper.

[3]) Publication no. 2 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

2. *Methods and results.*

First of all we used the method which has already been described by Booij and Bungenberg de Jong. To a certain volume of oleate solution we added the substance to be investigated, dissolved the substance (when necessary by gently heating), and measured the amount of KCl needed for the appearance of a coacervate layer of 50 %. Then we see that the concentration of KCl needed is practically always less with the alcohols than with the blank, and the difference between the KCl concentrations in both cases may be given as a measure for the activity of the substance added. The low solubility of the higher alcohols presents some difficulties when applying this method. It was, however, very well possible to measure the influence of the normal alcohols (fig. 1). Indeed,

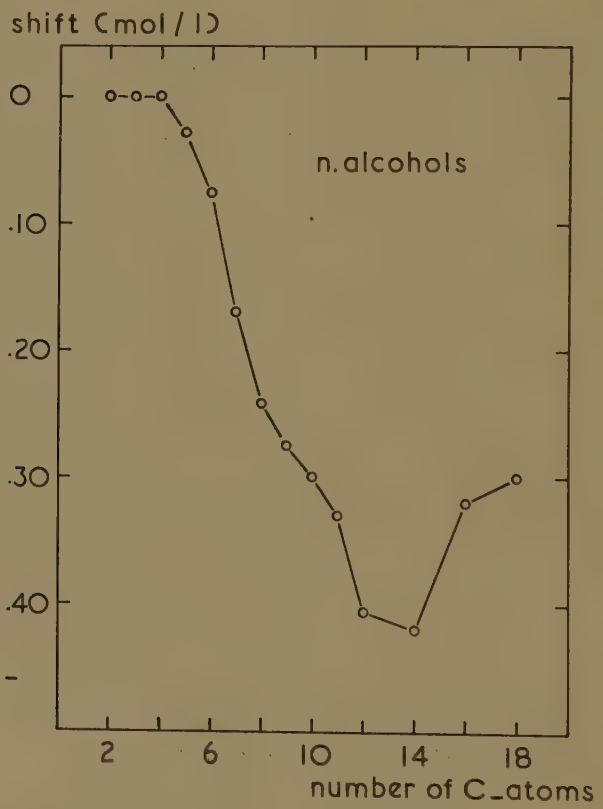

Fig. 1. Condensing action of 0.5 mmol/l of the normal alcohols on an oleate coacervate (ordinate = shift of the coacervation curve expressed in KCl concentration).

at a chain length of 14 C-atoms the activity shows a maximum, decreasing with cetyl and octadecyl alcohol. Of course, the smaller activity of the last mentioned alcohols is not established wholly beyond doubt. It might be possible that the alcohols with 16 and 18 C-atoms

did not completely dissolve, resulting in an apparently lower activity. We had to develop another method to settle this question.

In choosing a new method we made use of the following fact. If we measure the influence of the smaller alcohols we see that propyl alcohol behaves in an extraordinary fashion. First it decreases the volume of a given coacervate (in other words the concentration of KCl needed for coacervation is lowered), afterwards the reverse action takes place (fig. 2). There is a moment (at approximately 0.15 mol/l) when propyl-

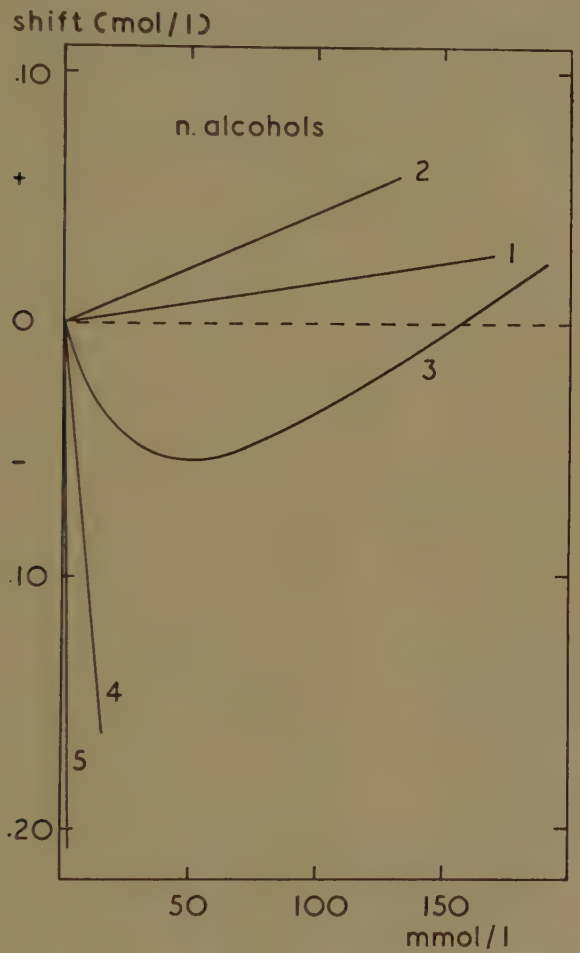

Fig. 2. The influence of the short normal alcohols on the oleate coacervate. Note the curve for *n*-propylalcohol. Abcis = concentration of the alcohol added, ordinate = shift of the coacervation curve expressed in KCl concentration.

alcohol has seemingly no influence on the coacervate. This phenomenon provided us with the necessary tool for developing a new method for measuring the influence of substances difficultly soluble in water.

We start with a small amount of propylalcohol, exactly enough to give an end concentration of 0.15 mol. Then we add the quantity wanted

of a high alcohol which dissoves rapidly in the propyl alcohol. Finally the oleate solution is poured into the flask and thus we get a solution of the high alcohol in the oleate without trouble. This method proved very useful, even for organic substances which are completely insoluble in water.

It goes without saying that the internal structure of the coacervate has undergone some change as a result of the addition of the propyl alcohol. We can see this immediately when comparing fig. 2 and fig. 3. Here the experiments differ only in that in the last case the alcohols investigated were brought into a known amount of propyl alcohol. Then propylalcohol will show only an opening action. The opening action of ethyl and methyl alcohol is strengthened, while the condensing influence of butyl- and amylalcohol is somewhat less.

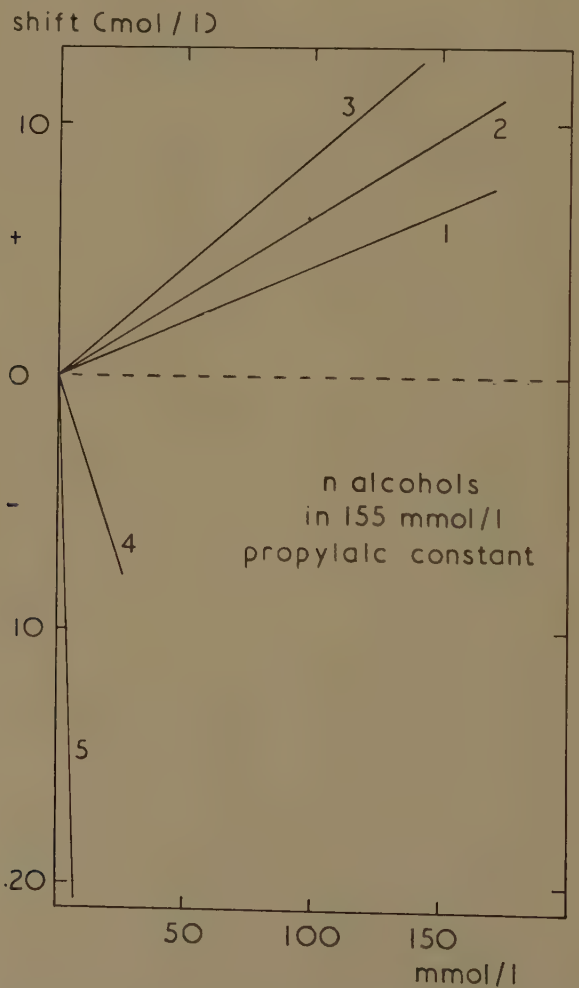

Fig. 3. Influence on an oleate coacervate of the short normal alcohols when previously dissolved in a constant quantity of *n*-propylalcohol. Any opening action is strengthened, any condensing action weakened (cf. fig. 2).

Now it is possible to measure the influence of a whole series of normal alcohols again, with practically the same result as in the more direct method (fig. 4). The maximum of condensing activity at 14 C-atoms appears once more and with this method it can evidently not be ascribed

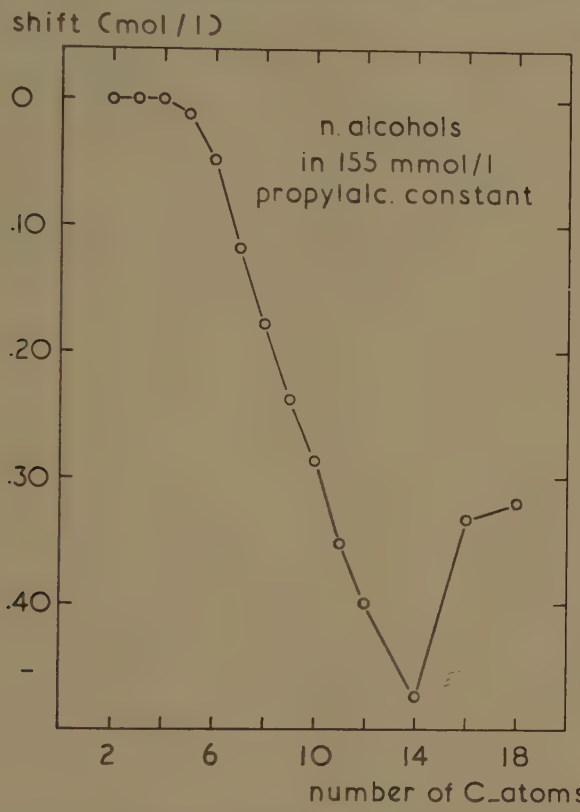

Fig. 4. Condensing action on the oleate coacervate of the normal alcohols (0.5 mmol/l) when dissolved in a known quantity of *n*-propylalcohol (cf. fig. 1).

to a low solubility of the higher alcohols. It seems logical to think of a relation of this phenomenon with the facts which the study of the fatty anions revealed, (the difference between the maximum of activity of the alcohols — 14 C-atoms — and the minimum of the fatty anions — 15 C-atoms — might be the result of the fact that in the latter case one carbon atom is more or less incorporated in the polar group). In the latter case the opening action was seen as a disturbing effect, being at a minimum when the carbon chains of the acting molecule and the substratum are equal. This disturbance must be ascribed to the ionised carboxyl group. When we now introduce into the soap system a substance with the same number of carbon atoms, but with a non-ionised group, the "condensing" influence of the carbon chain will be the dominating force of the molecule introduced. The result will be that this condensing action is highest when the chain is approximately

as long as the length of the oleate (this means, of course, that the "effective length" of the oleate chain is less than 18 carbon atoms). This idea might be illustrated in the following way (Booij, 1949). In a soap solution the soap anions tend to associate; the long carbon chains will be driven out of the water, and the longer the chain, the lower the critical concentration of micelle formation in a homologous series. Two opposing forces make themselves felt in the resulting micelle, a mutual attraction of the carbon chains by London-v. d. Waals forces (B) and a repelling action (A) of the charged carboxyl groups (fig. 5). A sufficient

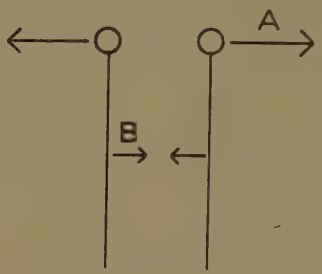

Fig. 5. Opposing forces between oleate anions in the micelle state. The shape of the resulting micelle is dependent on the equilibrium between these two forces.

amount of KCl will decrease the dissociation of the anionic groups so that the type of micelle might change (from small micelles to large micelles of the same thickness) and coacervation results (fig. 6). One might suppose that coacervation becomes possible at a certain size of the micelles. This size is, as a matter of fact, dependent on the relation between the volumes of the part of the micelles in between the flat surfaces (I_1) and the annular part at the outside (I_2, see fig. 6).

Each factor which stabilizes the curved surfaces of the micelles will — at a given amount of soap — shift the equilibrium between I_1 and I_2 towards I_2. This means that more and smaller micelles will appear,

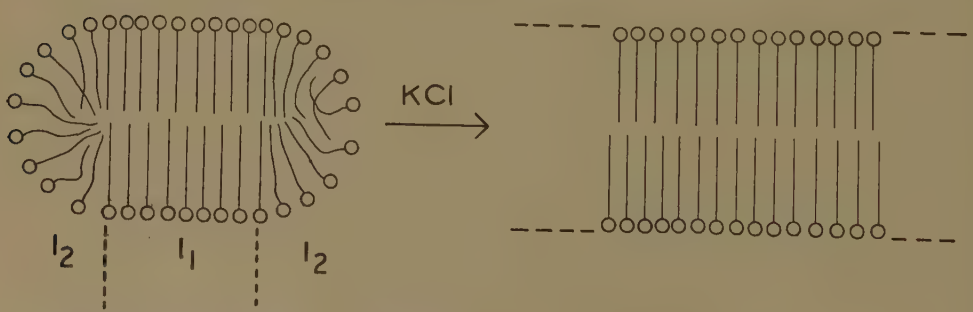

Fig. 6. If force A (fig. 5) is weakened by the addition of a high concentration of KCl the micelles will grow flatter. It is supposed that coacervation results at a certain (statistical) size of the micelles (I_1 = volume of the cylindrical part of the micelle, I_2 = volume of the annular part).

and that consequently more KCl is needed to attain the size of the micelles which is necessary for the beginning of coacervation.

Fig. 7 shows that fatty anions of a shorter chain length may indeed be expected to stabilize the curved surfaces, as here the proportion

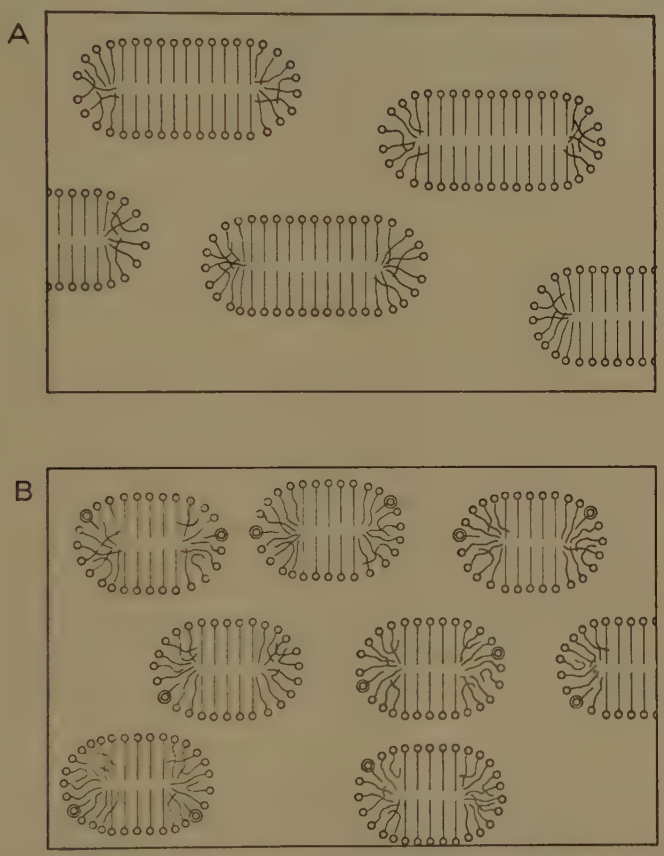

Fig. 7. A certain KCl concentration will give oleate micelles of a certain mean size (A). When adding a shorter fatty anion (at the same KCl concentration) this size will decrease, so that more KCl must be added to give the original dimensions of the micelles. This picture is very schematic; it is likely that the micelles will be much larger when coacervation sets in. For clearness' sake the "heads" of the shorter anions were drawn with a double circle.

between force A and force B (cf. fig. 5) is larger than in the case of the oleate anion. Thus smaller micelles are formed (= opening action) and more KCl is required to coacervate the system.

On the other hand alcohols will show the opposite behaviour as the proportion between forces A and B decreases and now a micelle may consist of more material (the flat area increases). It is obvious that this factor is especially strong when the micelles are of considerable size already. Then the volume of the curved part of the micelles is low as compared with that of the cylindrical part and a slight change may

produce a large effect. Figs. 7 and 8 do not try to give an idea of the real shape of the micelles. We are inclined to think that they are much bigger (so relatively flatter) when coacervation begins.

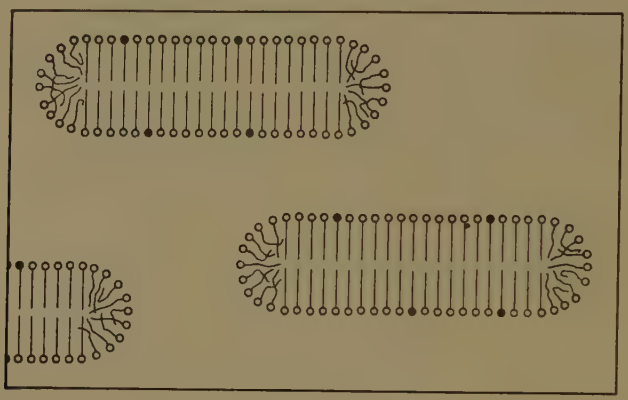

Fig. 8. Starting from the situation of fig. 7A, addition of a long alcohol (chains with black circles) results in larger micelles. In other words, less KCl is needed to give the original situation, which means that the coacervation curve shifts to lower concentrations of KCl.

Force A may be virtually weakened as the introduction of the alcohol separates some negative groups. Moreover, the possibility of an attraction of the OH group to the carboxyl groups must not be neglected.

A further consequence of the hypothesis on the mechanism of action of the alcohols is that we assume that the OH group stays in the polar part of the micelles.

3. *The uptake of alcohols into the oleate micelles.*

It is evident that an alcohol will exert its action only when it has been taken up into the micelle. The same question arose as regards the action of fatty anions on the oleate coacervate (Booij and Bungenberg de Jong). A study of the action of these fatty anions on oleate solutions of different concentration showed that a very large part of caprylate stays in the equilibrium liquid, but that the following members of the homologous series are ever more taken up into the micelles, so that from laurate upwards practically all the anions added will be found within the micelles.

As the polar group of the alcohols is much weaker than that of the fatty anions, it is expected that an alcohol with a chain length of less than 12 C-atoms will already be taken up for 100 % into the micelles. This was proved to be the case indeed (fig. 9). If the alcohol is taken up into the micelles totally, the amount of alcohol needed to produce a certain shift of the KCl line will be proportional to the amount of micelles present (so with the soap concentration). In the other case a

certain equilibrium-concentration of the alcohol will be found. We see that the equilibrium-concentration decreases in the homologous series, to become practically zero with n. octylalcohol. We may conclude that from octylalcohol upwards the alcohols are taken up wholly into the

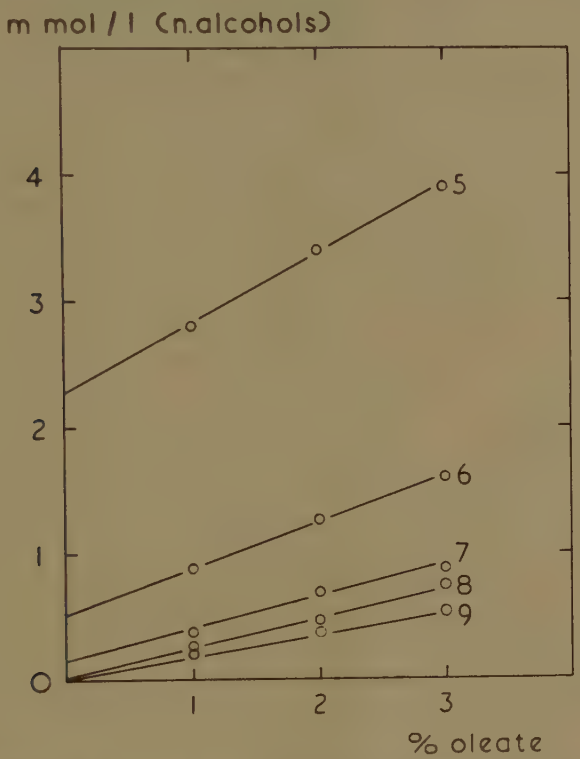

Fig. 9 The quantity of normal alcohols needed to shift the KCl curve a certain amount varies for different concentrations of the oleate solution (standard solutions: 1, 2, and 3 %; in the experiments the concentrations become approximately one quarter of these values). From this diagram one can deduce that the equilibrium concentration of the alcohols (= value for an oleate concentration of zero) decreases with increasing chain length, to become practically zero for *n*-octylalcohol.

micelles. Thus the conclusion is reached that we might explain only the left side of figs. 1 and 4 by the ever increasing amount of alcohol taken up into the micelles. The maximal activity of myristyl alcohol must find its explanation in the fact that its carbon chain fits exactly in the orderly building of the micelles (see preceding section).

A closer comparison of figs. 1 and 4 discloses some typical differences between the two experiments. In the propylalcohol experiment the highest alcohols act somewhat stronger (here the solubility in the soap system alone plays a great part), while the shorter alcohols (5—9 C-atoms) have a smaller condensing action. It might be that the propylalcohol interferes to some extent with the taking up of these alcohols in the

micelles. Another possibility — the internal structure of the micelles is changed by the propylalcohol — must not be excluded, of course.

There is a marked difference in the homologous series of normal alcohols and fatty anions as regards the action of the molecules taken up into the micelles as a comparison of figs. 10 and 9 will show. Fig. 10

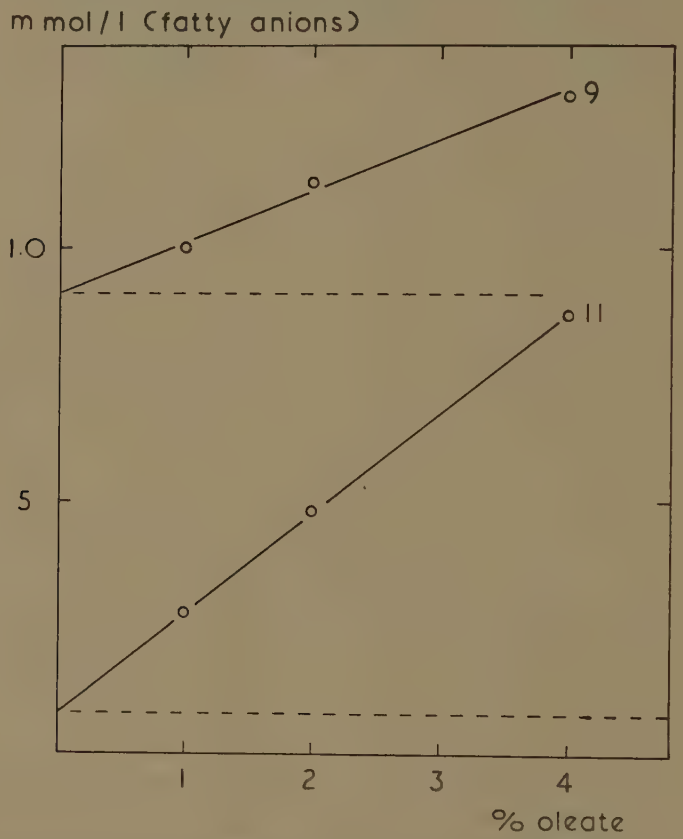

Fig. 10. In the homologous series of the fatty anions the equilibrium concentration decreases with increasing chain length. Note, however, the difference in slope of the lines in figs. 9 and 10.

demonstrates the amounts of nonylate and undecylate required to produce a certain shift of the KCl coacervation curve in different concentrations of oleate (see Booij and Bungenberg de Jong, 1949). Here the lines become steeper at higher lengths of the added ions.

The alcohols, however, do not show this phenomenon. There the lines grow less steep in the homologous series. This difference would be expected indeed from the hypothesis on the mechanism of action (see figs. 7 and 8). In the case of the alcohols the influence on force A and B (fig. 5) works in the same direction. This means that lengthening of the carbon chain will give an ever increasing ("condensing") action, provided the chain will not grow longer than the effective chain length of the

oleate. In fig. 9 this phenomenon is seen as an ever decreasing steepness of the lines; of the molecules taken up less is needed to reach a certain effect when the carbon chain is lengthened.

With the anions the situation is as follows (fig. 7). Introduction of a dissociated carboxyl group gives a strong increase of force A, which will be counteracted by the carbon chain (increase of B). Here the counteraction grows stronger as the carbon chain increases, now giving a minimum of action when the carbon chain is as long as the effective length of the oleate. Thus a shorter anion — when taken up into the micelle — has a stronger ("opening") action than a longer one (before the minimum). In the normal experiments this phenomenon is masked by the fact that the shorter anions are practically not taken up in the micelles.

Summary.

1. The condensing action of the normal alcohols on an oleate coacervate shows a maximum at 14 C-atoms.

2. This maximum of condensing activity agrees very well with the minimum of opening activity of the fatty anions (15 C-atoms).

3. It is explained by assuming that an alcohol with 14 C-atoms fits exactly in the orderly built oleate micelles, thus giving a maximal contribution to the LONDON-VAN DER WAALS forces between the carbon chains.

4. This picture leads to the idea that coacervation is connected with a change of the oleate micelles from small flat micelles with rounded sides to very big flat micelles.

5. Part of the fact that the activity in the homologous series of alcohols increases with the addition of a carbon atom can be explained by the unequal distribution of the alcohols between the micelles and the equilibrium liquid. The smallest alcohols are found in the equilibrium liquid, from octylalcohol onwards the alcohols are taken up into the micelles totally.

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A COHOMOLOGY THEORY WITH HIGHER COBOUNDARY OPERATORS. II

VERIFICATION OF THE AXIOMS OF EILENBERG-STEENROD *)

BY

SZE-TSEN HU

(Communicated by Prof. L. E. J. BROUWER at the meeting of November 26, 1949)

1. *Introduction.*

EILENBERG and STEENROD have axiomatised the concepts of homology and cohomology theories (see [1] [1]) and a forthcoming book of them). Under the restricted sense of EILENBERG-STEENROD, a cohomology theory defined on a class of pairs (X, X_0) and continuous maps is a function $H^m(X, X_0)$ defined for every pair (X, X_0) in the class and every integer $m \geqq 0$ with values in the class of abelian groups such that:

(i) If $f\colon (X, X_0) \to (Y, Y_0)$ is a continuous map in the class, there is an *induced homomorphism*

$$f^*\colon H^m(Y, Y_0) \to H^m(X, X_0).$$

(ii) For each pair (X, X_0) in the class and $m \geqq 0$, there is a *coboundary homomorphism*

$$\delta\colon H^m(X_0) \to H^{m+1}(X, X_0).$$

These concepts satisfy the following axioms.

Axiom I. *If f is the identity map on (X, X_0), then f^* is the identity automorphism on $H^m(X, X_0)$.*

Axiom II. *If $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (Y, Y_0) \to (Z, Z_0)$, then $(gf)^* = f^*g^*$.*

Axiom III. *If $f\colon (X, X_0) \to (Y, Y_0)$, then $f^*\delta = \delta f_0^*$, where $f_0 = f|X_0$.*

The foregoing axioms are sometimes called the *algebraic axioms.*

Axiom IV. (Homotopy Axiom) *If X is a compact Hausdorff space and X_0 is a closed subset of X and if the continuous maps f, g: $(X, X_0) \to (Y, Y_0)$ are homotopic relative to $\{X_0, Y_0\}$, then $f^* = g^*$.*

Axiom V. (Exactness Axiom) *Given (X, X_0) and the identity maps*

$$i\colon (X_0, \phi) \to (X, \phi), \quad j\colon (X, \phi) \to (X, X_0),$$

*) This work was done under Contract N7—ONR—434, Task Order No. III, Navy Department, Office of Naval Research, U.S.A.

[1]) Numbers in brackets refer to the bibliography at the end of the paper.

the groups and homomorphisms

$$H^0(X, X_0) \xrightarrow{j^*} H^0(X) \xrightarrow{i^*} \ldots \xrightarrow{j^*} H^m(X) \xrightarrow{i^*} H^m(X_0) \xrightarrow{\delta} H^{m+1}(X, X_0) \xrightarrow{j^*} H^{m+1}(X) \xrightarrow{i^*} \ldots$$

form an exact sequence, [3, p. 687], *called the cohomology sequence of the pair* (X, X_0).

Axiom VI. (Excision Axiom) *If* U *is an open set whose closure is contained in the interior of* X_0, *then the identity map*

$$\mu\colon (X-U,\ X_0-U) \leftarrow (X,\ X_0)$$

induces, for each m, *an isomorphism onto:*

$$\mu^*\colon H^m(X, X_0) \rightarrow H^m(X-U, X_0-U).$$

Axiom VII. (Dimension Axiom) *If* X *is a space consisting of a single point* x_0, *then* $H^m(X) = 0$ *for every* $m \neq 0$.

In our note I, [2], a generalisation of the classical cohomology theory is given by introducing coboundary operators of higher order. It is natural to ask if there are analogous axioms which hold good for all of the (p, q)-cohomology groups. The purpose of the present note is to carry out such an investigation.

The algebraic axioms are proved in § 4. The exactness axiom is proved in § 6 and the excision axiom in § 7. For the dimension axiom, the m-dimensional (p, q)-cohomology groups of a single point are computed. This shows that, in general, the dimension axiom does not hold. It remains open whether or not the homotopy axiom is satisfied.

2. *The coboundary homomorphisms.*

Throughout the present note, we assume G to be a discrete abelian group such that the order of every element of G divides $\theta(p, q)$ for a given pair of positive integers p and q, [2, §§ 4 and 5].

Let (X, X_0) be a given pair. We shall define for each integer m a homomorphism

$$\delta^p\colon H^m_{(p,q)}(X_0, G) \rightarrow H^{m-p}_{(q,p)}(X \bmod X_0, G),$$

called the *coboundary homomorphism* on the group $H^m_{(p,q)}(X_0, G)$ as what follows.

Let $\iota\colon X_0 \rightarrow X$ be the identity map. It follows from a statement of Spanier, [4, p. 411], that the induced cochain transformation, [2, § 7],

$$\iota^\#\colon C^m(X, G) \rightarrow C^m(X_0, G)$$

is onto. Therefore, for each element $e \in H^m_{(p,q)}(X_0, G)$ there is a cochain $c^m \in C^m(X, G)$ such that $\iota^\# c^m$ is a cocycle of order p and represents the element e.

$$\delta^p c^m \in Z^{m+p,q}(X \bmod X_0, G). \tag{2.1}$$

Proof. Since $\delta^q \delta^p c^m = 0$ by our assumption on the group G, $\delta^p c^m \in Z^{m+p,q}(X, G)$. Since $\iota^\# c^m$ is a cocycle of order p, we have

$$\iota^\# \delta^p c^m = \delta^p \iota^\# c^m = 0.$$

Hence, $\delta^p c^m \in C^{m+p}(X \bmod X_0, G)$. This completes the proof.

(2.2) *$\delta^p c^m$ represents an element $\delta^p e$ of the group $H^{m-p}_{(q,p)}(X \bmod X_0, G)$, which depends only on e.*

Proof. Suppose that $d^m \in C^m(X, G)$ be another cochain such that $\iota^\# d^m$ is a cocycle of order p and $\iota^\# d^m$ represents the element e. Then it follows that $\iota^\# c^m - \iota^\# d^m$ is a coboundary of order q, i.e. there exists a cochain $a^{m-q} \in C^{m-q}(X_0, G)$ such that

$$\iota^\# c^m - \iota^\# d^m = \delta^q a^{m-q}.$$

Since the homomorphism

$$\iota^\#;\ C^{m-q}(X, G) \to C^{m-q}(X_0, G)$$

is onto, there is a cochain $b^{m-q} \in C^{m-q}(X, G)$ such that $a^{m-q} = \iota^\# b^{m-q}$. Hence we obtain

$$\iota^\# (c^m - d^m - \delta^q b^{m-q}) = 0,$$

i.e. the cochain $c^m - d^m - \delta^q b^{m-q}$ belongs to the subgroup $C^m(X \bmod X_0, G)$. Taking its p-th coboundary, we obtain

$$\delta^p c^m - \delta^p d^m \in B^{m+p,p}(X \bmod X_0, G).$$

This completes the proof.

Now, the following theorem is obvious.

(2.3) *The correspondence $e \to \delta^p e$ is a homomorphism*

$$\delta^p:\ H^m_{(p,q)}(X_0, G) \to H^{mp}_{(q,p)}(X \bmod X_0, G).$$

3. *Induced homomorphisms of the groups.*

By a map f of a pair [2] (X, X_0) into a pair (Y, Y_0), we mean a (continuous) map $f: X \to Y$ such that $f(X_0) \subset Y_0$. Let f be a given map of (X, X_0) into (Y, Y_0) and denote by $f_0 = f|X_0$ the partial map on X_0 into Y_0. Let

$$\iota_1:\ X_0 \to X, \qquad \iota_2:\ Y_0 \to Y$$

be the identities. Then we have the following diagram of cochain transformations

$$\begin{array}{ccc} C^m(Y, G) & \xrightarrow{f^\#} & C^m(X, G) \\ \downarrow{\scriptstyle \iota_2^\#} & & \downarrow{\scriptstyle \iota_1^\#} \\ C^m(Y_0, G) & \xrightarrow{f_0^\#} & C^m(X_0, G) \end{array}$$

[2] See [2, § 8].

Since $f\iota_1 = \iota_2 f_0$, the following relation is an direct consequence of (7.4) of our Note I:

$$\iota_1^\# f^\# = f_0^\# \iota_2^\#. \tag{3.1}$$

The following properties can be easily verified:

$$f^\# (C^m (Y \bmod Y_0, G)) \subset C^m (X \bmod X_0, G), \tag{3.2}$$

$$f^\# (Z^{m,p} (Y \bmod Y_0, G)) \subset Z^{m,p} (X \bmod X_0, G), \tag{3.3}$$

$$f^\# (B^{m,q} (Y \bmod Y_0, G)) \subset B^{m,q} (X \bmod X_0, G). \tag{3.4}$$

As a consequence of (3.3) and (3.4), $f^\#$ induces a homomorphism

$$f^*: H^m_{(p,q)} (Y \bmod Y_0, G) \to H^m_{(p,q)} (X \bmod X_0, G),$$

called the homomorphism induced by the map f. If X_0 and Y_0 are vacuous, then this homomorphism reduces to the following homormophism

$$f^*: H^m_{(p,q)} (Y, G) \to H^m_{(p,q)} (X, G)$$

of the absolute groups. Hence the partial map $f_0 = f|X_0$ induces a homomorphism

$$f_0^*: H^m_{(p,q)} (Y_0, G) \to H^m_{(p,q)} (X_0, G)$$

of the groups of the subspaces.

4. *The algebraic axioms.*

The following two axioms are immediate consequences of (7.3) and (7.4) of our Note I, [2].

Axiom I. *If* $f: (X, X_0) \to (X, X_0)$ *is the identity map, then the induced homomorphism* f^* *is the identity automorphism of the group* $H^m_{(p,q)}(X \bmod X_0, G)$.

Axiom II. *If* $f: (X, X_0) \to (Y, Y_0)$ *and* $g: (Y, Y_0) \to (Z, Z_0)$, *then* $(gf)^* = f^* g^*$.

We are going to prove the following axiom:

Axiom III. *If* $f: (X, X_0) \to (Y, Y_0)$ *and* $f_0 = f|X_0$, *then* $f^* \delta^p = \delta^q f_0^*$.

Proof. Let $e \in H^m_{(p,q)} (Y_0, G)$ be an arbitrary element. Choose a cocycle of order p $a^m \in Z^{m,p} (Y_0, G)$ which represents e. [Let

$$\iota_1: X_0 \to X, \qquad \iota_2: Y_0 \to Y$$

be the identities. Since $\iota_2^\#$ is onto, we may choose a cochain $c^m \in C^m (Y, G)$ such that $\iota_2^\# c^m = a^m$. Then the element

$$f^* \delta^p e \in H^{m-p}_{(q,p)} (X \bmod X_0, G)$$

is represented by the cocycle $f^\# \delta^p c^m$ of order q. On the other hand, the element $f_0^* e$ is represented by the cocycle $f_0^\# a^m$ of order p. Since

$$\iota_1^\# f^\# c^m = f_0^\# \iota_2^\# c^m = f_0^\# a^m$$

by (3.1), the element $\delta^p f_0^* e$ is represented by $\delta^p f^\# c^m$. Since $f^\# \delta^p = \delta^p f^\#$, our proof is complete.

5. *Mayer cochain complexes.*

A sequence of groups and homomorphisms is termed a *homomorphism sequence* if it can be indexed from the integers so that, if G_r and h_r denote respectively the group and homomorphism with index r, then

$$h_r;\ G_r \to G_{r+1}; \qquad (r = \ldots, -1, 0, 1, \ldots).$$

A *Mayer cochain complex* K is a homomorphism sequence $K = \{G_r, h_r\}$ such that $h_{r+1}h_r = 0$ for all $r = \ldots, -1, 0, 1 \ldots$, [3, p. 684].

For an arbitrary integer m, we construct a homomorphism sequence

$$K(m, p, q; X, G) = \{G_r, h_r\}$$

by taking

$$G_r = C^{m+[(r+1)/2]p+[r/2]q}(X, G),$$

$$h_r = \begin{cases} \delta^p, & \text{if } r \text{ is even,} \\ \delta^q, & \text{if } r \text{ is odd,} \end{cases}$$

where the symbol $[x]$ denotes the greatest integer not exceeding x. Since $\delta^p\delta^q = 0 = \delta^q\delta^p$, we have

(5.1) *$K(m, p, q; X, G)$ is a Mayer cochain complex.*

To every group G_r of the Mayer cochain complex $K(m, p, q; X, G)$, we associate with the subgroup

$$F_r = C^{m+[(r+1)/2]p+[r/2]q}(X \bmod X_0, G)$$

of G_r, where X_0 is a given subspace of X. It follows from (8.1) of our Note I, [2], that

$$h_r(F_r) \subset F_{r+1}$$

for each integer r. Thus we obtain another Mayer cochain complex

$$K(m, p, q; X \bmod X_0, G) = \{F_r, h_r\}.$$

The following statement is an immediate consequence of the definition of a subcomplex, [3, p. 685].

(5.2) *$K(m, p, q; X \bmod X_0, G)$ is a subcomplex of the Mayer cochain complex $K(m, p, q; X, G)$.*

Let us use the following abridged notations:

$$K^m = K(m, p, q; X, G),$$
$$K^m_* = K(m, p, q; X \bmod X_0, G).$$

Let $H^r(K)$ denote the r-dimensional cohomology group of the Mayer cochain complex K, [3, p. 685]. Then clearly we have

$$H^r(K^m) = \begin{cases} H^{m+rp/2+rq/2}_{(p,q)}(X, G), & \text{(if } r \text{ is even),} \\ H^{m+(r+1)p/2+(r-1)q/2}_{(q,p)}(X, G), & \text{(if } r \text{ is odd);} \end{cases}$$

$$H^r(K^m_*) = \begin{cases} H^{m+rp/2+rq/2}_{(p,q)}(X \bmod X_0, G), & \text{(if } r \text{ is even),} \\ H^{m+(r+1)p/2+(r-1)q/2}_{(q,p)}(X \bmod X_0, G), & \text{(if } r \text{ is odd).} \end{cases}$$

6. *The exactness axiom.*

Let (X, X_0) be a given pair. Denote by Φ the empty set and consider the identity maps

$$i;\ (X_0, \Phi) \to (X, \Phi), \quad j;\ (X, \Phi) \to (X, X_0).$$

The maps i and j induce homomorphisms:

$$i^*;\ H^m_{(p,q)}(X, G) \to H^m_{(p,q)}(X_0, G),$$
$$j^*:\ H^m_{(p,q)}(X \bmod X_0, G) \to H^m_{(p,q)}(X, G).$$

Further, we have defined in § 2 the coboundary homomorphisms

$$\delta^p;\ H^m_{(p,q)}(X_0, G) \to H^{m-p}_{(q,p)}(X \bmod X_0, G).$$

Thus, for an arbitrary integer m, we obtain a homomorphism sequence:

$$\ldots \xrightarrow{i^*} H^{m-q}_{(q,p)}(X_0, G) \xrightarrow{\delta^q} H^m_{(p,q)}(X \bmod X_0, G) \xrightarrow{j^*} H^m_{(p,q)}(X, G) \xrightarrow{i^*}$$
$$H^m_{(p,q)}(X_0, G) \xrightarrow{\delta^p} H^{m-p}_{(q,p)}(X \bmod X_0, G) \xrightarrow{j^*} H^{m-p}_{(q,p)}(X, G) \xrightarrow{i^*} \ldots$$

called the (m, p, q)-*cohomology sequence* of the pair (X, X_0) over G.

Exactness axiom. *The (m, p, q)-cohomology sequence of a pair (X, X_0) over G is exact*, i.e. *the kernel of each homomorphism is identical with the image of the preceding.*

Proof. Let ι: $X_0 \to X$ be the identity map, then the induced cochain homomorphism

$$\iota^{\#}:\ C^m(X, G) \to C^m(X_0, G)$$

and its kernel is the subgroup $C^m(X \bmod X_0, G)$. Hence, it follows from Noether theorem that $C^m(X_0, G)$ is isomorphic with the factor group

$$C^m(X, G)/C^m(X \bmod X_0, G).$$

Therefore, the Mayer cochain complex

$$K^m_0 = K(m, p, q;\ X_0, G)$$

might be considered as the quotient complex K^m/K^m_*, [3, p. 685]. Then, our axiom follows from a general theorem of Kelley-Pitcher, [3, p. 688].

7. *The excision axiom.*

Excision axiom. *If U is an open set of X whose closure is contained in the interior of X_0, then the identity map*

$$\mu:\ (X-U,\ X_0-U) \to (X,\ X_0)$$

induces onto isomorphisms

$$\mu^*:\ H^m_{(p,q)}(X \bmod X_0, G) \to H^m_{(p,q)}(X-U \bmod X_0-U, G)$$

for every integer m.

Proof. According to SPANIER, [4, p. 418], the induced cochain transformation

$$\mu^{\#}\colon C^m(X \bmod X_0, G) \to C^m(X - U \bmod X_0 - U, G)$$

is an onto isomorphism. Since $\delta^p\mu^{\#} = \mu^{\#}\delta^p$ and $\mu^{\#}\delta^q = \delta^q\mu^{\#}$, our axiom follows immediately. Q.E.D.

8. *The groups for a single point.*

In the present section, we shall be concerned with the topological space X which consists of only a single point x_0. We are going to compute the m-dimensional (p, q)-cohomology group $H^m_{(p,q)}(X, G)$ in terms of the coefficient group G and the integers m, p, q.

Since X consists of a single point x_0, X^{m+1} consists of the point $x_0^{m+1} = (x_0, \ldots, x_0)$ only. Hence

$$C^m(X, G) = \Phi^m(X, G) \approx G.$$

The onto isomorphism is given by associating to each m-function Φ of $\Phi^m(X, G)$ the element $g = \Phi(x_0^{m+1})$ of G.

For an arbitrary m-function $\Phi \in \Phi^m(X, G)$, it is easy to see that

$$\delta^p \Phi(x_0^{m+p+1}) = \theta(p, m+1)\, \Phi(x_0^{m+1}).$$

Let kG denote the subgroup of G which consists of all elements of the form kg, $g \in G$; and let Gk denote the subgroup of G consisting of all elements $g \in G$ such that $kg = 0$.

The following easily verified identity

$$\theta(p, m+1)\, \theta(q, m-q+1) = \theta(p, q)\, \theta(p+q, m-q-1)$$

shows that $\theta(q, m-q+1)\, G$ is a subgroup of $G\,\theta(p, m+1)$. Then, an elementary consideration will give the following isomorphisms onto:

$$A^m_{(p,q)}(X, G) \approx \begin{cases} 0, & (m < 0), \\ G\,\theta(p, m+1), & (0 \leqq m < q), \\ G\,\theta(p, m+1)/\theta(q, m-q+1)\, G & (m \geqq q). \end{cases}$$

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A TABULATION OF THE MEDIAN TEST FOR UNEQUAL SAMPLES

BY

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(Communicated by Prof. M. W. WOERDEMAN at the meeting of Dec. 17, 1949)

In a previous paper (WESTENBERG, 1948) a significance test was developed, applicable to samples taken from continuous populations of any frequency distribution. In designing this test, the following postulate was taken as a starting point.

In any continuous population, the chance of a variable in any specimen not exceeding a certain critical value must have some definite value, p. Hence in a sample of N specimens, the number of specimens, n, in which the variable does not exceed the abovementioned critical value, will follow the so called binomial frequency law. This statement is expressed in the formula, specifying the probability of obtaining a certain value of n,

$$P_n = \frac{N!}{n!\,(N-n)!} \cdot p^n \cdot (1-p)^{N-n}.$$

According to the same reasoning, the chance of $n+\triangle$ specimens having the same property occurring in a sample of N_1 specimens. is given by

$$P_{n-\angle} = \frac{N_1!}{(n-\angle)!\,(N_1-n-\angle)!} \cdot p^{n+\triangle} \cdot (1-p)^{N_1-n-\triangle}.$$

Similarly the probability of $n-\angle$ specimens of the same kind occurring in another sample of N_2 specimens, taken from the same population, is given by

$$P_{n-\angle} = \frac{N_2!}{(n-\triangle)!\,(N_2-n+\angle)!} \cdot p^{n-\angle} \cdot (1-p)^{N_2-n+\triangle}.$$

The probability of the simultaneous and independent occurrence of these two events, or rather of the occurrence of a difference $2\ \triangle$ with fixed values of n, N_1, and N_2, can be proved to be

$$P_{\triangle;\,n,N_1,N_2} = \frac{\dfrac{N_1!\cdot N_2!}{(n+\triangle)!\,(N_1-n-\triangle)!\,(n-\triangle)!\,(N_2-n+\triangle)!}}{\dfrac{(N_1+N_2)!}{(2n)!\,(N_1+N_2-2n)!}}.$$

This problem has been solved by R. A. FISHER (1935) (c.f. WESTENBERG, 1947) and in its turn constitutes the mathematical basis of the significance test of the median.

For our purpose, we take the median of the pooled samples as the critical value which is not exceeded in $(n + \triangle)$ specimens of the first, and in $(n - \triangle)$ specimens of the second sample. Thus n assumes the value $\frac{1}{4}(N_1 + N_2)$ because the number of smaller individuals in the combined samples will then be

$$(n+\triangle)+(n-\triangle) = \{\tfrac{1}{4}(N_1+N_2)+\triangle\}+\{\tfrac{1}{4}(N_1+N_2)-\triangle\} = \\ = \tfrac{1}{2}(N_1+N_2).$$

In the first sample we have $\frac{1}{2}N_1$ individuals on the smaller side of the median of this sample, and $\frac{1}{4}(N_1 - N_2) + \triangle$ specimens on the smaller side of the median of the combined samples. Hence in the first sample the number of individuals between the two medians amounts to

$$\delta = \tfrac{1}{4}(N_1+N_2)+\triangle-\tfrac{1}{2}N_1 = \tfrac{1}{4}(N_2-N_1)+\triangle$$

For the case in which $N_1 = N_2$, we have

$$\delta = \tfrac{1}{4}(N_2-N_1)+\triangle = \triangle$$

and for this reason δ and $\triangle$ were not distinguished in our previous paper on this subject, dealing with equal samples only.

Computation reveals that the probability is maximal for a situation in which $\triangle = \frac{1}{4}(N_1 - N_2)$, coinciding with $\delta = 0$. Moreover the distribution of δ is symmetrical with respect to N_1 and N_2. Hence $|\delta|$ will not be affected by interchanging N_1 and N_2. Therefore a tabulation of $|\delta|$ is most suitable for practical use.

Having substituted $\frac{1}{4}(N_1+N_2)$ for n in the formula specifying $P_{\triangle;n,N_1,N_2}$, we can calculate in what proportion of cases, $\triangle$ will not be exceeded, when the two samples are taken at random from the same population. [1]) From these calculations we can easily arrive at a tabulation of $|\delta|$ for different levels of significance, depending on N_1 and N_2, regardless of interchanging N_1 and N_2. For $N_1 = N_2$ we have only to adopt the $\triangle$-values in our previous paper. Thus we arrive at the following table.

In the foregoing paper, dealing with equal samples only, the minimal number of specimens that could result in the samples displaying a difference on some given level of significance, was established. To this end, $P_{\triangle;n,N}$ was computed for $\triangle = n = \frac{1}{2}N$, various values of N being taken; and by means of graphical interpolation the values of N were found that would correspond to several given (unilateral) tail error values.

In the present extension for unequal samples the same problem is dealt with by computing $P_{\triangle;n,N_1,N_2}$ for $\triangle = n = \frac{1}{4}(N_1+N_2)$, and varying values of N_1 and N_2. In each series of calculations, N_1 is kept constant and N_2 varied, and the ensuing probabilities plotted against

---

[1]) It is clear, that the same reasoning could have been based on the frequencies of the specimens in which the variable does exceed the critical value, e.g. $\frac{1}{4}(N_1 + N_2)$.

$N_1 = 6$		10	20	50	100	200	500	1000	2000	
							$\lvert\delta\rvert =$	26,5		
								28,1		
								30,2		
2000								33,0		2000
								37,2		
								41,2		
								49,3		
								18,9	21,7	
								20,0[5]	23,1	
								21,5	24,8	
1000								23,4[5]	27,0	1000
								26,5	30,7	
								29,3	33,7	
								35,0	40,4[5]	
							13,5	15,5	16,9	
							14,3	16,5	18,0	
							15,4	17,7	19,3	
500							16,7	19,3	21,0	500
							18,8	21,8	23,8	
							20,8	24,0	26,2	
							24,9	28,7	31,4	
						8,7	10,3	11,1	11,6	
						9,2	10,9	11,8	12,3	
						9,9	11,7	12,6	13,2	
200						10,7	12,8	13,7	14,3	200
						12,1	14,4	15,6	16,2	
						13,3[5]	15,9	17,1	17,8	
						15,9	18,9	20,4	21,2	
					6,3	7,2	8,0	8,3	8,5	
					6,7	7,6	8,5	8,8	9,0	
					7,1[5]	8,2	9,1	9,5	9,7	
100					7,7	8,9	9,9	10,3	10,5	100
					8,6[5]	10,0	11,1	11,5	11,8	
					9,5	11,0	12,2	12,7	13,0	
					11,4	13,1	14,6	15,1	15,4	
				4,6	5,2	5,7	6,0	6,1[5]	6,3	
				4,9	5,5	6,0	6,4	6,5	6,6	
				5,2	5,9	6,4	6,8	7,0	7,1	
50				5,6	6,4	7,0	7,4	7,5	7,6	50
				6,3	7,1	7,8	8,3	8,5	8,6	
				6,9	7,9	8,6	9,1	9,3	9,4	
				8,1	9,3	10,2	10,8	11,0	11,1	
			3,1	3,6	3,8[5]	4,0	4,1	4,1	4,1	
			3,2	3,8	4,1	4,2	4,3	4,3	4,3	
			3,4	4,0	4,3	4,5	4,6	4,6	4,6	
20			3,7	4,4	4,6	4,8	4,9[5]	5,0	5,0	20
			4,2	4,9	5,2	5,3	5,5	5,6	5,6	
			4,6	5,3	5,5	5,9	6,0	6,1	6,1	
			5,3	6,2	6,7	6,9	7,0	7,1	7,1	
		2,3	2,6	2,8	3,0	3,0	3,0[5]	3,0[5]	3,0[5]	
		2,4	2,7	3,0	3,1	3,1	3,2	3,2	3,2	
		2,5	2,9	3,2	3,2[5]	3,2[5]	3,4	3,4	3,4	
10		2,7	3,1	3,4	3,5	3,5	3,6[5]	3,6[5]	3,6[5]	10
		3,1	3,4	3,7	3,9	3,9[5]	4,0	4,0	4,0	
		3,4	3,7	4,1	4,3	4,3	4,3	4,3	4,3	
		3,8	4,4	4,6	4,8	4,9	4,9	5,0	5,0	
	5 %	$\lvert\delta\rvert =$ 1,9	2,1	2,2	2,4	2,4	2,5	2,5	2,5	2,5	
	4 %	2,0	2,2	2,3	2,5	2,5	2,6	2,6	2,6	2,6	
	3 %	2,1	2,3	2,4	2,6	2,6[5]	2,7	2,7	2,7	2,7	
$N_2 = 6$	2 %	2,2	2,4	2,6	2,7	2,8	2,9	2,9	2,9	2,9	$N_2 = 6$
	1 %	2,5	2,6	2,9	—	—	—	—	—	—	
	$\frac{1}{2}$ %	2,7	2,9	—	—	—	—	—	—	—	
	$\frac{1}{10}$ %	3,0	—	—	—	—	—	—	—	—	
Tailerror		$N_1 = 6$	10	20	50	100	200	500	1000	2000	

the corresponding values of N_2. Graphical interpolation then leads to the following table.

Tail error	$N_1 = 6$	10	20	50	100	200	500	1000	2000
5 %	3,4	3,6	3,9	4,1	4,2	4,2	4,3	4,3	4,3
4 %	3,6	3,8	4,2	4,4	4,5	4,6	4,6	4,6	4,6
3 %	3,8	4,1	4,5	4,8	4,9	5,0	5,0	$5,0^5$	$5,0^5$
2 %	4,1	4,5	4,9	5,3	5,5	5,6	5,6	5,6	5,6
1 %	4,6	5,1	5,7	6,2	6,4	6,6	6,6	6,6	6,6
$\frac{1}{2}$ %	5,1	5,7	6,4	7,0	7,4	7,5	7,5	7,6	7,6
$\frac{1}{10}$ %	6,0	6,9	8,0	8,9	9,4	9,7	9,9	9,9	9,9

This table is shown diagrammatically in fig. 1 Points of equal tail error value are connected by a smooth curve, and show the limits of the area outside which no significance can be attributed to a difference

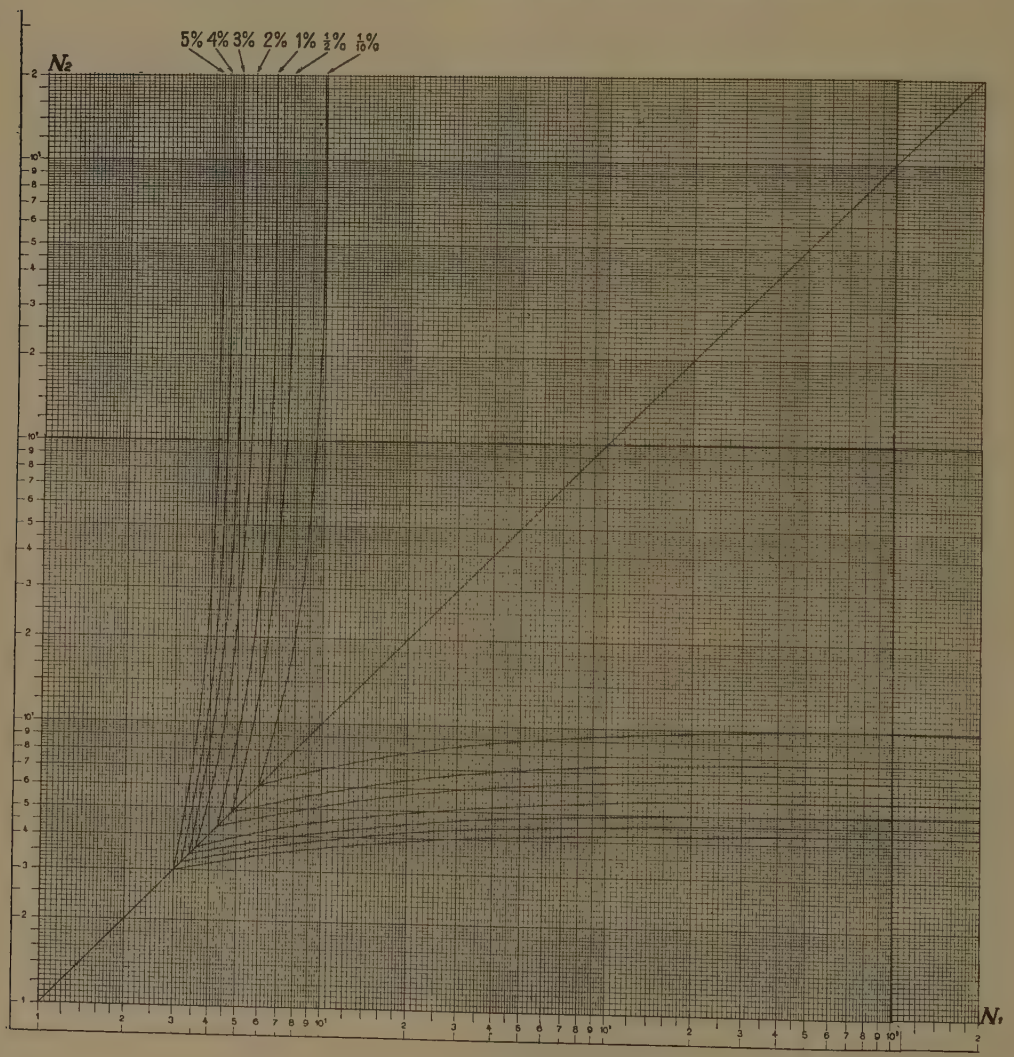

Fig. 1. Area of significance.

between two samples. These boundaries are given for several levels of significance. Since N_1 and N_2 are interchangeable in the computation of the probabilities $P_{\Delta \geq n, N, N_1}$, the diagram is symmetrical with respect to the line for $N_1 = N_2$.

In dealing with the median test for equal samples, all confidence limits were plotted as a graph, which enabled an easy interpolation for any desired value of N. The present extension for different values of N_1 and N_2 would require a three dimensional construction for the same purpose. In order to avoid excessive complexity, this is depicted for the 1 % significance level only (Fig. 2). This representation does not, however,

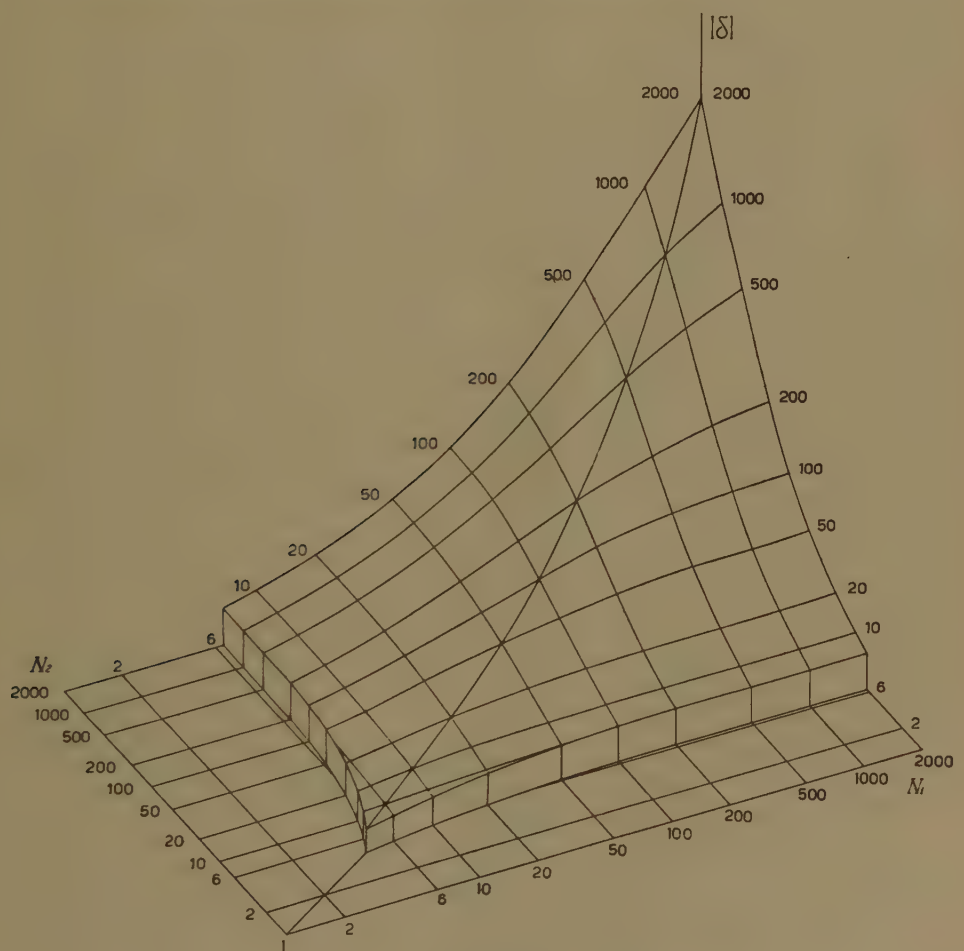

Fig. 2. 1 % significance level.

allow a direct interpolation for intermediate values of N_1 and N_2, for which successive interpolations have to be carried out in a number of parallel planes, and finally in a transverse plane, as can be seen from fig. 3.

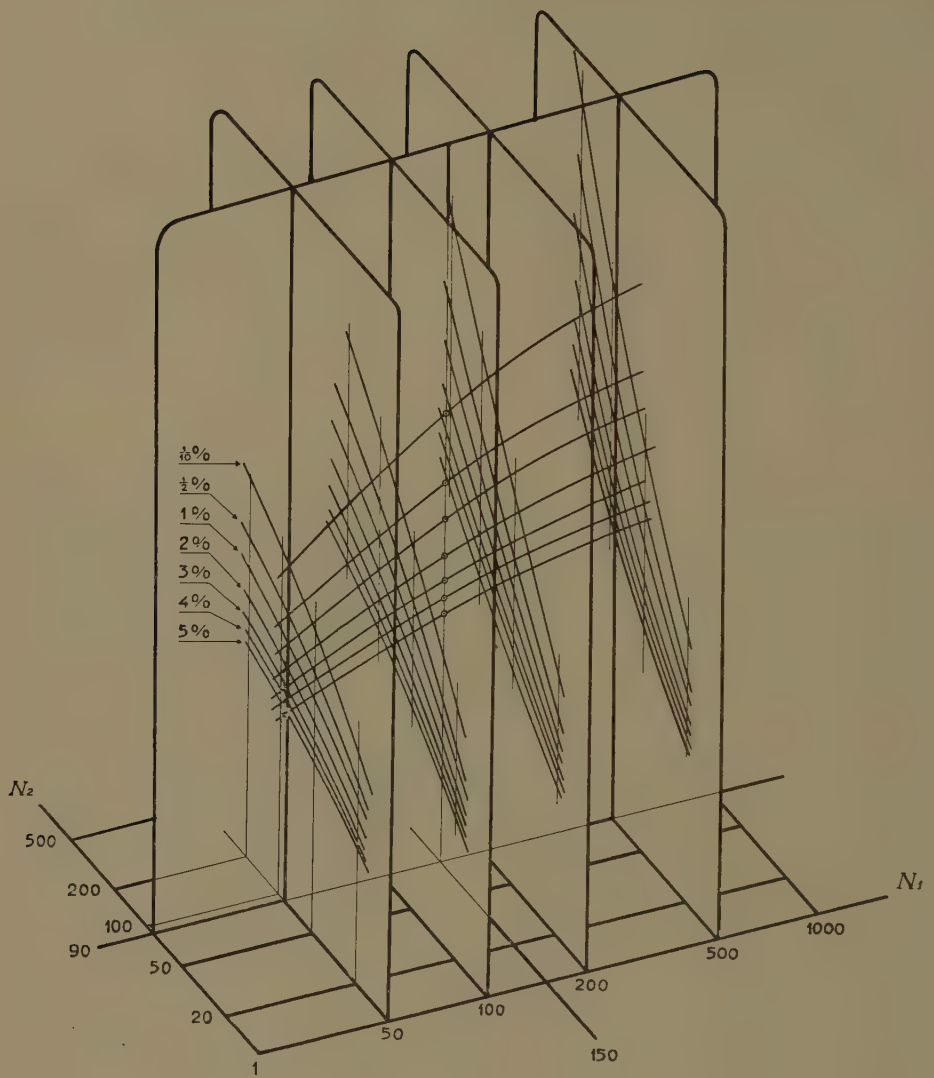

Fig. 3. Interpolation for $N_1 = 150$ and $N_2 = 90$.

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ON THE IDENTITY OF ONCHOCALANUS NUDIPES C. B. WILSON, 1942

BY

W. VERVOORT

(Communicated by Prof. H. BOSCHMA at the meeting of Dec. 17, 1949)

In 1942, in a posthumous paper by C. B. WILSON on Copepods collected during the last cruise of the research vessel "Carnegie", the description was published of a new species of the genus *Onchocalanus* G. O. Sars, 1905, which species was named *Onchocalanus nudipes*. The trivial name was chosen because of the structure of the 5th pair of legs, which has no cover of strong hairs, as is commonly found in species of *Onchocalanus*. During a revision of the species of this genus, rather considerable differences were found between the structures usually met with in *Onchocalanus* and those attributed to *O. nudipes* by WILSON. The shape of the body in that species is quite different from the type found in the species of *Onchocalanus*, which are known at present. In *O. nudipes* the body is ovoid — it is elongated in *Onchocalanus* — whilst the head and the 1st thoracic segment, as well as the 4th and 5th thoracic segments are completely separated. In *Onchocalanus* the division between head and 1st thoracic segment is very weakly indicated, while the 4th and 5th thoracic segments are completely fused. Besides, there are distinct differences in the structure of the 2nd antennae, oral appendages and legs, as can be easily seen from a comparison of the drawings and descriptions by WILSON with those of a true *Onchocalanus* by WITH (1915) or SARS (1925). Unfortunately WILSON's descriptions of the oral appendages of *O. nudipes* are short and his drawings of these appendages, the structure of which is of fundamental importance for the taxonomy of Calanoid Copepods, are too vague and schematic to be fully reliable. Moreover, the structure of the legs, especially of the 5th pair, differs so much from the type found in *Onchocalanus*, that the systematical position of WILSON's species becomes questionable.

According to WILSON's notes material of his new form — about 40 specimens in various stages of development were found in samples from the tropical Pacific — was deposited in the U.S. National Museum (Smithsonian Institution) and through the kindness of Mr. PAUL L. ILLG, Associate Curator, Division of Marine Invertebrates, the Rijksmuseum van Natuurlijke Historie at Leiden acquired 4 females in various stages of development and one young male. In addition I had the opportunity to study an adult male (which sex has not been described or mentioned

by WILSON), received as a loan from the Smithsonian Institution. The specimens are labelled:

"*Onchocalanus nudipes* Wilson. Off N.W. coast of South America, 1°32′ S, 82°16′ W, Nov. 8, 1928. Station 40, surface, last cruise of

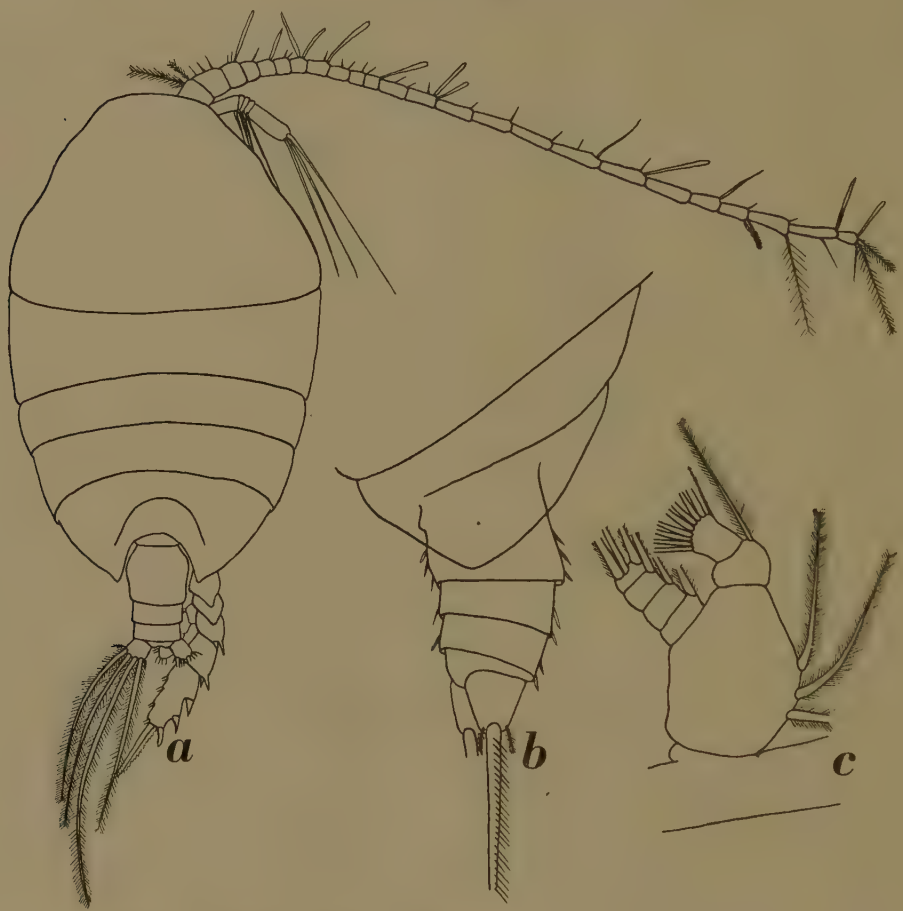

Fig. 1. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection, adult female. a, whole animal in dorsal view, left 1st antenna, left 4th leg and right furcal setae omitted; b, adbomen, lateral view from left side; c, left mandibular palpus, posterior surface. a, × 35; b, × 90; c, × 165.

Carnegie. C. B. WILSON det. Type". The material comprises one adult female, one adult male, a female and a male in the Vth copepodid stage, a female in the IVth and a female in the IIIrd copepodid stage. The adult female and the young male were stained with azurine, the appendages removed, and parts as well as the rest of the animals mounted on slides.

A study of WILSON's material revealed that the form described by that author as *Onchocalanus nudipes* is identical with *Phaenna spinifera* Claus, 1863, a well known, although not so common form. The de-

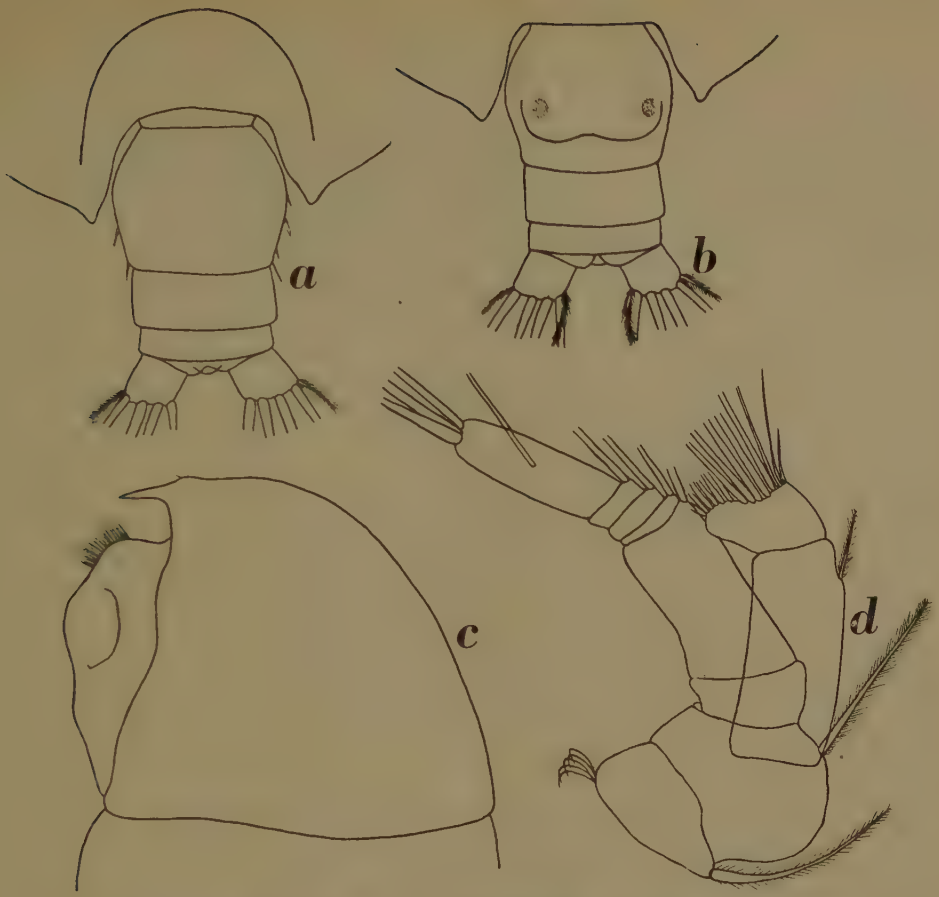

Fig. 2. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection, adult female. a, abdomen, dorsal aspect; b, abdomen, ventral aspect; c, head, lateral view from left side, 1st antennae and oral appendages removed to show the buccal field; d, 2nd antenna, posterior surface. a, b, × 90; c, × 65; d, × 165.

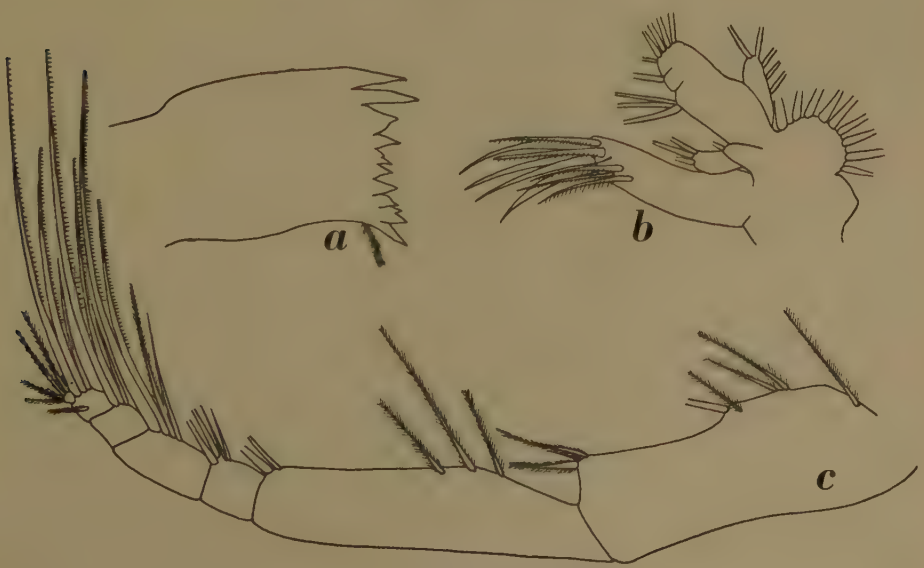

Fig. 3. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection, adult female. a, left manducatory plate, interior surface; b, left 1st maxilla, posterior surface; c, left maxilliped, anterior surface. a, × 340; b, c, × 150.

scription by Wilson is partly based on an adult female, partly on a male in the Vth copepodid stage. Wilson's drawings of the 2nd antenna (1942, fig. 72), 1st maxilla (l.c., fig. 73), 2nd maxilla (l.c., fig. 74) and maxilliped (l.c., fig. 75) all are very inaccurate with regards to the number of setae on the various lobes and joints. The figures of the 3rd and 4th pairs of legs apparently have been changed in Wilson's paper; fig. 81 represents the 4th leg, fig. 82 the 3rd. The 5th pair of legs, fig. 83, is that of a male in the Vth copepodid stage. The name *Onchocalanus nudipes* C. B. Wilson must be considered a synonym of *Phaenna spinifera* Claus, 1863.

The following description is taken from the adult female mentioned above; some additional information on the young male is also given here. I have compared the present specimens with representatives of *Phaenna spinifera* in the Dana and Siboga collections, all specimens originating from the tropical Pacific. I failed to notice differences in structure, but there is some variability in length. This can also be ascertained from a comparison of the length measurements by various authors who have recorded *Phaenna spinifera*. Exceptionally large specimens were described by With (1915), the females of which measured 2.9 mm.

Female, adult stage. Total length 2.07 mm. Proportions of cephalothorax and abdomen as 112 : 26. Body robust, almost globular, in dorsal view broadly ovoid in outline. Head and 1st thoracic segment completely and very distinctly separated. Line of separation between the other thoracic segments very distinct, with the exception of the line between the 4th and 5th thoracic segments, which is only visible on the dorsal surface (fig. 1a). The body appears to be slightly constricted at the lines of separation of the various thoracic segments, so that these segments are well marked. Head in dorsal aspect broadly rounded, very slightly dilated in the oral region. Head laterally also broadly rounded, without a protuberance as described by With (1915, fig. 79), provided with a distinct, bifurcated rostrum, of which each ramus is acutely pointed and directed straightly downwards. Frontal organ just visible at base of the rostrum, carrying 2 small hairs. The buccal field is very prominent; the epistoma covered by numerous long hairs (fig. 2c). Postero-lateral thoracic margin slightly produced in dorsal view; in lateral aspect it covers about half the length of the genital segment of the abdomen (fig. 1b); extreme tip rounded.

Abdomen very short, 4-segmented, the 4th (anal) segment completely telescoped into the 3rd. The various segments have, with the furca, the following proportional length:

segment	1+2	3	4	5	furca	
	44	19	15	4	18	= 100.

Genital segment slightly longer than broad, swollen on both sides; laterally with the indication only of a genital swelling at about half the length. Genital segment ventrally with a broadly rounded genital flap,

covering the orifices of two small, scarcely visible receptacles (figs. 2a, b). Abdominal segments with some scattered hair-like scales of characteristic appearance. Furcal joints about as long as broad (fig. 2a), with 4 strong, subequal marginal setae. 2nd internal seta of both sides elongated,

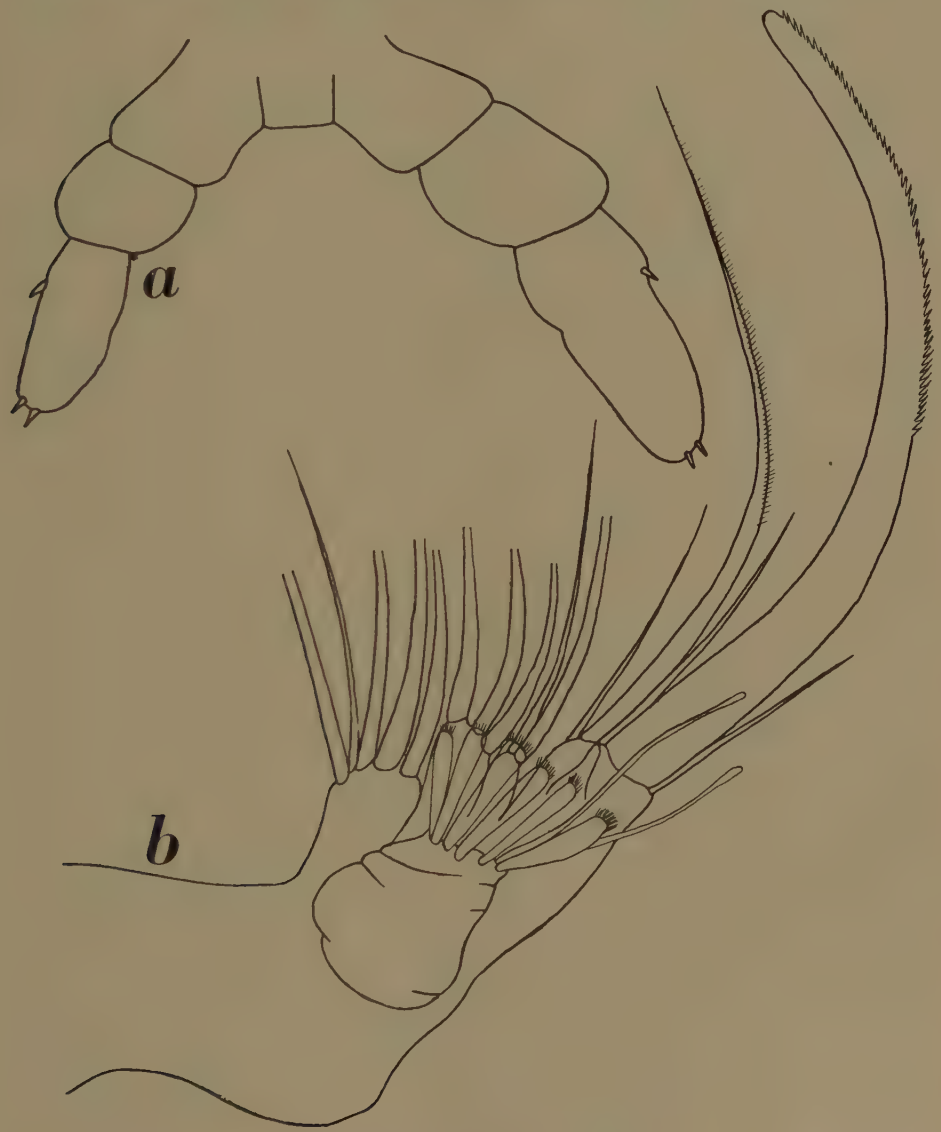

Fig. 4. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection, a, male, Vth copepodid stage, 5th pair of legs, anterior surface, left leg represented on the right side; b, adult female, right 2nd maxilla, interior surface. a, × 320; b, × 350.

about $2\frac{1}{2}$ — 3 times as long as the abdomen, the other setae $1\frac{1}{2}$ — twice as long as the abdomen. In addition each furcal ramus carries a small

external and a small internal (ventral) seta. All marginal setae, the long as well as the small, are densely plumose. The anal operculum is only just visible at the base of the furca.

1st antennae directed laterally, slightly longer than the total body-length. They are 24-jointed; the 8th and 9th joints are completely fused; the 1st and 2nd joints are imperfectly separated. The various joints have the following proportional lengths:

1	2	3	4	5	6	7	8+9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
54	30	20	24	27	25	24	37	20	16	24	36	43	60	64	64	61	60	55	52	63	55	58	28	= 1.000.

The arrangement of the setae is as illustrated in fig. 1a.

2nd antenna with endo- and exopod widely different in length. The endopod is about as long as the first two exopodal segments (fig. 2d). 1st basal joint small, with a single, plumose internal seta and a patch of curved hairs at the external corner. 2nd basal joint also with a single internal seta. 1st endopodal joint $2\frac{1}{2}$ times as long as wide; the internal margin carries a small seta at about 1/5 the length from the apex. 2nd endopodal joint small, 8 setae on the internal, 6 on the external lobe. 1st and 2nd exopodal joints separated, the 1st small, the 2nd about twice as long as wide, with an external, apical seta. There are 3 additional exopodal joints, each with a single seta, and an elongated apical joint, carrying 3 apical setae and a smaller seta at about the middle of its length.

Mandibular palpus small; 2nd basal joint squarish, with 3 strong, internal setae (fig. 1c). Endopod 2-jointed, 1st joint with a single internal seta, placed along the slightly swollen internal margin; the 2nd joint with 8 setae. Exopod 5-jointed, the first 4 joints with a single seta, the terminal joint with 2 setae. Mandibular plate much elongated, cutting edge armed with about 10 very acute teeth and a small, plumose basal seta (fig. 3a).

1st maxilla small; 1st inner lobe elongated, carrying 6 strong, more or less plumose setiform spines and 4 smaller setae, one of which is very small and delicate (fig. 3b). 2nd inner lobe with a single seta, 3rd inner lobe with 3 setae. 1st outer lobe with 9 strong, plumose setae. Exopod with 5 setae. 2nd basal joint with 3 setae; endopod with totally 6 setae, the jointing between the endopodal joints is indistinct, but apparently there is 1 seta on the 1st, 1 on the 2nd and 4 on the 3rd (terminal) joint.

2nd maxilla strongly developed (fig. 4b). The 1st basal joint carries 4 lobes; on the 1st lobe there are 4 setae of about equal length and one smaller seta, on the 2nd lobe there are 3 equal setae, on the 3rd 2 sub-equal setae, and on the 4th one strong, spinulose seta and a much smaller and shorter seta. The denticles on the last mentioned strong seta on the 4th lobe are minute. 2nd basal joint terminating into a single (5th) lobe, which carries a strong, curved, spiniform seta. This seta is much longer than the other setae on this limb; there are rather coarse denticles

along the convex border, from the apex down to about half its length. In addition there are 2 much smaller setae on the 5th lobe. The jointing of the endopod is very indistinct; there are apparently 6 brush-shaped sensory appendages and 2 much longer, filiform appendages. The extremities of these sensory filaments, however, are covered by a mucous layer, so that I could scarcely observe their structure, although their number undoubtedly is 8. There is no seta along the external margin of the 2nd maxilla.

The 2nd maxillipeds are comparatively long and slender; they normally are directed forwards and reach beyond the apex of the head (fig. 3c). The basal joints are slightly longer than has been indicated by GIESBRECHT (1892, p. 295). The 1st basal joint is about $2\frac{1}{2}$ times as long as wide, the 2nd $3\frac{1}{2}$ times as long as wide. On the 1st basal joint 1 seta is found on the 1st lobe, 2 on the 2nd, 2 on the 3rd and 2 setae and a spiniform seta on the 4th or apical lobe. 2nd basal joint with 3 setae along the slightly swollen internal border and 2 apical setae. The 1st endopodal joint has 4 setae, the 2nd 4, the 3rd 3, the 4th 3 + 1 and the 5th (apical) joint 4 setae. One of the setae on the 2nd, 3rd and 4th joints is rather long and strong: each of these is slightly curved and carries acute spinules along the internal border. Each of the above mentioned joints carries a second spinulose seta, which, however, is much smaller. The 2nd endopodal joint is slightly longer than has been figured by GIESBRECHT, it is about twice as long as wide.

1st pair of legs with the 1st basal joint short and wide, no hairs along external or internal margins (fig. 5a). 2nd basal joint with a small, curved external spine, internal margin with long, fine hairs and, near the insertion of the endopod, a curved, plumose seta. Endopod 1-jointed, about as long as the first two exopodal joints. External margin of the endopod swollen and set with a fringe of hair-like spinules. There are 6 marginal, plumose setae, 4 of which are found along the internal border, 2 are apical. Exopod 3-jointed; 1st joint without internal setae, but with a strong and straight external spine. 2nd exopodal joint with internal seta and strong, straight external spine. 3rd exopodal joint with 3 internal setae and a setiform endspine. The latter shows a slight bend at about 1/3 of its length from the base; the external lamella is small and indistinctly dentate. In addition there is a strong external marginal spine, at the base of which a smaller spine is observed. The external margin of the 3rd exopodal joint shows a slight incision or notch at about 1/3 of its length. All external marginal spines have minute teeth along their internal borders.

1st basal joint of the 2nd pair of legs with a strong, plumose internal seta (fig. 5b). 2nd basal joint with a rather strong, curved external spine. Endopod 2-jointed; 1st joint with internal seta and strong, downwardly directed external spine. 2nd endopodal joint with 1 external, 2 terminal and 2 internal setae. The posterior surface of that joint has 4 strong

Fig. 5. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection, adult female. a, 1st left leg; b, 2nd left leg; c, 3rd left leg; d, 4th left leg. Of all legs the posterior surface has been figured. a, × 150; b, c, d, × 75.

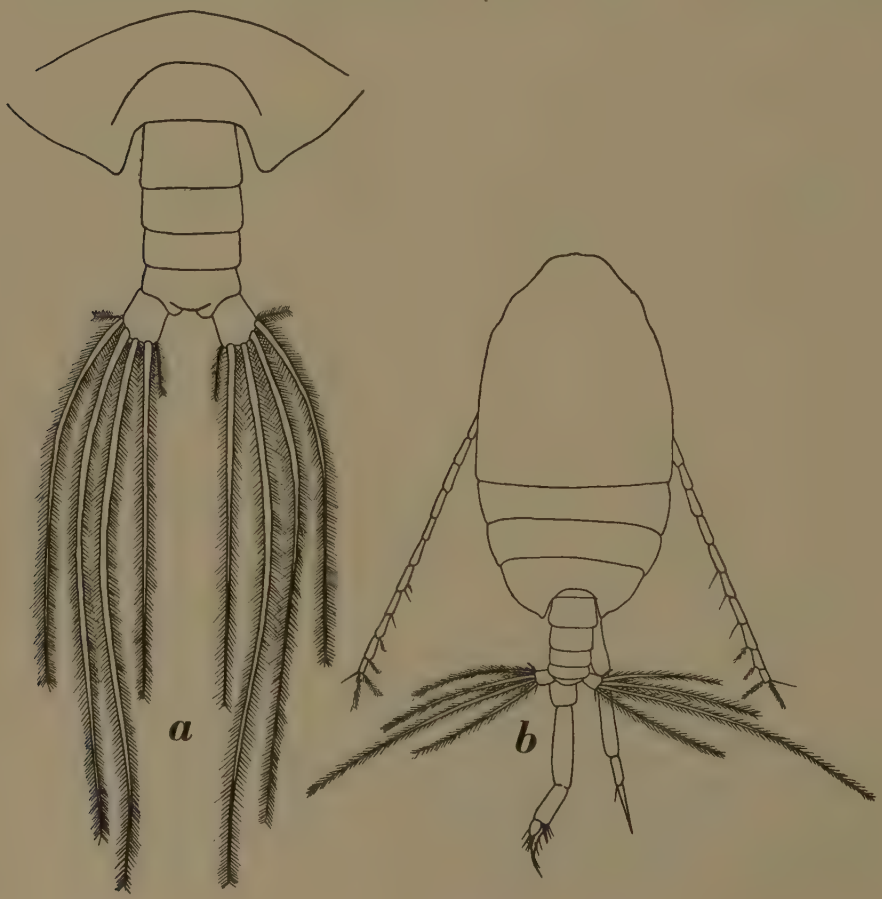

Fig. 6. *Phaenna spinifera* Claus (= *Onchocalanus nudipes* C. B. Wilson) from "Carnegie" collection. a, male, Vth copepodid stage, abdomen in dorsal view; b, adult male, dorsal aspect. a, × 75; b, × 45.

and 6 slightly smaller teeth, arranged in two coronulae. Exopod 3-jointed; 1st exopodal joint with an internal seta, external spine curved, with smaller tooth at its base. 2nd exopodal joint with a strong internal seta and a strong, curved external spine, at the base of which 2 smaller teeth are found. 3rd exopodal joint as long as the first two joints together. Internal margin with 4 plumose setae, external margin with 3 strong teeth, the two proximal each with 1 smaller basal tooth, the 3rd (distal) tooth with two smaller teeth of different size at its base. Endspine straight, as long as the 3rd exopodal joint. External lamella rounded, set with sharp, short teeth, about 23 of which are present.

Basal joints of the 3rd pair of legs as in the 2nd, external spine on the 2nd joint small and slightly curved (fig. 5c). Endopod 3-jointed, the first two joints each with a strong, downwardly directed, external spine and an internal seta. 3rd endopodal joint with 1 internal, 2 terminal and 2 external setae. 2nd endopodal joint with 2 strong and 4 smaller teeth on the posterior surface: 3rd endopodal joint with 4 strong teeth on that surface. All teeth are arranged in coronulae. Exopod as in the 2nd pair: at the base of the external spine of the 1st joint there are 2 smaller teeth. All external spines minutely denticulated along the external border. Apex of terminal spine on 3rd exopodal joint curved outwards: serrated lamella with slight incurvation near the base of the spine. The teeth along the lamella number 27; they are slightly coarser than those on 2nd feet.

Legs of the 4th pair almost as those of the 3rd (fig. 5d). The internal margin of the 1st basal joint carries a small plumose seta, which seems to have escaped GIESBRECHT'S attention. On the 2nd endopodal joint there are 6 strong spines on the posterior surface, arranged in one coronula: the other endopodal joints are nude. 2nd exopodal joint with a transverse row of small, hair-like spinules on the posterior surface, close to the articulation with the 1st exopodal joint. 3rd exopodal joint slightly longer than in the other pairs, endspine also slightly longer. Serrated lamella smoothly curved, with rather coarse marginal teeth, 26 of which are present.

Fifth pair of legs completely absent.

The adult male measures 1.89 mm. This sex has not been described by WILSON; the adult male of *Phaenna spinifera*, however, was carefully described and figured by GIESBRECHT (1892). As the present specimen closely agrees with GIESBRECHT'S account I refrain from giving a description here; for its external appearance I refer to fig. 6b.

The male in the Vth copepodid stage measures 1.92 mm. It differs in external appearance from the adult female only in the shape of the abdomen and the presence of an incompletely developed 5th pair of legs. I have not studied the oral appendages and legs in detail, with the exception of the 5th legs. The abdomen is incompletely developed (fig. 6a); there are four segments, but the first two segments, which

become genital segment after the final moult, still show no trace of receptacles. The anal segment is distinct and has a rounded anal flap. The furca is as in the adult female. The 5th legs are slightly asymmetrical, the left leg is slightly longer than the right (fig. 4a). There is a basal joint on both sides, more or less coalescent with the chitinous ridge at the ventral portion of the cephalothorax. There is a well separated intermediate joint on both sides, which carries no internal or external spines. The terminal joint on the left side is about twice as long as broad and has the indication of a line of fusion at about 1/3 of its length from the base. A small external spine is found some distance above that line; in addition there are two minute apical spines. The right terminal joint is slightly shorter than that on the left side; it carries one external spine at about 1/3 of its length from the base and two rather strong apical spines. The general shape of the appendage is in accordance with WILSON's figure of the 5th pair of legs of *Onchocalanus nudipes* (l.c., fig. 83), but three apical teeth are figured by WILSON.

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PRELIMINARY DESCRIPTIONS OF TWELVE NEW SPECIES OF PALAEMONID PRAWNS FROM AMERICAN WATERS (CRUSTACEA DECAPODA)

BY

L. B. HOLTHUIS

(Communicated by Prof. H. BOSCHMA at the meeting of Dec. 17, 1949)

During a study of the American Palaemoninae of the collections present in the U.S. National Museum, Washington, D.C., and the Rijksmuseum van Natuurlijke Historie, Leiden, Holland, for a revision of the American species of that subfamily, representatives of 12 species were found, which proved to be new to science. Of these species, like of most other American Palaemoninae, extensive descriptions and figures will be given in the above mentioned revision, which now is ready for the press.

In the Siboga monograph, which deals with the Palaemoninae, and which is now being printed, I have given a list of all the species of this group known to me. For completeness's sake I thought it advisable to include in this list also the new American species. As, however, it seems highly undesirable to me to use their names as nomina nuda, the following preliminary descriptions are given so as to validate these names.

The descriptions given in this paper are kept as concise as possible, since more extensive descriptions will be published later.

Macrobrachium inca nov. spec.

Adult male:

The rostrum is rather short and straight, reaching to the end of the antennular peduncle. The rostral formula is:

$$\frac{2-3)\ 10-14}{2-4}.$$

The carapace is smooth, but covered with short hairs. No tubercles are present.

The second legs are unequal in size, but about equal in shape. The fingers of the larger leg are 3/5 to 4/5 of the length of the palm. They possess one large tooth on the cutting edge, with small denticles behind it. The palm generally is more than twice as long as high. The carpus is about 7/9 of the length of the palm, about as long as the merus and about thrice as long as broad. All joints are covered with numerous club-like spinules. The palm and the basal part of the fingers are densely

pubescent, only the extreme dorsal part of the palm being naked. Carpus and merus too are pubescent, the merus is somewhat swollen. Small spinules are present on all the joints of the last three pairs of legs.

Distribution. The species is an inhabitant of fresh waters of N. Peru and Ecuador.

Type. The holotype is a male from Rio Mochè near Salavery, N. Peru. It is preserved in the U.S. National Museum.

Macrobrachium rathbunae nov. spec.

Adult male:

The rostrum is straight and rather high, reaching slightly beyond the end of the antennular peduncle. The rostral formula is:

$$\frac{1-2)\ 9-10}{3-6}.$$

The carapace is smooth, only bearing some hairs in the anterolateral region.

The second legs are equal in shape and subequal in size. The fingers are 1/2 to 4/7 as long as the palm. The fingers bear one large tooth on the cutting edge, behind which tooth there are some small denticles. The palm is about 5 to 6 times as long as high. The carpus is longer than the palm, but shorter than the chela, it is about 6 times as long as high, and 4/3 to 5/4 as long as the merus. All the joints are covered with small spinules. The fingers are entirely covered by a thick pubescence. No pubescence is present on the palm or any of the other joints. The last three pairs of legs have all the joints densely covered with spinules.

Distribution. The species lives in fresh water of the west coast of America from Panama to Ecuador.

Type. The holotype originates from Hog Creek Valley, San José Island, Archipielago de las Perlas, Gulf of Panama. It is preserved in the U.S. National Museum.

Macrobrachium transandicum nov. spec.

Adult male:

The rostrum is high and straight, reaching to or slightly beyond the antennular peduncle. The rostral formula is:

$$\frac{5-7)\ 13-16}{3-4}.$$

The carapace is smooth.

The second legs are unequal in size but about equal in shape. The fingers each bear one large tooth on the cutting edge, which posteriorly as well as anteriorly of this tooth bears several smaller teeth. The fingers are about 0.7 times as long as the palm. The latter is about 4 times as long as high. The carpus is slightly shorter than the palm, being about

thrice as long as broad and 4/3 as long as the merus. All joints bear numerous small spinules. The chela is naked (except for a short pubescence along the cutting edges like in most other species of this genus), but the carpus and merus are pubescent underneath.

The last three pairs of legs are covered with numerous small spinules.

Distribution. Fresh water of W. Colombia (Pacific slope).

Type. The holotype originates from the Rio Telembi near San Lorenzo, S.W. Colombia, it is preserved in the U.S. National Museum.

Macrobrachium occidentale nov. spec.

This species is very closely related to *Macrobrachium heterochirus* (Wiegmann), of which it might be considered the representative in W. America. *M. occidentale* differs from *M. heterochirus* by having generally 5 or 6 teeth of the rostrum placed behind the orbit, these teeth occupying less than 1/3 of the dorsal length of the carapace (rostrum excluded). The first legs of the present species have the carpus decidedly less than twice as long as the chela. The right and left second legs of the adult male are much more different in shape. The fingers of the larger leg are gaping and have the cutting edge provided with 5 to 8 denticles, which are placed almost up to the tip and are of almost equal size. The carpus is more robust than in *M. heterochirus*. The smaller second leg is less elongate than in *M. heterochirus*.

Distribution. The species is known from fresh waters of the west coast of Central America from Guatemala to S. Panama.

Type. The holotype orginates from Rio de los Esclavos, just S. of Cuilapa, Guatemala. It is preserved in the U.S. National Museum.

Macrobrachium crenulatum nov. spec.

Adult male:

The rostrum is straight and reaches about the end of the antennular peduncle. The rostral formula is:

$$\frac{4-6)\ 11-14}{3-4}.$$

The carapace is smooth.

The second legs are strongly unequal in shape and size. The larger leg has the fingers about as long as the palm. The fingers generally are curved and gape. One large tooth is present in the proximal part of the cutting edge. Behind this tooth there are some small denticles. Anteriorly of the large tooth the cutting edge is crenulated. The palm is compressed, it is about twice as long as high. Longitudinal rows of strong spines are present on the outer surface and the lower margin of the palm, leaving an ill-defined smooth region in the middle. Both fingers are densely covered with small spinules which are not placed in distinct longitudinal

rows. Small spinules are present on the inner surface of the palm. There is no conspicuous pubescence on the outer surface of the palm, but the inner surface, especially in its ventral part, bears a long and dense pubescence. The carpus is twice as long as high and strongly narrowed near the base. It is shorter than the palm and also shorter than the merus. The merus is somewhat thickened in the middle. Longitudinal rows of spinules, but no pubescence, are visible on the carpus and merus. The smaller leg has the fingers about 1.5 times as long as the palm, they are gaping. There are some denticles in the proximal part of the cutting edges. The gap between the fingers is filled by stiff inwards directed setae. The carpus is longer than the palm and shorter than the merus. The spinulation is as in the large leg, but less strong.

The last three pairs of legs bear no spinules except those on the posterior margin of the propodus, while in the third and fourth legs there is a row on the posterior margin of the merus.

Distribution. The species inhabits fresh waters of the West Indies, Venezuela and E. Panama.

Type. The holotype originates from the Pejebobo River, E. Panama and is preserved in the U.S. National Museum.

Macrobrachium hancocki nov. spec.

This species is very closely related to the previous, from which it differs in a few small, but constant characters. In the first place the rostrum is shorter, usually not reaching beyond the middle of the last joint of the antennular peduncle. The carpus of the first leg is decidedly less than twice the length of the chela. The outer surface of the fixed finger of the second leg bears in the basal part a few spinules, arranged in only two or three rows, while in *M. crenulatum* very numerous, rather irregularly placed spinules are present there.

Distribution. The species lives in fresh water of W. America from Costa Rica to Colombia, Cocos Island and the Galápagos Archipelago.

Type. The holotype originates from Esparta, Rio Barranca, Costa Rica, and is preserved in the collection of the U.S. National Museum.

Remarks. This species and the previous have been recorded in literature by some authors as *Macrobrachium* (or *Palaemon*) *olfersii* (Wiegmann), but certainly are distinct from that species.

Palaemon gladiator nov. spec.

A species of *Palaemon* which is very close to *Palaemon ritteri* Holmes, but differs from it by having the rostrum much more slender and ending in a narrow unarmed sword-like tip. Furthermore the sixth abdominal segment is much longer than in *P. ritteri*. The anterolateral spine of the basal segment of the antennular peduncle is much smaller, the free portion of the shorter ramus of the antennular flagellum is less than

twice as long as the fused portion (in *P. ritteri* it is much more than twice as long), the carpus of the second legs in the new species is longer than the palm, whereas it is shorter than the palm in *P. ritteri*. And finally the last three pereiopods are more slender in the new form.

Distribution. The species is known from Clipperton Island and rom the Galápagos Archipelago.

Type. The holotype originates from Academy Bay, Indefatigable Island, Galápagos Archipelago (Allan Hancock Expeditions). It is preserved in the collection of the U.S. National Museum.

Palaemon peruanus nov. spec.

This new species is erected for the specimen identified with some doubt by RATHBUN (Proc. U.S. Nat. Mus., vol. 38, 1910, p. 561, pl. 53 fig. 1) with *Palaemon ritteri*. Like *Palaemon gladiator* it differs from *P. ritteri* by having the carpus of the second legs longer than the palm. The rostrum is shorter than that of *P. ritteri* and much shorter than that of *P. gladiator*, from which it differs furthermore by not ending in a slender sword-like tip. The anterior margin of the basal segment of the antennular peduncle is produced much farther forwards than in *P. ritteri*. In the shape of the sixth abdominal segment and the antennular flagellum it shows most resemblance to *P. ritteri*. The legs of this new species reach distinctly farther forwards than in either *P. ritteri* of *P. gladiator*.

Distribution. The species is only known from the holotype, which originates from salt creeks at La Palisada near Tumbes, N. Peru, and is preserved in the U.S. National Museum.

Palaemon hancocki nov. spec.

This new species is very close to *Palaemon gracilis* (SMITH) but differs from that species by possessing 12 to 16 ventral teeth on the rostrum (*P. gracilis* 9—12), and by having the last joint of the third maxilliped shorter than the penultimate (it is as long as the penultimate in *P. gracilis*). In the new species the second legs are more slender than in *P. gracilis*, the fingers are shorter than the palm and the carpus is twice as long as the chela. Furthermore the species attains a much larger size than *P. gracilis* does.

Distribution. The species inhabits fresh water of Colombia and Ecuador.

Type. The holotype is a specimen from the Guayas River, Ecuador (Allan Hancock Expeditions). It is preserved in the U.S. National Museum.

Palaemon schmitti nov. spec.

The present species strongly resembles *Palaemon tenuipes* (HENDERSON) from India and *Palaemon hastatus* Aurivillius from West Africa. It agrees with these two species and differs from all other species of *Palaemon*

by having the dactyli of the last three pairs of legs extremely long and slender, being distinctly longer than the propodus and carpus combined. *Palaemon schmitti* agrees with *Palaemon tenuipes* in almost all respects, the main difference being found in the rostral formula. This formula for *P. schmitti* runs as follows:

$$\frac{1-2)\ 3-5+1}{7-9}.$$

For *Palaemon tenuipes* the rostral formula is:

$$\frac{1-3)\ 4-6+1}{2-6}.$$

The rostral formula of *Palaemon hastatus* is:

$$\frac{1-3)\ 7-11+1}{3-6}.$$

Distribution. The species inhabits estuaries of large rivers in Surinam (= Dutch Guiana).

Type. The holotype originates from the mouth of the Surinam River near Resolutie, Dutch Guiana. It is preserved in the Rijksmuseum van Natuurlijke Historie at Leiden.

Palaemonetes ivonicus nov. spec.

The rostrum is straight and reaches slightly beyond the antennular peduncle. The rostral formula is:

$$\frac{1)\ 6-10}{3}.$$

The teeth are placed rather regularly over the upper margin of the rostrum. The branchiostegal spine is placed much below the branchiostegal groove and is so far removed from the anterior margin of the carapace that its tip just falls short of that margin.

The anterior margin of the basal segment of the antennular peduncle is strongly produced forwards, it reaches to the end of the second segment. The free part of the shorter ramus of the upper antennular flagellum is twice as long as the fused part. The scaphocerite has the outer margin slightly convex, the final tooth is far overreached by the lamella.

The second pereiopods have the fingers slightly shorter than the palm. The cutting edge of the dactylus bears two, that of the fixed finger one tooth. The carpus is 1.6 times as long as the chela and about 1.8 times as long as the merus.

Distribution. The species is only known from the 2 type specimens, which originate from Ivon at the Beni River, N. Bolivia. They are preserved in the U. S. National Museum.

Palaemonetes schmitti nov. spec.

The rostrum is straight and very deep. It reaches slightly beyond the scaphocerite. The rostral formula runs as follows:

$$\frac{2)\ 11-13}{2-4}.$$

The branchiostegal spine is placed on the anterior margin of the carapace just below the branchiostegal groove.

The anterior margin of the basal segment of the antennular peduncle is convex and reaches as far forwards as the anterolateral spine. The free part of the shorter ramus of the upper antennular flagellum is about twice as long as the fused portion. The scaphocerite has the outer margin straight or slightly concave. The final tooth reaches almost to the end of the lamella.

The second legs have the fingers 0.6 to 0.9 times as long as the palm. The cutting edge of the dactylus bears 2, that of the fixed finger 1 tooth. The carpus is about as long as the palm and 3/4 of the length of the merus.

The species is closely related to *Palaemonetes vulgaris*, but may be distinguished by the much deeper rostrum, which has the upper margin straight and provided with more teeth (11 to 13 instead of 8 to 11). The telson of the present form is shorter and its legs are distinctly longer than in *P. vulgaris*.

Distribution. The species apparently is a marine form and up till now is found only in W. Panama.

Type. The holotype originates from the Upper Chamber, East side, Miraflores Locks, Panama Canal, and is preserved in the U.S. National Museum.

ON THE CRINOID NATURE OF TIMOROCIDARIS SPHAERACANTHA WANN.

BY

R. LAKEMAN

(Communicated by Prof. H. A. BROUWER at the meeting of Dec. 17, 1949)

Timorocidaris sphaeracantha WANN. is one of the most common petrefacts in the permian fossiliferous layers at Timor.

This petrefact, as shown in fig. 1 consists of a hemispherical body,

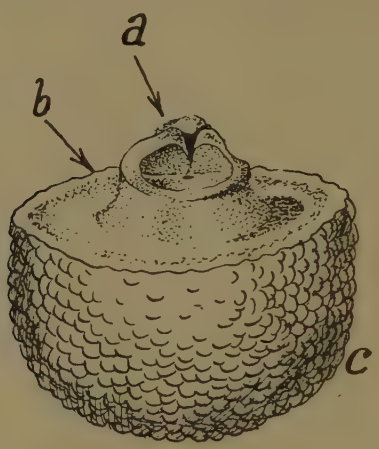

Fig. 1. *Timorocidaris sphaeracantha* WANNER. a. joint-facet. b. joint-cone. c. hemisphere (schematic, $2^1/_2 \times$ nat. size).

with on the flat side of the hemisphere a joint-cone, bearing three joint-facets. Some specimens showing aberrant shapes, were described by WANNER (1940) under the names *Timorocidaris pistilliformis*, *T. fusiformis*, *T. baculiformis*, etc. As the number of these specimens is insignificant as compared with the number of specimens found of *Timorocidaris sphaeracantha* WANN., the author only uses the name *Timorocidaris* in the following, when *Timorocidaris sphaeracantha* WANN. is meant.

The examination of a number of *Timorocidaris* specimens (coll. L. 21 and 22) has led the author to the conclusion that the place of this fossil in the biological system, as proposed by WANNER in 1940 is not in accordance with all the peculiarities which these specimens show.

The object of this article is to suggest a new theory on the nature of this petrefact.

This problematic petrefact presumably is an axillary primibrach of an unknown crinoid, belonging to the *Poteriocrinitidae*. [1])

[1]) *Poteriocrinitidae* S. L. according to ROEMER or WACHSMUTH & SPRINGER.

The author is greatly indebted to Prof. H. A. Brouwer, Director of the Geological Institute of the University of Amsterdam, who has put the extensive collection of this Institute at his disposal.

To Prof. J. Wanner who has been so kind to discuss the nature of *Timorocidaris* with the author, and who has given most valuable advice, and to Prof. M. G. Rutten who also took a great interest in this matter, the author wishes to express his sincerest gratitude.

For a thorough description of *Timorocidaris* the reader is referred to J. Wanner, Palaeontographica, Supplement band IV, 1940.

General remarks

The variation in shape of *Timorocidaris*, makes it difficult to decide which we must take as the standard-form. About 55 to 60 % of the specimens have a granular surface of distinct nature. This is considered the standard surface. 75 % have a joint-cone with three joint-facets; this joint-cone is also considered as standard type. Apart from the joint-cone 70 % of the specimens are hemispherical. If again we take this shape as the standard type, it appears that only 20 % of the specimens possess these three properties combined, though generally deviations are small. This illustrates how variable these petrefacts are as to their shape.

Epizoa have grown on 2 % of the petrefacts, and are of considerable interest. They grow principally on the periphery, and seldom on the middle of the hemisphere. Sometimes we find them on the flat side growing up to the facets. The epizoa never grow on the facets however, these always remain free, which proves definitely that the epizoa attached themselves only to the living specimens.

The place of Timorocidaris in the zoological system

Bather (1920) has observed that the first question to be answered is whether *Timorocidaris* is a crinoid or an echinoid, either possibility being favoured by several arguments.

Supporting the echinoid nature of *Timorocidaris* are:

a. the shape;

b. the fact that one specimen in the collection of the Geological Institute in Amsterdam has a joint-cone resembling the joint of an echinoid spine.

Sub *a.* *Timorocidaris* sometimes resembles very much the spines of *Tylocidaris vexilifera* Schlüter, (Schlüter 1892, p. 126).

Sub *b.* Only one example has been found showing an echinoid spine joint. Therefore it is dangerous to use this, perhaps coincidental shape as an argument in favour of the echinoid nature of *Timorocidaris*.

There are several facts supporting the crinoid character of *Timorocidaris*:

a. The specimens nearly always have a joint-cone with joint-facets of distinctly crinoid character.

b. Following BATHER (1928), we can only account for these three facets, if each facet has one movable part attached to it.

c. In the joint-pyramid canals are found. The occurrence of canals near the joint of echinoid spines is an unknown feature, but it does occur with crinoids.

d. The body is one in structure, and cannot be divided into spine-axis and -periphery. The echinoid spine also consists of one crystal, but here a structural difference between axis and periphery exists.

e. If *Timorocidaris* were part of an echinoid, the fossil concerned would most probably be rather large. No other skeleton parts belonging to *Timorocidaris* have ever been found. Thick plates would have been fossilised together with *Timorocidaris*, however.

f. It has been observed with echinoids (MORTENSEN 1928), that spines sometimes branch; never, that two spines, each with a separate joint, grow together.

g. In *Timorocidaris* the surface of the hemisphere may project a little past the distal end of the joint-pyramid. If *Timorocidaris* had been an echinoid spine, a long joint-cone on the plate of this echinoid would have been necessary. Otherwise the attachment of *Timorocidaris* to the echinoid is impossible. Even if the echinoid plate had a rather protruding joint-cone there still would have been little space for movement of this spine.

It is obvious that the arguments in favour of the crinoid nature of *Timorocidaris* are greater in number, and more convincing. BATHER (1920) was the first to come to the conclusion that these petrefacts must be of crinoid nature. Afterwards WANNER (1940) was convinced by BATHER's arguments.

Timorocidaris as an element of the crinoid cup

This appears indeed to be the most likely explanation, though by this solution it is not settled what *Timorocidaris* really has been. BATHER (1920) has made the following suggestion: "To the three facets of *Timorocidaris* three arms were attached. The hemisphere originated from radials, basals, and possibly infrabasals, by the fusing of these three circlets of crinoid plates". According to WANNER (1940) *Timorocidaris* originated only from radials. Thus it is more acceptable that the different parts have fused to one crystal. It is difficult indeed to accept, that *Timorocidaris* has originated from something like 11 different parts.

BATHER (1920) suggested that the body was situated between the arms, considering the hemisphere as an anchor. WANNER, considering the bottom too uneven and too oozy for this mechanism, does not believe in BATHER'S theory that this anchor kept the crinoid in a perpendicular position.

WANNER (1940, p. 230) mentioned moreover the possibility that

Timorocidaris is not the spine of an echinoid, but a similar part of a crinoid. Such crinoid spines are also known, e.g. *Scyphocrinus* SPRINGER (1917). These Scyphocrinus spines outwardly resemble *Timorocidaris claveaformis* WANN. SPRINGER (1917) mentioned that these spines are attached on warts on the plates of *Scyphocrinus*, and thus are attached by one joint only. The *Scyphocrinus* spines do branch, contrary to other crinoid spines as in *Anthroacantha*. In spite of this great conformity WANNER rejects the assumption that *Timorocidaris* is a crinoid spine,

Fig. 2. Specimen of *Timorocidaris* showing disturbances in growth on the flat side of the hemisphere; seen from above. The joint-cone is visible in the centre. (Palaeontographica Suppl. IV, Abt. IV, Table XV, fig. 24. Spec. Br. Polaeont. museum Bonn. Nat. size).

as the arguments against the echinoid nature of this petrefact also hold good against this assumption.

Besides the arguments mentioned under *a-g*, WANNER (1940, p. 229) puts forward two more arguments to prove the crinoid nature of *Timorocidaris:*

1. The specimen of fig. 2 shows that its growth has been disturbed between the joint-cone and the periphery.
2. The epizoa found on *Timorocidaris* disprove the echinoid nature of these petrefacts. If *Timorocidaris* were an echinoid spine, WANNER cannot imagine why these epizoa are not limited to the outer surface of the hemisphere, but also grow on its sides, and on the flat lower surface. If *Timorocidaris* had formed echinoid spines, there would not have been any space for the epizoa between the spines mutually, and between the spines and the echinoid body. Some epizoa even protrude considerably sideways. According to WANNER this proves that during life, *Timorocidaris* was not closely surrounded by other specimens.

Sub 1. The specimen of fig. 2 proves that *Timorocidaris* has not been a crinoid cup. The three, probably thin arms, that we must assume in this case, could not be the cause of disorders in growth on the flat side of the hemisphere. If the specimen of fig. 2 does show disorders in growth, this speaks against the assumptions that *Timorocidaris* is a crinoid cup or anchor, or an echinoid spine.

Sub 2. The simultaneous occurrence of laterally flattened specimens, and of specimens with epizoa growing considerably sideways, can be explained by the assumption that *Timorocidaris* occurred in rows, but

not as a kind of pavement on a larger body. In this case *Timorocidaris* could freely develop in one direction, but not in the direction perpendicular to this one. The great number of laterally flattened specimens is in support of this assumption.

Considering the arguments *a-g* we note that:

Sub a. This argument only proves the crinoid nature of *Timorocidaris* but does not tell us anything about the mode of occurrence.

Sub b. This argument disproves the occurrence of *Timorocidaris* on one joint-cone of a cup-plate, but does not prove that this petrefact could not in another way be part of a crinoid. In the latter case *Timorocidaris* must have been attached to independently movable skeleton elements.

Sub c. What has been said sub *a*, holds good for this argument also.

Sub d. This argument does not speak against the assumption of *Timorocidaris* being part of a crinoid, on the contrary.

Sub e. This argument speaks against the occurrence of *Timorocidaris* on another organism with large cup-plates. We will refer to this later on.

Sub f. The twinning of *Timorocidaris* is an indication against the echinoid nature. With crinoids the fusing of skeleton elements is not unusual.

Sub g. The attachment of a specimen with a protruding rim to a cup-plate is improbable. If we assume that *Timorocidaris* was attached to three independently movable skeleton-elements, this difficulty is eliminated. Moreover, in those specimens in which the rim protrudes far, it shows signs that it was prevented from growing further by a body facing it.

From the preceding it appears that the possibility of *Timorocidaris* being a movable part of a crinoid ought to be considered more closely.

The *Poteriocrinitidae* have skeleton elements supplied with three joint-facets of exactly the same nature as those of *Timorocidaris*, but larger, e.g. the axillary primibrachials (1 Ax). This is the first plate above the cup, belonging to the arm, and which is followed immediately

Fig. 3. Axillary primibrachial of a *Poteriocrinites*. The darts point to the facets. Two of the three facets are visible on this figure. (Spec. L 25. Geol. Inst. Amsterdam. $1\frac{1}{2}\times$ nat. size).

by two branches of the arm. This plate has two facets on its top, giving rise to the two branches, and is itself attached to the underlying radial, also by means of a facet, (fig. 3). This axillary primibrachial often has

at its distal end a sculptured elevation, and shows a striking resemblance with the specimen of fig. 4, interpreted by WANNER (1940) as

Fig. 4. Juvenile form of *Timorocidaris*. The darts point to two of the three facets, visible on this figure. (Specimen Wf. coll. WANNER. Palaeontographica Suppl. IV, Abt. IV, Table XVII, fig. 11, 3.2 × nat. size).

a juvenile form. If we assume that *Timorocidaris* were axillary primibrachials, we get a crinoid as shown in fig. 5 and 6.

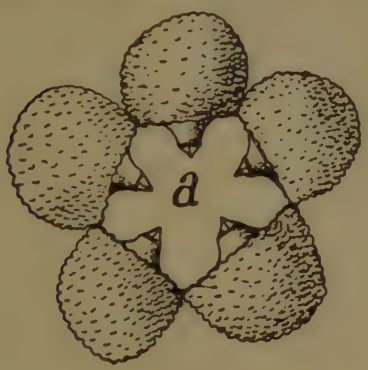

Fig. 5.

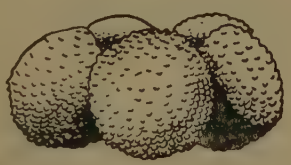

Fig. 6.

Reconstruction of the hypothetical crinoid, a. place of cup. Fig. 5 seen from above, and fig. 6 side view.

A great many features of *Timorocidaris* favour this hypothesis, e.g.:

a. A striking resemblance exists between the structure and the arrangement of the joint-facets of *Timorocidaris* and of the axillary primibrachials of some crinoids. Many of the primibrachials of the *Poteriocrinitidae* show at their distal end a sculptured elevation, as has been said before. If this sculptured elevation were larger with regard to the joint-facets, it would show a great resemblance to the juvenile specimen of *Timorocidaris* represented by fig. 4. In this specimen the joint-cone is also missing, and the facets are in this case much larger in comparison to the body. Here the sculpture of the hemisphere is almost completely missing also and these three features correspond more with those of the crinoid primibrachials than with those of adult *Timorocidaris* specimens. The three facets of *Timorocidaris*, however, do not show much difference in size mutually, whereas the facets of the axillary primibrachials of the *Poteriocrinitidae* do show difference in size. The only indication that the three facets of *Timorocidaris* are not quite

equivalent is perhaps the frequent occurrence of specimens in which the three joint-facets do not form a pyramid with an equilateral triangle at its base. This triangle often shows one angle of about 90°, the other angles being about 45°. The facets of the primibrachial of the *Poteriocrinitidae* often show the same arrangement.

b. A number of specimens of *Timorocidaris* has been found with a flat side. In this flat side we sometimes see imprints of other *Timorocidaris* specimens. WANNER has made a plaster cast of the imprints of the specimen of fig. 7. This cast proved that the imprints were caused

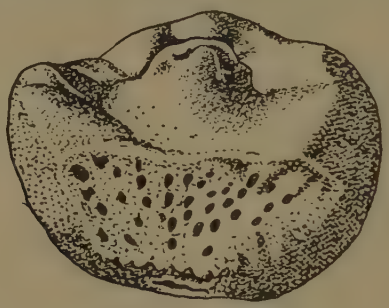

Fig. 7. Specimen Wt (coll. WANNER. Polaeontographica Suppl. IV, Abt. IV, Table XVI, fig. 8), with imprints on one side of the hemisphere. These are caused by the surface sculpture of a neighbouring specimen of *Timorocidaris* growing on the same cup. (2.9 × nat. size).

by surface sculpture of another specimen of *Timorocidaris*. We may follow WANNER in assuming that these two specimens were lying side by side on the bottom of the sea, and were themselves squeezed between other objects, during a considerable period of their lives. Or we may assume that these specimens could not get out of each others way, because they were attached to a common object. The latter solution seems the most probable.

c. In some cases the imprints are developed as circular furrows. These arcs form parts of concentric circlets, whose centre is situated near the top of the joint-pyramid. This is not surprising if we assume that both specimens have pivoted upon their joint-pyramids.

d. WANNER (1940, p. 232) remarked that facetless specimens are biologically incomprehensible if we assume an armless crinoid, for in this case the crinoids have strongly developed orals. These orals are missing, and we cannot understand how the animal could take in food. With crinoids, it is not unusual that one or more of the arms are missing. Evidently the axillary primibrachials have not developed facets in this case, as one joint was sufficient to attach them to the crinoid-cup, no facets being necessary for attachment of the branches of the arm. Presumably the axillary primibrachial was not obliterated itself, but only served as protection of the crinoid, and flexibility of the joint was not necessary in this case.

e. About 50 % of the flattened specimens developed only two facets. In these cases the facets are always situated in such a way that they correspond to the remaining direction of movement, fig. 8. If one of

Fig. 8. Specimen L 22 seen from above. The joint-cone visible in the centre of the figure has only two facets. The orientation of the facets is in accordance with the only possible direction of movement (indicated by the darts) (coll. Geol. Inst. of Amsterdam, 1½ × nat. size).

the facets is missing, we must assume that one of the branches of the arm is also missing, presumably also because of lack of space.

The last two characteristics are illogical if we assume that *Timorocidaris* has been a freely living body on the bottom of the sea. The remaining possibility is that *Timorocidaris* is part of a crinoid, probably belonging to the *Poteriocrinitidae*. The difficulty arises that the cup, however, has never been found. According to WANNER, (personal communication) it is no exception that skeleton-parts were found in great quantities in Timor, while other elements of the fossil to which these parts belong have seldom or never been found; e.g. *Miocidaris permicus*, 540 spines on three cup-plates, and crinoidal stem-fragments which do not belong to any known crinoid-cup.

The author believes that *Timorocidaris* was a strongly differentiated axillary primibrachial serving for the protection of a crinoid consisting of thin plates.

The hypothesis that the crinoid-cup consisted of thin plates explains two phenomena:

a. The cup plates have never been found, as during sedimentation these thin plates were not deposited at the same locality as the large *Timorocidaris* specimens.

b. The joint-facets of *Timorocidaris* are equivalent because the weight attached to them was equivalent, (i.e. the cup was about equivalent in weight to the arms).

The conception of the author is that *Timorocidaris* served also for the floating of this free-living crinoid.

Conclusion

Timorocidaris sphaeracantha WANN. should be understood to be a

strongly differentiated part of a crinoid, corresponding to the axillary primibrachials occurring with the *Poteriocrinitidae*, which often resemble these *Timorocidaris* specimens in organisation and in structure. Therefore we assume that *Timorocidaris sphaeracantha* WANN. belongs to a crinoid of the family of the *Poteriocrinitidae*.

In one of his letters to the author, Prof. J. WANNER suggested some other possibilities regarding the organisation of *Timorocidaris*.

a. *Timorocidaris* is a differentiated centrodorsal plate as occurring with *Somphocrinus mexicanus*, (Peck 1948, Journal of Palaeontology, vol. 22, Nr. 1).

In this case the three elements attached to the facets of *Timorocidaris* represent radials. These radials of *Somphocrinus* have a restricted mobility. In their lower parts, they were not attached to each other.

The five joints of the centrodorsal plate of *Somphocrinus*, however, show a Cyathocrinidal character.

b. *Timorocidaris* is a peculiarly differentiated peripheral plate of a crinoidal proboscis. The proboscis, which was distally broadened, was mushroom-shaped like the proboscis of the *Hydreionocrinidae*. WANNER compares *Timorocidaris* in particular with the proboscis-plates of *Teratocrinus bulbosus* (WANNER 1940, page 238). These plates show a remarkable conformity with the twinned specimens of *Timorocidaris*.

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ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. VIII [1])

a. *Influence of the n. primary alcohols C_1—C_6 on the elastic behaviour of the 0.6 % oleate system.*

b. *The use of $G - \lambda$, $G - 1/\lambda$ and $G - n$ diagrams in discussing the results obtained with the homologous series of the alcohols.*

BY

H. G. BUNGENBERG DE JONG AND H. J. VAN DEN BERG

(Communicated at the meeting of January 28, 1950)

1. *Introduction.*

Section 2 deals with the action of aliphatic alcohols $C_1 - C_6$ on the 0.6 % oleate system, an investigation wholly analogous to that on the 1.2 % oleate system in Part VI. We will then compare the results obtained with the 0.6 % and the 1.2 % systems, which comparison seemed of importance in connection with the findings in Part III that the damping in the 0.6 % system shows quite another character than in the 1.2 % system.

In the next sections we will consider the question if there exist specific differences between the various alcohols, with regard to the nature of the action excerted on the elastic properties of the oleate system by the alcohol molecules bound to the oleate.

2. *Influence of the first six terms of the n. primary alcohols on the elastic behaviour of the 0.6 % oleate system at the KCl concentration of minimum damping.*

The oleate system used for the following experiments had the composition 0.6 g oleate per 100 ml 1.43 N KCl + 0.12 N KOH, which electrolyte composition corresponds to the minimum damping of the elastic oscillations. [2]) We used exactly half filled spheres of nominally 500 ml and followed the same method as described in detail in Part VI of this series (cf. ibid. section 1 and 3). [3]) For exciting the rotational

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[1]) Part I has appeared in these Proceedings **51**, 1197 (1948), Parts II–VI in these proceedings **52**, 15, 99, 363, 377, 465 (1949), Part VII in these Proceedings **53**, 7 (1950).

[2]) As our stock of Na oleinicum medicinale pur. pulv. Merck was nearly exhausted and could not possibly be replenished, we decided to use for the experiments in this section the large volume of the oleate system, which had already served in previous experiments (on the dependence of T and Λ on R, see Part III, section 3, of this series).

[3]) In one respect we deviated from this method for investigating the influence of ethanol and methanol. These alcohols exert their full influence at concentrations

oscillation we made use of the contrivance described in Part VII of this series.

The results have been given in Table I, and in fig. 1. In this figure

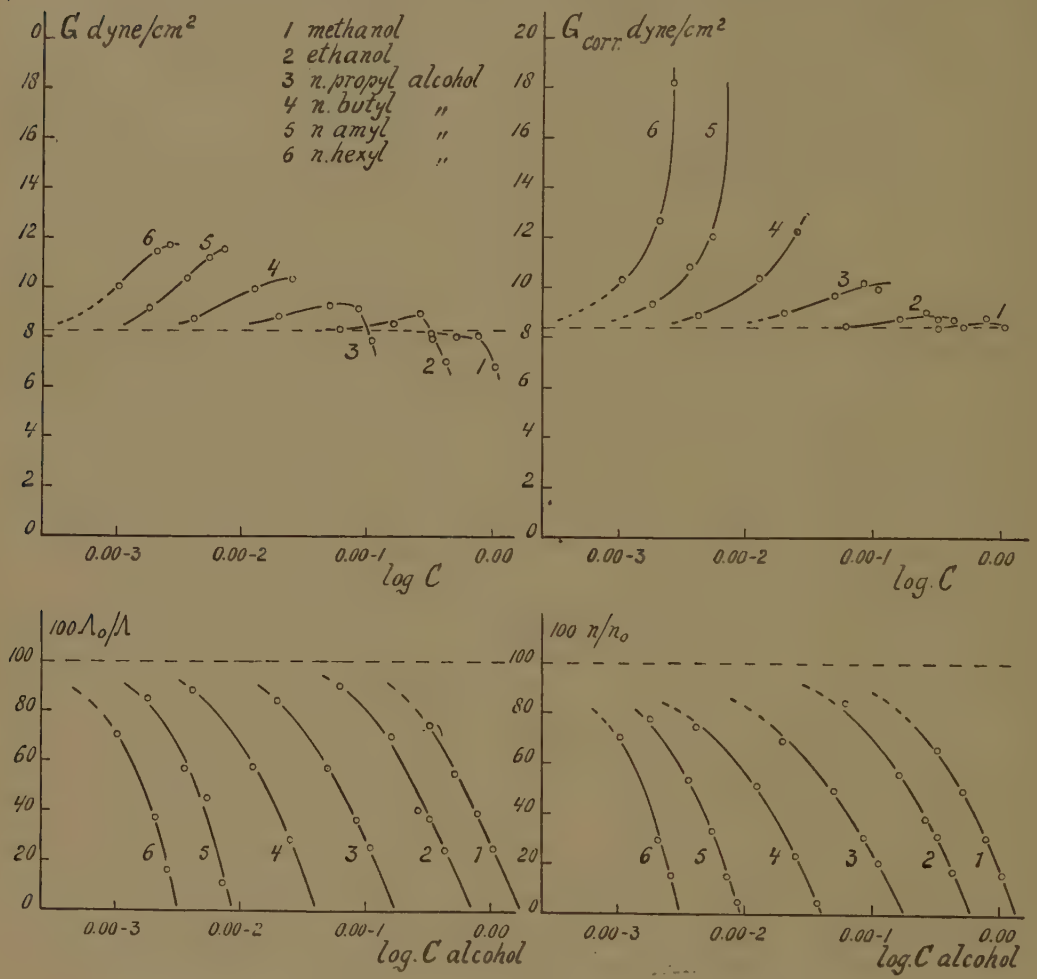

Fig. 1

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which correspond to an increase in volume of the elastic system of one or several per cent., which again necessitates a number of corrections (because of changes in the degree of filling, in the oleate concentration and in the KCl + KOH concentration).

The necessity of applying all these corrections was made redundant by the use of an auxillary mixture, which apart from a relatively large, known alcohol content, had the same composition as the elastic system as regards the oleate, the KCl and the KOH concentration. First a certain volume of this auxillary mixture was added to the oleate system, the contents of the vessel thoroughly mixed and by removing then a same volume as had been added, the degree of filling, the oleate, KCl and KOH concentrations have not been altered and we have only introduced a certain known quantity of alcohol.

TABLE I

Influence of n. primary alcohols on the elastic behaviour of the 0.6 % oleate system at the KCl concentration of minimum damping (at 15°)

Alcohol	conc. moles/l	log. conc.	G dyne/cm²	G corr. for Λ	$100\ G/G_0$	Λ	n	$100\ \Lambda_0/\Lambda$	$100\ n/n_0$
methanol	blank	$-\infty$	8.22	8.35	100	0.758	19.6	100	100
	0.321	0.51−1	8.11	8.35	100.0	1.026	12.9	73.9	65.8
	0.517	0.71−1	8.01	8.42	100.8	1.378	9.7	55.0	49.5
	0.775	0.89−1	8.00	8.75	104.8	1.928	6.0	39.3	30.6
	1.034	0.01	(6.78)	(8.31)	(99.5)	2.98	3.3	25.5	16.8
ethanol	blank	$-\infty$	8.24	8.36	100	0.751	19.6	100	100
	0.0617	0.79−2	8.31	8.46	101.2	0.835	16.5	89.9	84.2
	0.164	0.21−1	8.50	8.74	104.5	1.073	11.0	70.0	56.1
	0.266	0.42−1	8.89	9.03	108.0	1.851	7.5	40.6	38.3
	0.327	0.51−1	7.91	8.71	104.2	1.991	6.2	37.7	31.6
	0.429	0.63−1	(6.99)	(8.64)	(103.3)	3.05	3.4	24.6	17.3
n. propyl alcohol	blank	$-\infty$	8.31	8.43	100	0.762	19.5	100	100
	0.0200	0.30−2	8.78	8.96	106.3	0.906	13.5	84.1	69.2
	0.0501	0.70−2	9.21	9.64	114.3	1.338	9.7	57.0	49.7
	0.0852	0.93−2	9.12	10.14	120.3	2.096	6.0	36.4	30.8
	0.110	0.04−1	(7.84)	(9.87)	(117.1)	3.012	4.0	25.3	20.5
n. butyl alcohol	blank	$-\infty$	8.28	8.40	100	0.762	19.6	100	100
	0.00417	0.62−3	8.69	8.85	105.3	0.862	14.6	88.4	74.5
	0.0125	0.10−2	9.89	10.33	123.0	1.324	10.0	57.6	51.0
	0.0250	0.40−2	(10.32)	(12.18)	(145.0)	2.674	4.5	28.5	23.0
	0.0375	0.57−2	—	—	—	—	1.0	—	5.1
n. amyl alcohol	blank	$-\infty$	8.25	8.37	100	0.765	19.7	100	100
	0.00182	0.26−3	9.14	9.32	111.4	0.905	15.2	84.5	77.2
	0.00364	0.56−3	10.35	10.83	129.4	1.339	10.5	57.1	53.3
	0.00546	0.74−3	11.17	12.00	143.4	1.702	6.5	44.9	33.0
	0.00728	0.86−3	(11.53)	(25.4)	(303.5)	6.89	3.0	11.1	15.2
	0.00850	0.94−3	—	—	—	—	1.0	—	5.1
n. hexyl alcohol	blank	$-\infty$	8.27	8.39	100	0.759	19.6	100	100
	0.00105	0.02−3	10.06	10.36	123.5	1.077	13.8	70.5	70.4
	0.00210	0.32−3	11.42	12.65	150.8	2.050	5.8	37.0	29.6
	0.00263	0.42−3	(11.68)	(18.2)	(217.2)	4.70	3.0	16.1	15.3

mean value of the blanks: $G_0 = 8{,}38$; $\Lambda_0 = 0.760$; $n_0 = 19.6$.

we give two representations of the functional relation between G and the logarithm of the alcohol concentration, which differ in the way G has been calculated from the experimental data (we shall return to this below). In figure 1 we give further $1/\Lambda$ and n in per cent. of the blank values.

We miss in this fig. 1 a graph for $100\ \lambda/\lambda_0$, because it is not possible

to calculate a relaxation time from the experimental data. Cf. Part III of this series in which it was shown, that the 1.2 % and the 0.6 % oleate system differ as regards the functional relation between the logarithmic decrement Λ and the radius of the sphere R (in the 1.2 % system Λ is proportional to R; in the 0.6 % system Λ is independent of R). The quantitative relationships found in Part III gave strong support to the conception that the oscillations in the 0.6 % oleate system are damped in consequence of slipping along the boundary of the glass vessel. This case has been dealt with theoretically by J. M. BURGERS [4]). In a subsequent publication of the same author [5]) it was shown that in the case of slipping along the wall instead of the usual one quite another correction of the period had to be applied for calculating G, as the former holds good only in the case of damping as a consequence of relaxation. He further pointed out, that there are certain indications in our experiments that, besides the supposed slipping relaxation too may play a rôle.

Since then we have made direct observations of the oscillations of very small air bubbles in the 0.6 % *oleate system in the immediate vicinity of the glass wall and came to the surprising conclusion that no slipping along the glass wall could be detected.* [6]) *We could not but come to the conclusion, that the explanation why in the* 0.6% *oleate system Λ is independent of R has not yet been given.*

Returning to the question how we are to calculate G in the present experiments, we must certainly not apply the correction given theoretically by BURGERS, as slipping along the wall is absent. We may follow two different ways:

a. use the uncorrected T values themselves, which gives the G values depicted in the upper left graph in fig. 1;

b. apply the usual correction, $T_{corr} = T/\sqrt{1 + (\Lambda/2\pi)^2}$, before calculating G, which gives the G values depicted in the upper right graph in fig. 1.

We will now compare the results laid down in fig. 1 with the analogous results obtained in Part VI with the 1.2 % oleate system. For this purpose the corresponding figure is once more inserted here as fig. 2 (see also Table II). [7])

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[4]) J. M. BURGERS, these Proceedings, **51**, 1211 (1948).

[5]) J. M. BURGERS, these Proceedings, **52**, 113 (1949).

[6]) We hope to return to these experiments in a later communication in this series.

[7]) Fig. 2 given here differs from fig. 2 given in Part VI by the course of the ends to the right of the G curves in particular for methanol and ethanol (in Part VI markedly bent upwards, here slightly bent downwards).

This difference results from omitting here an extra correction of T which we thought in Part VI to be necessary for highly damped oscillations (reduction of T to the isochronous level, compare Part VII, section 4). This correction we now

We may now first ask if their are any indications which point to a totally different character of the damping in the 0.6 % and the 1.2 % system. On comparing the analogous graphs (referring to $1/\Lambda$ and n) we see for both systems the same characteristics. All alcohols decrease

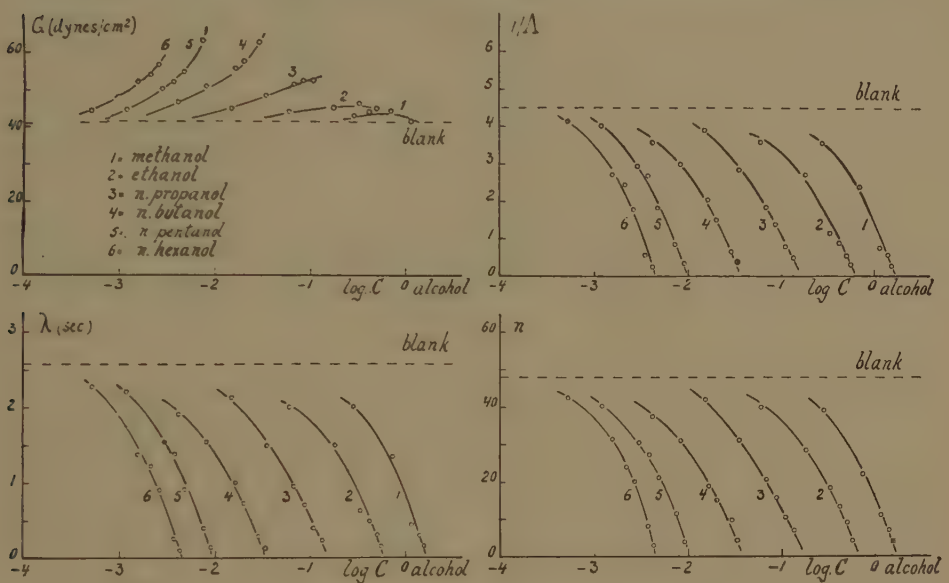

Fig. 2

these quantities and this decreasing action is becoming greater in a highly considerable measure every time the alcohol is lengthened with one carbon atom. In Table III we give for the 0.6% and the 1.2% system (taken from a table in Part VI, section 3) the logarithms of the alcohol concentrations corresponding to a decrease of $1/\Lambda$ and n down to 50 % of their original value.

For the 0.6 % and the 1.2 % oleate systems are the absolute values of the logarithms of the concentrations for each alcohol of the same order of magnitude. The same applies for the mean logarithmic differences

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consider as erroneous, for the example given in Part VII was a slightly damped oscillation, and correction factors following from this example may not be applied to strongly damped oscillations (in which the isochronous level is reached proportionally sooner). Therefore all values of G in Table II (and also in Table I) have been recalculated from the experimental T values themselves (of course corrected in the usual way for Λ). Still for the highest damped oscillations (e.g. n below 6) a correction of the kind used in Part VI is probably necessary, the omitting of which may be the cause of the slightly bent downward ends of the G curves for methanol and ethanol. Therefore these downwards bent ends of the curves in fig. 1 and 2 must possibly be considered as not exact, but they come nearer to the true course of the curves than the markedly bent upward ends of the curves in fig. 2 of Part VI.

TABLE II

Influence of n. primary alcohols on the elastic behaviour of the 1.2 % oleate system at approximately the KCl concentration of minimum damping (at 15°)

Alcohol	Conc. moles/l	log. conc.	G dyne/cm²	Λ	$1/\Lambda$	λ sec.	n
methanol	blank	$-\infty$	40.8	0.222	4.50	2.56	48.2
	0.28	0.45−1	42.5	0.279	3.58	1.99	39.1
	0.73	0.86−1	43.9	0.415	2.41	1.33	22.0
	1.15	0.06	40.9	1.31	0.76	0.42	11.0
	1.42	0.15	(39.4)	1.755	0.57	0.31	7.0
	1.56	0.19	(38.8)	3.64	0.27	0.15	4.0
ethanol	blank	$-\infty$	41.1	0.219	4.57	2.60	48.1
	0.061	0.79−2	44.1	0.277	3.61	1.99	39.8
	0.181	0.26−1	44.8	0.368	2.72	1.50	28.3
	0.33	0.52−1	45.6	0.859	1.16	0.63	18.2
	0.416	0.62−1	43.7	1.12	0.89	0.47	13.2
	0.503	0.70−1	44.7	1.78	0.56	0.31	9.0
	0.506	0.75−1	(42.6)	3.20	0.31	0.16	4.0
n. propyl alcohol	blank	$-\infty$	40.5	0.222	4.50	2.57	48.0
	0.0153	0.18−2	44.5	0.255	3.92	2.13	42.1
	0.0357	0.55−2	47.8	0.350	2.86	1.49	31.1
	0.069	0.84−2	50.9	0.540	1.85	0.94	20.4
	0.086	0.93−2	51.8	0.715	1.40	0.70	15.4
	0.110	0.04−1	51.8	1.26	0.79	0.39	10.5
	0.136	0.13−1	(58.2)	1.92	0.52	0.24	6.8
n. butyl alcohol	blank	$-\infty$	40.6	0.222	4.50	2.57	47.9
	0.00414	0.62−3	46.1	0.280	3.57	1.91	37.6
	0.0083	0.92−3	50.6	0.332	3.01	1.54	31.0
	0.0166	0.22−2	55.1	0.491	2.04	0.99	18.7
	0.0207	0.32−2	57.0	0.670	1.49	0.72	15.0
	0.0289	0.46−2	62.2	1.59	0.63	0.28	9.7
	0.0328	0.52−2	(67.4)	3.04	0.33	0.13	4.0
n. amyl alcohol	blank	$-\infty$	40.8	0.220	4.55	2.60	47.8
	0.00119	0.08−3	44.1	0.249	4.02	2.21	40.2
	0.00298	0.47−3	49.7	0.338	2.96	1.54	30.5
	0.00382	0.58−3	51.2	0.369	2.71	1.38	26.9
	0.00476	0.68−3	53.9	0.549	1.82	0.91	21.2
	0.0074	0.87−3	62.2	1.185	0.84	0.39	11.6
	0.0092	0.96−3	(65.7)	2.95	0.34	0.15	4.0
n. hexyl alcohol	blank	$-\infty$	41.2	0.224	4.46	2.54	47.9
	0.00052	0.72−4	44.2	0.241	4.15	2.28	42.4
	0.00157	0.20−3	51.7	0.367	2.72	1.38	31.5
	0.00217	0.34−3	53.5	0.407	2.46	1.23	23.8
	0.00262	0.42−3	56.2	0.561	1.78	0.87	20.1
	0.00374	0.57−3	(64.8)	1.89	0.53	0.24	8.0
	0.00426	0.63−3	—	4.03	0.25	0.10	3.0

mean values of the blanks: $G_0 = 40.8$; $\Lambda_0 = 0{,}222$; $\lambda_0 = 2{,}57$; $n_0 = 48{,}0$

TABLE III

Logarithms of the alcohol concentrations corresponding to a decrease of $1/\Lambda$ and n down to 50% of their original value.

Oleate concentr.		n.hexyl alcohol	n.amyl alcohol	n.butyl alcohol	n.propyl alcohol	ethanol	methanol
0.6 %	$1/\Lambda$	0.23 − 3	0.67 − 3	0.21 − 2	0.80 − 2	0.39 − 1	0.78 − 1
	n	0.20 − 3	0.58 − 3	0.11 − 2	0.69 − 2	0.31 − 1	0.70 − 1
	mean logarithmic difference	0.41	0.54	0.59	0.61	0.39	
1.2 %	$1/\Lambda$	0.32 − 3	0.63 − 3	0.15 − 2	0.74 − 2	0.34 − 1	0.85 − 1
	n	0.35 − 3	0.64 − 3	0.12 − 2	0.73 − 2	0.37 − 1	0.84 − 1
	mean logarithmic difference	0.30	0.50	0.60	0.62	0.49	

which in both cases (at least when going from methanol to amyl alcohol) correspond to ratios of approximately 3 or 4.

All this gives us a strong conviction that the damping in the 0.6 % oleate system is also a consequence of relaxation, though by a unknown circumstance Λ is independent of R, so that a relaxation time λ cannot be calculated.

We are strengthtened in this conviction if we now turn to a comparison of the G graphs of fig. 1 with the G graph of fig. 2. We observe that there is a striking similarity (viz. all the alcohols increase G only, though with methanol this is so slight that it comes in the order of the experimental errors) only if we now take the G graph in fig. 1 to the right. As already stated above, this latter graph represents G as computed from T after applying the usual correction for Λ, viz. that correction which was also applied for calculating the G values in fig. 2. The drooping ends of the G curves upper left in fig. 1 we therefore consider to be the result only of non correcting the period before computing G. In this connection it may be mentioned that the G curves for methanol and ethanol in fig. 2 would also show such downward bent curls intersecting the blank level, if G was calculated from the uncorrected T values.

3. *$G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams, an alternative method of representing graphically the experimental results obtained with the homologous series of the alcohols.*

In studying the action of organic substances on the elastic properties of the oleate system at constant KCl concentration, one must make a difference between the gross concentration of the organic substances at which a certain effect is obtained, and the action per bound molecule of the organic substance.

If we consider the hydrocarbons — which are on the whole practically insoluble in water — all molecules added will be taken up by the oleate micelles. The different positions of the curves, obtained with a number of hydrocarbons (see fig. 5 in Part VII) then indicate differences in the action of the hydrocarbon molecules bound to the oleate micelles.

If we compare, however, the lower terms of the homologous series of the n-primary alcohols — these being soluble in water — this no longer holds.

In this case only part of the added alcohol molecules are bound to the oleate micelles, the other part being left free in the medium and constituting a true equilibrium concentration characteristic for the distribution of the alcohol molecules between micelles and medium. If the quantity of bound molecules is small in comparison with the total quantity of molecules added (the latter representing the gross concentration), the different positions of the curves (laid out as a function of the gross concentration) will in the first place give information about the distribution of the alcohol molecules between micelles and medium.

And indeed, if we compare the terms $C_1 - C_5$, we find values of 3 to 4 (compare section 2) for the ratios of the gross concentrations of two successive terms needed to reach a same change in λ (or $1/\Lambda$ or n). This is reminiscent of values occurring in TRAUBE's rule. This shows that even for C_5 the bound molecules are only a small fraction of the total amount of molecules added, or in other words: From C_1 up tot C_5 the gross concentrations represent practically equilibrium concentrations.

Therefore if we consider the figures 1 and 2 and deduce from them a series of "increasing activity" of the alcohols: $6 > 5 > 4 > 3 > 2 > 1$, it must be well understood that this series indicates only that the ease with which the alcohols are taken up by the oleate micelles increases rapidly from right to left every time the carbon chain in lengthtened with one carbon atom. Therefore graphs of the kind of fig. 1 and 2, in which G or λ or $1/\Lambda$ or n are plotted as functions of the alcohol concentrations, cannot give expression to possible differences in the nature of the action exerted by the bound molecules of the various alcohols.

For the latter purpose we must evidently look for a graphical representation of the experimental results, in which the concentrations of the various alcohols are no longer used. Now the measurements always give two kinds of quantities: a. G, the elastic shear modulus, and b. quantities which refer to the damping (λ, Λ, n). This circumstance opens the possibility of representing the experimental results by diagrams in which these two kinds of quantities are used as ordinates and abscissa.

In such diagrams we can plot the actually determined values of G and of λ (or of $1/\Lambda$ or of n) as such, or expressed in percents of the blank G and λ (or $1/\Lambda$ or n) values. The latter method will be followed here, as it is to advantage in view of comparison of the results obtained with the 0.6 % and the 1.2 % oleate system.

Fig. 3 gives the results for the 0.6 % oleate system, using the data of table I. In these graphs we see the curves going upwards to the left. They all take their origin at the point with an ordinate 100 and an abscissa 100. This point therefore represents the blank oleate system.

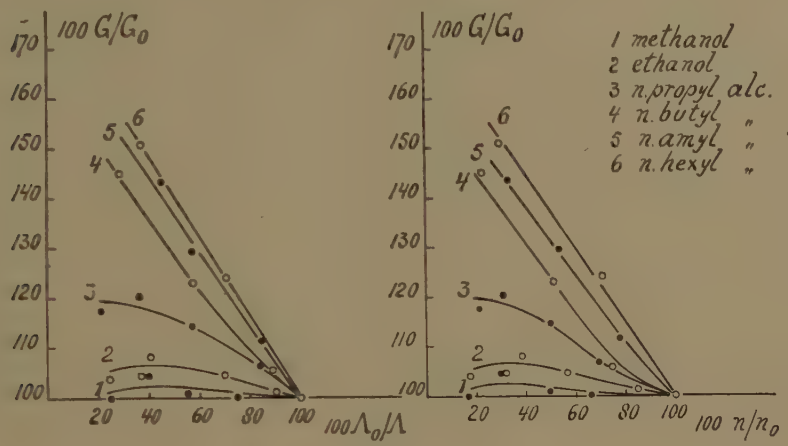

Fig. 3

We observe that their are six different curves each representing the measuring points of one of the n primary alcohols. The increase in G at the same decrease of $1/\Lambda$ or of n is seen to diminish in the order

$$6 > 5 > 4 > 3 > 2 > 1 \qquad \text{(0.6 \% oleate at 1.43 N KCl)}$$

In the same way we have laid out in fig. 4 the results obtained with the 1.2 % oleate system (data of Table II, expressed in per cents of the corresponding blanks). The graphs are fairly similar to those of fig. 3.

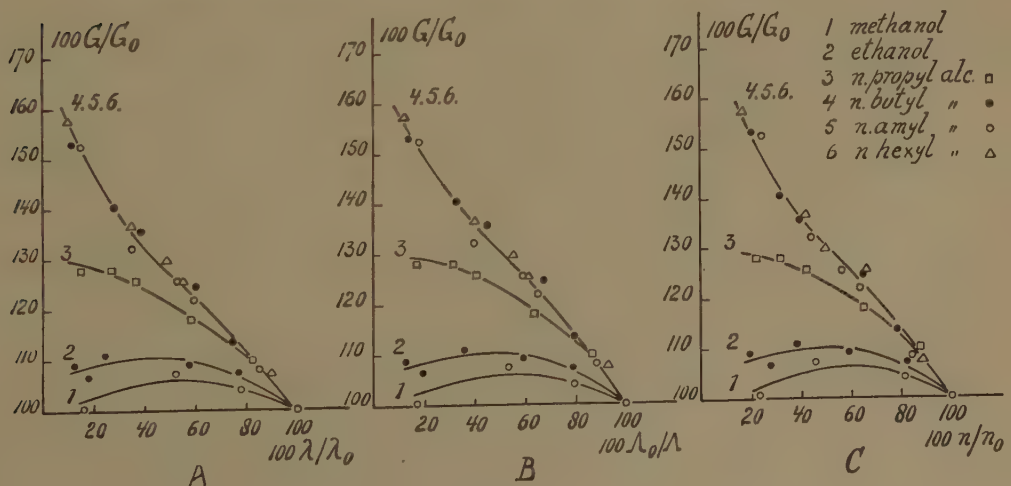

Fig. 4

The position of the curves in this percentual representation is not very much different, though the absolute values of G, $1/\Lambda$ and n are very different in the 0.6 and 1.2 % oleate systems. The sequence of the curves is in principle the same:

$$6,\ 5,\ 4 > 3 > 2 > 1 \qquad (1.2\ \%\ \text{oleate at } 1.52\ \text{N KCl}).$$

The expected spreading of the curves for 6, 5 and 4 does not find expression here, probably because by some unknown reason [8]) these curves are lying so close together here, that their distance is of the same order of magnitude as the experimental error.

The $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams of fig. 3 and fig. 4 are representations in which the concentration of the alcohols no longer plays any part. They show us that for the successive terms of the homologous series the molecules bound to the oleate have not the same action, *but that a gradual shift in the nature of the action occurs as the carbon chain of the alcohol is lenghtened.* For if the nature of the action was exactly the same, the experimentally determined points of all alcohols would lay on one and the same $G - \lambda$ or $G - 1/\Lambda$ or $G - n$ curve. The next section will show us how to interpretate the figs. 3 and 4.

4. *Interpretation of the G—λ, G—$1/\Lambda$ and G—n diagrams of section* 3.

In Part VI of this series we have investigated the shift of the curves, which represent G and λ (or $1/\Lambda$ or n) as functions of the KCl concentration, by the addition of a number of constant quantities of n. hexylalcohol and of ethanol. In both cases the maximum of the λ (or $1/\Lambda$ or n) curve is displaced downwards and to the left, but in the case of n. hexylalcohol the ratio of the horizontal and the vertical component of this displacement is much larger than in the case of ethanol. Evidently this difference points to a different kind of action of the bound molecules of both alcohols because in the corresponding graphs (see Part VI, fig. 2 and fig. 4) the alcohol concentrations themselves are no variables.

It can be easily shown that there exists a close relation between the magnitude of the above mentioned ratios and the slope of the curves in the $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams of fig. 3 and 4.

For that purpose we consider the schemes A_2, A_1 and B of fig. 5 in which the full drawn curves represent G and λ (or $1/\Lambda$ or n) as functions of the KCl concentration for the blank oleate system and in which the dotted curves represent G and λ (or $1/\Lambda$ or n) as functions of the KCl concentration at certain (see below) constant concentrations of three different substances A_2, A_1 and B.

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[8]) It may be that the crowding together is caused by the circumstance, that in this 1.2 % oleate series the KCl concentration (1.52 N) is somewhat higher than that corresponding with the minimum damping of the blank (1.43 N).

The schemes A_2 and A_1 of fig. 5 relate to two substances of type A.[9]) In these schemes the ratio of the horizontal and the vertical component of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curve has different values. This ratio (ad/dc) is larger in the case of A_2 than in the

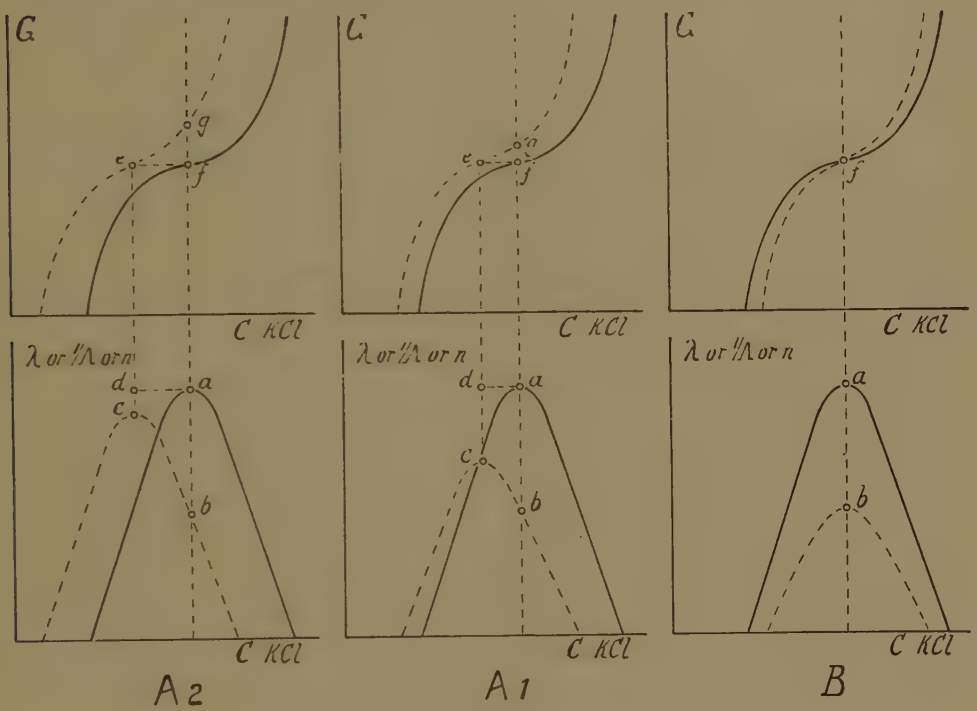

Fig. 5

case of A_1. The third scheme relates to a substance of type B, which is the limiting case of type A as here the horizontal component ad is zero.

In the schemes of fig. 5, we have further considered such concentrations of the substances A_2, A_1 and B, that the displaced λ (or $1/\Lambda$ or n) curve of each passes (with its right side, or with its maximum) through the same point b, which is situated on a vertical line drawn through the maximum a of the λ (or $1/\Lambda$ or n) curve of the blank.

This means, that we only consider such concentrations of the substances A_2, A_1 and B, where the decrease of λ (or $1/\Lambda$ or n), represented by the distance ab, is the same for all three substances.

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[9]) In Part VI we have already discerned three types of organic substances A, B and C, characterised by the direction of the horizontal component of the shift of the maximum of the λ (or $1/\Lambda$ or n) curve. Having type A when the horizontal component of this shift shows towards smaller KCl concentrations, type C, when it shows to the opposite direction and B when the horizontal component is zero.

We now direct our attention to the upper graphs which show the displacement of the G-curves. For this displacement two things are characteristic (cf Part VI and Part VII);

a. that the inflexion point situated on the G curve always corresponds to the maximum of the λ (or $1/\Lambda$ or n) curve, independent whether we consider the blank (points f and a) or the shifted curves (points e and c),

b. that the shift of the inflexion point on the G curve consist of an horizontal component fe only, which (see under a) is just as large as the horizontal component ad of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curve.

It may now be read from the figure, that the distance fg, that is the increase of G at the KCl concentration of minimum damping of the blank, is correlated with the ratio ad/dc.

This increase of G is larger for substance A_2 than for substance A_1, and the ratio ad/dc is indeed larger for A_2 than for A_1. For substance B the increase of G is zero (the points g and f are coinciding in this case) and the ratio ad/dc is here also zero (the points d and a coincide).

The above reasonings open the possibility to use the results of a much simpler form of experimentation — in which the influence of added organic substances at only one KCl concentration, namely the concentration of minimum damping of the blank are determined — to arrange a number of substances A_1, A_2 A_3 etc. into a series in which the ratio of the horizontal and the vertical component of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curve increases.

One procedure to use these results consists of reading from interpolation graphs, at which concentration a certain decrease (e.g. of 50 per cent.) of λ (or $1/\Lambda$ or n) is attained with each of the alcohols, and after that to read on analogous G- interpolation graphs, which G values correspond with these concentrations for each of the alcohols.

If the above procedure will have any real significance, we must find of course the same arrangement of the substances A_1, A_2, A_3, A_4, etc. if we choose for comparison other (constant) values of the decrease in λ (or $1/\Lambda$ or n).

A still simpler procedure consists of laying out the actual experimental results in a $G - \lambda$ diagram (or in a $G - 1/\Lambda$ or in a $G - n$ diagram). For in these diagrams we have at once a representation of all increases in G at all decreases in λ, (or in $1/\Lambda$ or in n). The interpretation of these diagrams is then very simple, provided we obtain a bundle of non intersecting curves: The order of increasing steepness of the curves will be the order in which the ratio of the horizontal and vertical component of the displacement increases.

Applying this to the diagrams of fig. 3 and 4, we may conclude that

the alcohols $C_1 - C_6$ may be arranged in one continuous series:

$$6 > 5 > 4 > 3 > 2 > 1$$

in which from right to left the ratio of the horizontal and vertical component of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curve increases.

Summary.

1. In connection with the divergent character of the damping in the 1.2% oleate system (Λ is proportional to R) and in the 0.6% oleate system (Λ is independent of R) — compare Part III of this series — the influence of the first six terms of the n. primary alcohols has been investigated on the elastic behaviour of the 0.6 % oleate system at the KCl concentration of minimum damping. The influence on G, $1/\Lambda$ and n is quite the same as in the case of the 1.2 % oleate system.

2. The results mentioned sub 1., combined with the finding that any slipping along the glass wall could not be observed, lead to the conclusion of the non validity of the interpretation (slipping along the boundary of the glass vessel) as was accepted in Part III of this series for: Λ is independent of R, characteristic of the 0.6 % oleate system.

3. Graphs in which G or λ or $1/\Lambda$ or n are plotted as functions of the alcohol concentrations (compare figures 1 and 2) bring mainly to epxression the differences between the successive terms of the homologous series with regard to the distribution of the alcohol molecules between oleate micelles and medium.

The alcohols may be arranged in the order $6>5>4>3>2>1$, in which from right to left the above distribution strongly shifts in favour of the oleate micelles.

4. $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams, being representations in which the concentration of the alcohols no longer plays any part, inform us on the nature of the action of the alcohol molecules bound to the oleate micelles. The graphs obtained (compare fig. 3 and 4) show that this action is not the same for the successive terms of the homologous series, but that a gradual shift occurs as the carbon chain of the alcohol is lengthened.

5. The gradual shift sub 4. consists of a change in the ratio of the horizontal and vertical component of the displacement of the maximum of the λ, $1/\Lambda$ or n curves (in the graphs representing λ, $1/\Lambda$ and n as functions of the KCl concentration). The alcohols may be arranged in the order $6>5>4>3>2>1$, in which from right to left the above ratio increases.

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THE FORMATION OF VORTEX SHEETS IN A SIMPLIFIED TYPE OF TURBULENT MOTION

BY

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(*Mededeling no. 64 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated at the meeting of Jan. 28, 1950)

1. *Introduction.*

The observation of a number of features of turbulent fluid motion has led to the conviction that an important part in this motion is played by vortices with axes in the direction of the flow. Such vortices become visible for instance in the so-called "secondary motion" appearing in rectilinear ducts with non-circular cross section [1]). They can also be seen when a heavy wind blows over a sandy beach: the sand taken up by the wind moves in thin layers or ribbons, constantly waving to and fro and folding and curling about the streamlines [2]). In the explanation of relations governing the turbulent friction in the boundary layer along a wall, vortex motion with the axis of rotation parallel to the direction of the main flow is responsible for the difference between the so-called "momentum transfer", proposed by PRANDTL, and the "vorticity transfer", considered by Taylor [3]).

TAYLOR has pointed out that vortices with their axes in the direction of the streamlines can be subjected to extension or contraction in consequence of local acceleration or deceleration of the flow. In the first case the vorticity in the core will increase; in the second case it will decrease. These two processes play a different part in the development of turbulence: the extension is responsible for the appearance of regions of high vorticity

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[1]) Compare e.g. L. PRANDTL, Proceedings IInd Intern. Congress for Applied Mechanics, Zürich 1926, p. 71; or "Führer durch die Strömungslehre" (Braunschweig 1942), p. 133.

[2]) Compare also R. A. BAGNOLD, The Physics of Blown Sand and Desert Dunes (London 1941), p. 176—179. The so-called "spectres aérodynamiques" obtained by RIABOUCHINSKY likewise give the impression that they are due to the effect of vortices with axes along the streamlines. See D. RIABOUCHINSKY, Spectres Aérodynamiques, Bull. Institut Aérodynamique de Koutchino, Fasc. III (Moscou 1909), p. 59 and seqq.; and also Fasc. IV (Moscou 1912), p. 98, where reference is made to such vortices.

[3]) Compare e.g. S. GOLDSTEIN, Modern Developments in Fluid Dynamics (Oxford 1938), vol. I, p. 206—213.

which are the seat of intensive dissipation, while contraction leads to the appearance of regions of low vorticity [4]).

Looking again to the layers of sand mentioned above, the idea comes to the mind that the drawing out of vorticity regions into thin sheets will be a more general process than the drawing out of vortex tubes into thin cores. The idea finds support through the observation of the curling ribbons or veils of smoke which can rise from a lighted cigarette, or of the veils formed by the vapour rising from a cup of hot water or tea. Even flames seem mainly to be formed of thin sheets. Perhaps in this connection one may also point to the luminous veils presented by "network nebulae" such as N.G.C. 6960 and 6992 in the constellation of Cygnus.

Judging by the generality of these phenomena, one may ask if it can be helpful for the understanding of certain features of turbulent flow, if a simplified picture could be developed illustrating the tendency towards the formation of vortex sheets. Since an adequate treatment of the complete hydrodynamic equations has not yet been found, one must be prepared to accept some drastic simplifications in order to arrive at a picture amenable to mathematical treatment. Now if the x-axis is taken in the direction of the main flow, the vorticity component in the x-direction is dependent on the velocity components in the directions of y and z. These same velocity components are responsible for the change of the cross section of vortices. Hence it will be necessary to have the components v and w and the coordinates y and z in the picture. We will, however, leave aside the component u. As the "divergence in the y, z-plane", $\partial v/\partial y + \partial w/\partial z$, in the actual flow is balanced by $\partial u/\partial x$, we are free to assume that this divergence may take any value and we need not introduce an equation of continuity for v and w. We shall then make a drastic simplification by *neglecting pressure gradients in the directions of y and z*. We shall also omit the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$ from the equations of motion for v and w. If the frictional terms provisionally are retained, these equations take the form:

$$(1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = \nu\left(\dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial w}{\partial t} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = \nu\left(\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) \end{cases}$$

With respect to the neglected terms the following may be observed: When we suppose that the velocity component u is equal to a constant amount U + a small correction θ, fluctuating about zero more or less periodically when we proceed along a streamline, the main parts

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[4]) G. I. Taylor, Proc. Roy. Society (London) A **164**, 15 (1938). — The case of a vortex tube drawn out into a thin core has also been considered by the present author in Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam, **43**, 11—12 (1940), and in "Advances in Applied Mechanics", vol. I (New York 1948), p. 198.

$U(\partial v/\partial x)$, $U(\partial w/\partial x)$ of the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$ can be absorbed into the terms $\partial v/\partial t$, $\partial w/\partial t$ by introducing a moving coordinate system. Further, since the pressure will be mainly determined by $-\varrho U\theta$, we may expect the neglected pressure gradients to be more or less out of phase with v and w. Formally we may introduce all neglected terms as a system of "impressed forces" on the right hand side of the equations. They represent a coupling of the v, w-field with the main motion u, and we might imagine that more or less periodically they will bring about a new initial state, the further history of which can be discussed with the aid of eqs. (1). It is of course not certain that by cutting up the problem in this way we shall arrive at an adequate picture of what happens in actual turbulent fluid motion, and it is possible that relationships appearing in the solution of eqs. (1) may considerably differ from corresponding relations in actual turbulence. However, the solutions of eqs. (1) show certain properties which, even if they may differ in details from what actually happens, nevertheless seem so striking that one is inclined to see in them an indication of effects which must be of importance.

We shall introduce still another simplification: we assume that ν is so small that in regions where the derivatives of v and w are of normal magnitude, the terms with the factor ν can be discarded. We are then left with the system:

$$\left\{\begin{aligned} &\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=0\\ &\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0 \end{aligned}\right. \tag{2}$$

2. *Elementary properties of the equations. — Application to a "homogeneous domain".* — The system (2) possesses characteristics determined by:

$$dy/dt = v \quad ; \quad dz/dt = w \tag{3}$$

Along a characteristic we have:

$$dv/dt = 0 \quad ; \quad dw/dt = 0 \tag{4}$$

Hence the characteristics are straight lines.

When the initial state of the field has been given, it is possible, at least in principle, to derive the future state of the field by means of a construction making use of the characteristics. Difficulties, however, arise when characteristics drawn in an y, z, t-space, should meet each other. In order to obtain insight into what may happen, we consider a domain in which v and w are initially given by the linear functions:

$$\left\{\begin{aligned} v &= A\,(y-\bar{y}) + B\,(z-\bar{z})\\ w &= C\,(y-\bar{y}) + D\,(z-\bar{z}) \end{aligned}\right. \tag{5}$$

Here $\bar{y}$ and $\bar{z}$ are constants, representing a point which may be either inside or outside the domain. Within this domain the vorticity and the

components of the tensor of the rate of deformation have values independent of the coordinates. We shall call such a domain a "*homogeneous domain*". — It is useful to note that the following treatment remains applicable when the coordinate axes are not rectangular.

In order to find the state of the field at a later instant, we suppose that A, B, C, D are functions of t and substitute expressions (5) into eqs. (2). Since the equations must be satisfied identically in y and z, we obtain:

$$(6) \qquad \begin{cases} \dot{A} + A^2 + BC = 0 \\ \dot{B} + AB + BD = 0 \\ \dot{C} + AC + CD = 0 \\ \dot{D} + BC + D^2 = 0 \end{cases}$$

From these we deduce:

$$\frac{\dot{A}-\dot{D}}{A-D} = \frac{\dot{B}}{B} = \frac{\dot{C}}{C} = -(A+D).$$

Hence if we write: $A - D = q$, we find: $B = c_1 q$, $C = c_2 q$ (where c_1 and c_2 are integration constants) and further $A + D = -\dot{q}/q$. It follows that:

$$A = -\tfrac{1}{2}(\dot{q}/q - q) \quad ; \quad D = -\tfrac{1}{2}(\dot{q}/q + q).$$

There is still one remaining equation of the system (6), for which we take:

$$\dot{A} + \dot{D} + A^2 + D^2 + 2BC = 0.$$

Substitution of the expressions found above leads to:

$$\frac{\ddot{q}}{q} - \frac{3\dot{q}^2}{2q^2} - \frac{1+4c_1c_2}{2}q^2 = 0.$$

This equation can be integrated by means of standard methods; its integral has the form: $q = 2\beta/N$, where:

$$(7) \qquad N = 1 + 2\alpha t + (\alpha^2 - \gamma)t^2 = (1+\alpha t)^2 - \gamma t^2$$

α and β being two further integration constants, while $\gamma = \beta^2(1 + 4c_1c_2)$. The full solution of (6) now becomes:

$$(8) \qquad \begin{cases} A = \dfrac{(\alpha+\beta) + (\alpha^2-\gamma)t}{N} \quad ; \quad B = \dfrac{2\beta c_1}{N} \\ D = \dfrac{(\alpha-\beta) + (\alpha^2-\gamma)t}{N} \quad ; \quad C = \dfrac{2\beta c_2}{N} \end{cases}$$

The following formulae serve to find the constants from the initial values of A, B, C, D:

$$(9) \qquad \begin{cases} 2\alpha = A_0 + D_0 & c_1 = B_0/2\beta \\ 2\beta = A_0 - D_0 & c_2 = C_0/2\beta \\ \gamma = \beta^2 + B_0 C_0 \end{cases}$$

We observe that $\alpha^2 - \gamma = A_0 D_0 - B_0 C_0$ and note that α and γ remain

invariant when the initial state of the field is described with respect to an arbitrary system of rectilinear coordinates, whether rectangular or not, provided the determinant of the transformation formulae has the value unity.

3. *Discussion of the result.* — A decisive part in the development of the field is played by the circumstance whether N can or cannot become zero for a positive value of t. The following cases must be distinguished.

(I) *N does not become zero for a positive value of t* in the cases:

(a) $\gamma < 0$

(b) $\gamma > 0$, provided $\alpha > 0$ and $\gamma < \alpha^2$.

In both cases the asymptotic state of the field is given by:

$$A \to t^{-1} \quad ; \quad D \to t^{-1} \quad ; \quad B \text{ and } C \text{ of order } t^{-2}.$$

Hence the asymptotic expressions for the velocity components are:

$$v = (y - \bar{y})/t \quad ; \quad w = (z - \bar{z})/t.$$

This means that in the end we have a symmetric divergent field; all vorticity originally present has become infinitely diluted.

In the transition case:

$$(b^*) \quad \alpha > 0 \quad ; \quad \gamma = \alpha^2$$

we have: $N = 1 + 2\,\alpha\,t$, so that A, B, C, D all become proportional to $1/(1 + 2\,\alpha\,t)$.

(II) *N does become zero for a positive value of t* (to be denoted by t_c) in the cases:

(c) $\gamma > 0$; $\alpha > 0$; $\gamma > \alpha^2$

(d) $\gamma > 0$; $\alpha < 0$; $\gamma > \alpha^2$

(e) $\gamma > 0$; $\alpha < 0$; $\gamma < \alpha^2$.

In all three cases we have:

$$t_c = 1/(\sqrt{\gamma} - \alpha) \tag{10}$$

In case (e) the equation $N = 0$ has a second positive root $t^* = 1/(-\sqrt{\gamma} - \alpha)$, which is greater than t_c.

In case (c) we have $A_0 + D_0 > 0$; the value of $A + D$ changes sign for $t = 2\,\alpha/(\gamma - \alpha^2)$. In cases (d) and (e) the value of $A + D$ is always negative.

If we consider the motion of the particles of the fluid, it is found that a particle which at the instant $t = 0$ had the position y_0, z_0 and velocities $v_0 = A_0\,(y - \bar{y}) + B_0\,(z - \bar{z})$, $w_0 = C_0\,(y - \bar{y}) + D_0\,(z - \bar{z})$, will have come to a point y_c, z_c at the critical instant t_c, such that

$$y_c - \bar{y} = y_0 - \bar{y} + v_0\,t_c \quad ; \quad z_c - \bar{z} = z_0 - \bar{z} + w_0\,t_c.$$

Making use of the value found for t_c we deduce:

$$y_c - \bar{y} = \frac{(\sqrt{\gamma} - \alpha + A_0)(y_0 - \bar{y}) + B_0 (z_0 - \bar{z})}{\sqrt{\gamma} - \alpha}$$

$$z_c - \bar{z} = \frac{C_0 (y_0 - \bar{y}) + (\sqrt{\gamma} - \alpha + D_0)(z_0 - \bar{z})}{\sqrt{\gamma} - \alpha}.$$

It is easily shown that the ratio:

$$\frac{z_c - \bar{z}}{y_c - \bar{y}} = \frac{C_0}{\sqrt{\gamma} + \beta} = \frac{\sqrt{\gamma} - \beta}{B_0} = \frac{2\beta c_2}{\sqrt{\gamma} + \beta}$$

is independent of the values of y_0 *and* z_0. This means that the whole domain in which the velocity components were given by the expressions (5), *has contracted into a single line* passing through $\bar{y}$, $\bar{z}$.

This result can be illustrated by observing that the two straight lines

$$\frac{z - \bar{z}}{y - \bar{y}} = \frac{2\beta c_2}{\sqrt{\gamma} + \beta} \quad ; \quad \frac{z - \bar{z}}{y - \bar{y}} = \frac{2\beta c_2}{-\sqrt{\gamma} + \beta}$$

are the only two straight streamlines of the initial field. They keep their positions for all values of t. When the description of the field is referred to these lines as coordinate axes, we obtain a new set of values of A, B, C, D as follows:

$$A_1 = \frac{\alpha + \sqrt{\gamma} + (\alpha^2 - \gamma)t}{N} \quad ; \quad B_1 = 0$$

$$D_1 = \frac{\alpha - \sqrt{\gamma} + (\alpha^2 - \gamma)t}{N} \quad ; \quad C_1 = 0.$$

It is found that in all three cases (*c*), (*d*), (*e*) the value of D_1 is always negative and becomes negative infinite for $t = t_c$. In the cases (*c*) and (*d*) the value of A_1 is positive, its limit for $t = t_c$ being $(\gamma - \alpha^2)/2\sqrt{\gamma}$. In case (*e*) A_1 is negative for $t < t_c$; the limiting value is given by the same expression. Hence in cases (*c*) and (*d*) we have convergence towards the first axis, leading to complete contraction of the field into this axis, whilst there is divergence away from the second axis. In case (*e*) there is convergence towards both axes, but the convergence towards the first axis operates more rapidly and again leads to complete contraction of the field into this axis at the instant t_c.

4. *Application to a patchwork of homogeneous domains.* — The application of the method of characteristics to the integration of eqs. (2) in the case of arbitrary initial conditions, is beset with difficulties. However, since we are engaged in the construction of a model, we suppose that the initial state of the field can be represented with sufficient accuracy by a patchwork of homogeneous domains. Applying the analysis of sections 2 and 3 to each separate homogeneous domain, we may find that after a certain lapse of time one of these domains will have contracted into a segment

of a straight line. This line segment will then represent a discontinuity for both components of the velocity; hence it is a line of sinks and at the same time a vortex line. We may consider it as the cross section of a vortex sheet, which is being extended in the direction perpendicular to the y, z-plane.

The length of the segment and the values of the velocities on both sides will depend on the original extent of the domain which has contracted into it. By way of example we suppose the initial distribution of v and w to be as indicated in fig. 1. In the interior region, bounded by the square $|y| + |z| = 1$, we have:

$$v_0 = -2y \quad ; \quad w_0 = y + z,$$

so that here: $A_0 = -2$; $B_0 = 0$; $C_0 = 1$; $D_0 = 1$, and $\alpha = -\frac{1}{2}$; $\beta = -\frac{3}{2}$; $\gamma = \frac{9}{4} > \alpha^2$. The original field thus shows convergence and vorticity. We find $t_c = 0{,}5$; the field contracts into a segment of the line $y = 0$.

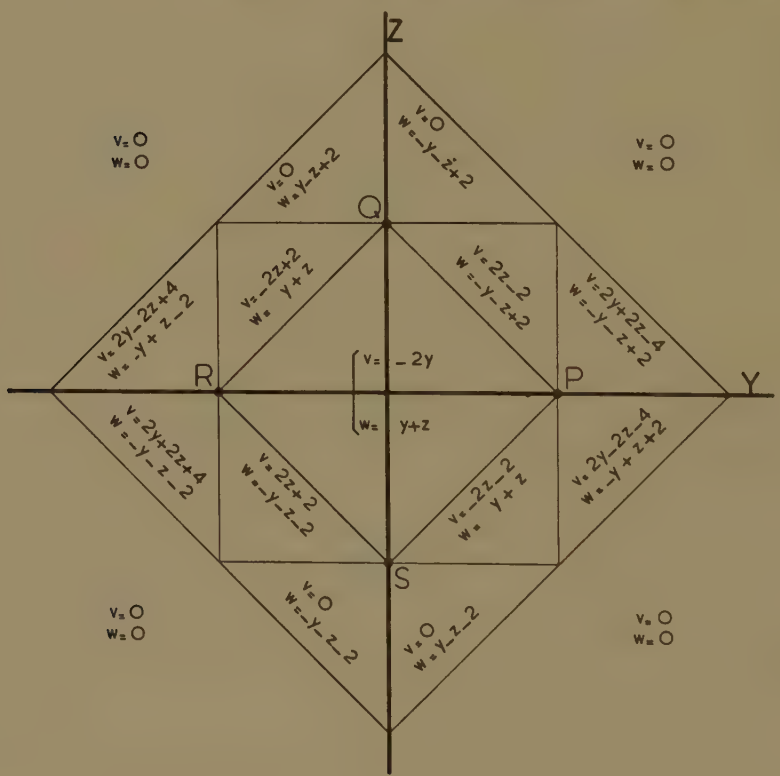

Fig. 1.

In the surrounding network of triangular areas the expressions for v and w have been chosen so as to make both components zero on the four sides of the square $|y| + |z| = 2$ and outside, while care has been taken that there are no discontinuities of v and w in the initial field (there are only discontinuities in their derivatives). With the aid of the formulae

of sections 2 and 3 it is found that there is convergence in the two upper triangles and in the two lower ones; however, since $t_c = 1$ for these domains, no difficulty will occur at the instant $t = 0{,}5$. For the other domains t_c is infinite or imaginary.

Some features of the state of the field at the instant $t = 0{,}5$ have been indicated in fig. 2. The outer square, which has retained its position, is

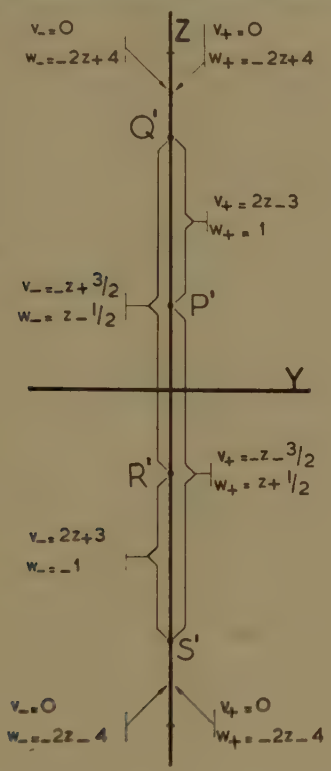

Fig. 2.

not pictured; the inner square has contracted into the segment of the z-axis from $z = +1{,}5$ to $z = -1{,}5$. The values of v and w on both sides of this segment have been inscribed in the diagram. — It is of interest to note the values of the following quantities:

segment :	$Q'P'$	$P'R'$	$R'S'$
$c = \frac{1}{2}(v_- + v_+)$:	$\frac{1}{2}z - \frac{3}{4}$	$-z$	$\frac{1}{2}z + \frac{3}{4}$
$\Delta = \frac{1}{2}(v_- - v_+)$:	$-\frac{3}{2}z + \frac{9}{4}$	$+\frac{3}{2}$	$+\frac{3}{2}z + \frac{9}{4}$
$\Gamma = w_+ - w_-$:	$-z + \frac{3}{2}$	1	$+z + \frac{3}{2}$

The quantity 2Δ, which measures the convergence into the segment, is essentially positive. The quantity Γ measures the strength of the vortex sheet.

5. *Structure and motion of the region of convergence.* — The "line (or segment) of convergence", obtained in the analysis of the preceding section

in reality must be considered as an approximation to a narrow region. For a more accurate picture we make use of the equations (1) in which the terms with ν have been retained.

We consider a small region surrounding a short segment of a "line of convergence", which we suppose to have a radius of curvature different from zero. We limit ourselves to a short interval of time. At the instant of observation we introduce a system of coordinates with the origin in the centre of the segment, the z-axis along the tangent and the y-axis in the direction of the normal. We can assume that derivatives with respect to y will be much larger than derivatives with respect to z in the region of convergence: the former will be of the order ν^{-1}, while the latter will be of normal magnitude. Since the region may be in motion, derivatives with respect to t can become large. However, for a short while we can introduce a moving system of auxiliary coordinates by means of the formulae:

$$y' = y - f(t; z) \quad ; \quad z' = z \quad ; \quad t' = t,$$

where we assume $\partial f/\partial z$ to be zero at the instant considered ($\partial^2 f/\partial z \partial t$ may be different from zero). We write $\partial f/\partial t = c$ and obtain:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad ; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad ; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - c\frac{\partial}{\partial y'}.$$

When these expressions are substituted into eqs. (1) it is possible to determine c in such a way that $\partial v/\partial t'$ becomes of normal order of magnitude at the instant considered. We assume that the same value of c also makes $\partial w/\partial t'$ to be of normal order of magnitude, which assumption is allowed provided the region of convergence displaces itself without a rapid change of internal structure. The equations will contain terms of various orders, the highest being of order ν^{-1}. If only these terms are retained, the equations reduce to:

$$\begin{cases} -c\dfrac{\partial v}{\partial y'} + v\dfrac{\partial v}{\partial y'} = \nu\dfrac{\partial^2 v}{\partial y'^2} \\ -c\dfrac{\partial w}{\partial y'} + v\dfrac{\partial w}{\partial y'} = \nu\dfrac{\partial^2 w}{\partial y'^2} \end{cases} \tag{11}$$

The first equation has the integral:

$$\tfrac{1}{2}(v - c)^2 - \nu(\partial v/\partial y') = \text{function of } z, \text{ say } g(z).$$

This expression must be valid through the region of rapid change of v and on both sides of it, where the derivative $\partial v/\partial y'$ takes values of normal magnitude, so that $\nu(\partial v/\partial y')$ can be neglected. Again using subscripts $_-$ and $_+$ to distinguish between values on the two sides of the region, we must have:

$$\tfrac{1}{2}(v_- - c)^2 = \tfrac{1}{2}(v_+ - c)^2 = g(z),$$

from which:

$$c = \tfrac{1}{2}(v_- + v_+) \tag{12}$$

The quantity c determines the velocity of displacement of the region of

convergence in the direction normal to its longitudinal extension. Since it can be a function of z (in the cases considered it is a linear function of z over a certain interval, passing then into another linear function, etc.), segments of the line of convergence will usually show a rotation simultaneously with their displacement. In general the line may deform in a complicated way.

The final expression for v becomes:

$$v = \tfrac{1}{2}(v_- + v_+) - \triangle \cdot \tanh(y'\triangle/2\nu) \tag{13a}$$

and the coresponding expression for w:

$$w = \tfrac{1}{2}(w_- + w_+) + \tfrac{1}{2}\Gamma \cdot \tanh(y'\triangle/2\nu) \tag{13b}$$

The integrals of $\nu(\partial v/\partial y)^2$ and $\nu(\partial w/\partial y)^2$ taken over the thickness of the region of convergence, have values $\frac{2}{3}\triangle^3$, $\frac{1}{6}\Gamma^2\triangle$ respectively; these values are independent of ν.

The results (13a) and (13b) lead to transition regions for v and w of a thickness of order ν. This is a consequence of our neglect of the effects of pressure gradients in the y, z-plane, which made it possible that fluid disappearing from the field is carried away in an extremely narrow region. In this respect our picture will deviate from the actual behaviour of a fluid. A picture more approaching actual conditions may perhaps be given by an example in which $u = U + A\,x$; $v = -A\,y$; $w =$ function of y and t: $p = -\frac{1}{2}\varrho\{(U - A\,x)^2 - (A\,y)^2\}$, U and A being constants. The equations for u and v are satisfied exactly. The equation for w has the form:

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2},$$

which outwardly is of the same type as before. The circumstance, however, that now v decreases linearly with y when we go towards $y = 0$, so that the "region of convergence" is of finite width, has the effect that the vorticity is much less concentrated than in the preceding case. Solutions derived from the above equation for various initial conditions indicate the appearance of a vortex region with thickness of order $\sqrt{\nu}$. The integral of $\nu(\partial w/\partial y)^2$ over the thickness of this region then becomes of the order $\sqrt{\nu}$. This point, however, requires further investigation.

Since the main result of this section is the expression (12) for c, we may leave aside the problem of the thickness of the vortex sheet and be content with the law for its displacement.

One may also assume a motion of this sheet in the tangential direction, determined by the value of $\frac{1}{2}(w_- + w_-)$. This is of importance mainly at the endpoints of its section with the y, z-plane. In between the endpoints the value of the normal velocity c is the more important quantity.

It will be evident that the displacement of the vortex sheet, taken into consideration together with the change of the domains of the field adjacent to it, as determined by eqs. (2), usually will entail a gradual change of the values of v_-, v_+, w_-, w_+, and thus also of c and of the vortex strength Γ.

6. *Statistical problems.* — We have arrived at a result which in principle enables us to follow the development of any field, the initial state of which is formed by a patchwork of "homogeneous domains" with rectilinear boundaries between them. We must distinguish between divergent domains and domains which are contracting; for those of the second class we must find out which one has the smallest value for t_c. Having determined the position of the segment into which this domain contracts, we must obtain the motion of this segment after the instant t_c from the fields of flow adjacent to it. Simultaneously we must investigate which domain is the next one to contract into a segment. From the instant when this has occurred, we must follow the motions of both segments.

Gradually the number of these segments will increase, until practically all homogeneous domains belonging to class II will have disappeared. The only domains left will be those of class I. As shown in section 3 all of these asymptotically approach to simple symmetric fields with $v = (y - \bar{y}_i)/t$, $w = (z - \bar{z}_i)/t$, each domain having its own values of $\bar{y}_i$, $\bar{z}_i$. These domains will be separated by a polygonal network of line segments, which are lines of sinks and of concentrated vorticity. It is probable that certain conditions must be satisfied at these segments which still must be investigated.

The lines of sinks and of concentrated vorticity (vortex sheets) are the objects of main interest in the field, and it is their motion which attracts attention. Neighbouring lines may approach to each other and finally may meet. In such a case the domain between them is gradually squeezed out of the field; the two line segments coalesce and from then onward form a single entity. Hence after an initial period of formation of lines of sinks and of concentrated vorticity, there comes a period in which the number of these lines is gradually reduced. This will entail a gradual increase of the average cross dimensions of the remaining homogeneous domains with divergent motion.

At the same time there will be dissipation of energy in the vortex sheets, which will be related to a gradual decrease of the energy content of the whole field. It is to be expected that the thickness of the regions of convergence or of the vortex sheets — which determines the limit to the finer structure of the field — will gradually increase likewise, although the rate may be different from the rate of increase of the coarse scale determined by the cross dimensions of the homogeneous domains.

The picture we have developed could be taken as a basis for a statistical treatment of the particular type of turbulence we have considered, in which the vorticity is concentrated into vortex sheets. We could start from an array of initial states, in order to find out which are the common characteristics of their development in course of time. The picture would refer to the case of decaying (or "free") turbulence, the initial state being continually produced anew by some outward agency, as is the case with

the turbulence of the air stream in a wind channel, passing through a screen. The main interest of the picture will be the motion and the gradual coalescence of the vortex sheets, and a means for attacking the problem would be to study the correlation between velocity components in different points of the field. This correlation is influenced by the chance for two points at a given distance from each other to belong or not to belong to the same homogeneous domain.

The construction of a statistical theory for a two-dimensional field of this type will require the consideration of many parameters, even when we restrict to the asymptotic state.

However, now we have arrived at the picture of vorticity concentrated into particular lines or sheets, moving and gradually coalescing, we may go a step further in our simplification and introduce a *one-dimensional picture*. We then consider a single variable v, which is a function of the time and of only one coordinate y, and is subjected to the equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} \tag{14}$$

Again regions in which $\partial v/\partial y < 0$ will behave differently from regions in which $\partial v/\partial y > 0$. The former contract into narrow domains and lead to the appearance of jumps in the curve of v. Such jumps will be of importance when we study mean values of the type $\overline{v(y)\, v(y+\eta)}$ with a fixed value of η. The equation moreover has the advantage that the corresponding relation for the energy of the field is simpler than it is for eqs. (1).

Properties of the solutions of the equation mentioned have been investigated in some previous papers. With regard to physical dimensions, it conforms to the type of the hydrodynamical equations; hence one may expect that similarity considerations of related character will be applicable. In a future communication it is intended to develop some statistical formulae referring to the asymptotic form of these solutions.

FURTHER EXPERIMENTS WITH OUR MECHANICAL HEART-LUNG-DEVICE

BY

J. JONGBLOED

(Communicated at the meeting of Jan. 28, 1950)

Using our mechanical heart-lung-system and our method of artificial circulation we performed a series of experiments in dogs to study the possibilities of intra-cardial operations.

The apparatus (2) (3) (fig. 1), which is designed and has sufficient capacity for use in man, consists of two units of six pumps, representing the artificial right and left heart respectively, an oxygenator and an automatic device preventing air to enter the arterial system. The oxygenator was, untill now, the most difficult part and the weakest point in an apparatus for artificial circulation. Our oxygenator or artificial lung, the spiral oxygenator, is of a quite new type, which has the great advantages that the oxygenation of 4 to 5 liters blood per minute is very good and that neither hemolysis nor foaming occurs, even if the blood circulates for hours through the apparatus. Our results with this new principle of artificial oxygenation are very satisfying.

Our method of extra-corporeal circulation (1) (2) (3) is very simple and can be applied with only very little damage to the subject. We introduce a catheter through the right external jugular vein into the vena cava superior close to the heart, and a second catheter high up in the vena cava inferior through the right femoral vein. Both catheters are connected with the artificial right heart, which sucks the blood from the venae cavae just before it would enter the heart. Then this venous blood passes the artificial lung and is oxygenated. The artificial left heart forces the blood into the femoral arteries in which T-canulas are inserted. A small part of the blood proceeds in the normal direction to the hind limbs, the greater part in abnormal direction in the aorta, and from here normally into the various arterial branches. In this way the whole circulation in the subject is maintained, the coronary and the bronchial circulation included.

As is formerly described (2) we tested the capacity of the apparatus in dogs with a bodyweight of about 32 kilograms, using only one half of the apparatus. As these experiments proved that half of the apparatus was sufficient to maintain life in dogs of 32 kilograms, we may readily assume that when the whole apparatus is used, the circulation and

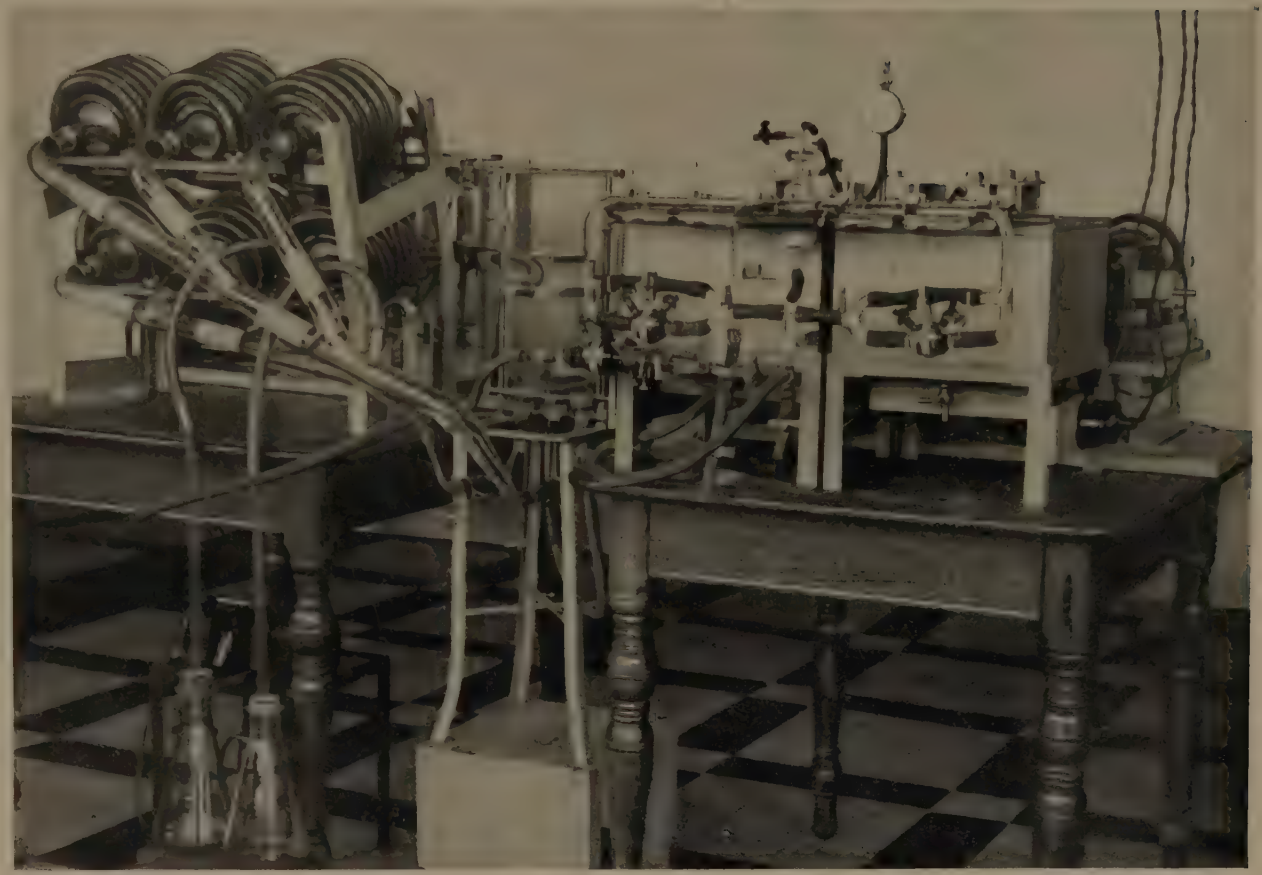

Fig. 1. Overall view of the apparatus.

oxygenation will be sufficient for a normal man of about 64 kilograms bodyweight.

Blood-pressure experiments.

From our experiments with artificial circulation in dogs (see below) it appears that the animals support this circulation for hours without a sign of early or later distress. This means that the circulation in the whole organism, especially in the heart and the nervous system is sufficient. Nevertheless we measured the bloodpressure in a carotid artery during artificial circulation in dogs. As the oxygenated blood is forced with a certain pressure by the artificial left heart into the femoral arteries, the pressure in the carotid artery will be less, and might possibly be too low for an adequate circulation of the brain. Our experiments, however, showed that the carotic pressure is very good. In one experiment we recorded e.g. during an artificial circulation of $1\frac{1}{2}$ hours, in a dog of 25 kilograms, a mean carotic pressure varying between 100 and 80 mm Hg, which is a normal pressure for a dog. During this experiment we stopped the mechanical device from time to time to see how the blood-pressure would react. It is clear that the organic heart immediately resumes its function if the blood is not sucked away from the caval veins: but we were anxious to see whether the blood-pressure is restored quickly. These experiments showed that an (artificial) carotic pressure of e.g. 100 mm fell to 30 or 40 mm the moment the artificial heart was stopped, but that this (now organic) pressure rose again to about 100 mm within about $\frac{1}{2}$ minute. This means that the organic heart remained in good condition.

We did not measure the pressure in the coronary arteries, but in our later experiments in which the heart was opened, it occurred now and then that a very small branch of a coronary artery was severed. In such cases we saw the blood spout from the vessel, which indicates that there was a real good pressure even in the small branches of the coronary system.

Re-animation experiments.

These experiments have little or nothing to do with artificial circulation, but as they were easily to perform with our apparatus, we did some in dogs.

The procedure runs as follows: a catheter is brought in the inferior caval vein and a canula in a femoral artery. Setting to work only the mechanical right-heart-pumps, practically all the animal's blood is aspirated, driven through the oxygenator and assembled in the reservoir. In this way the animal is practically totally bleeded. In most cases the respiration of the animal stops after some minutes; in some cases the heart also ceased to function (continuous check by electro-cardiographic recording). After the respiration has been absent for some 6 to 10 minutes

(or in the case the heart stops, immediately hereafter) the left mechanical heart-pumps are put to function and the (oxygenated) blood is pumped back into the animal through the femoral artery. If the stillstand of the respiration (resp. the heart) has not been too long, the animal revives and survives in quite normal condition. Thus the animal has been clinicaly dead for some time and is reanimated.

One of our experiments of this kind ran as follows.

Dog, greyhound, 25 kilograms.

Narcosis: morphine-kemithal; 140 mg heparine Vitrum.

2^{53} : Respiration: 14 per min.; mean blood pressure (art. fem.) 120 mm Hg.

3^{00}—3^{08} : In 8 minutes time nearly 2 liters blood are aspirated from the animal.

3^{09} : Respiration has practically ceased; now and then a gasp. Arterial blood pressure 20 mm or less.

3^{17} : Eyelid reflexes nearly imperceptible.

3^{18} : Stretching of the hindlegs; no more gasps.

3^{19} : Electro-cardiogram much deformed, very small and very slow; almost absent.

3^{20}—3^{25}: The blood is pumped back into the animal.

3^{23} : Respiration 14 per min.; deep.

3^{25} : Electro-cardiogram better; fast.

3^{30} : Eyelid reflexes very feeble.

3^{31} : Blood pressure 130 mm.

3^{40} : Respiration 20 per minute.

4^{33} : Eyelid reflexes normal; animal moves hig legs and opens his eyes.

6^{00} : The animal tries to stand and to walk.

7^{30} : The animal walks and drinks.

Next and following days: The animal (which has been „dead" for about 10 minutes) behaves normal.

The heartless dog.

This is more a curiosity than an experiment. In one of our experiments, whilst the artificial circulation was in function, the heart of the animal stopped rather suddenly, after the thorax had been opened. As this dog was lost for the experiment, we clamped the big vessels very close to the heart and removed the heart from the animal. Thanks to the artificial circulation the animal still „lived" and showed (in very deep narcosis and with curare) undiminished eyelid reflexes and active retraction of the tongue during a period of about one hour; after which period we finished the experiment.

Experiments on artificial circulation.

The experiments described above are more or less adventitious; we actually aimed to study the possibilities of intracardial operations. For

these operations an artificial circulation of about 40 minutes may be long enough, but we also performed circulation experiments of much longer duration.

The dogs supported artificial circulation lasting from 45 minutes to 4 hours very good; they recovered soon, and behave, months after the experiments, still normal. With one exception we did not observe any early or later distress. One of our dogs was killed 5 weeks after the experiment and examined thoroughly by the pathologist. (Prof. VAN NIEUWENHUYSE). He could not detect any abnormality which might have resulted from the artificial circulation. It is of special importance that the microscopic examination of the heartmuscle showed no alterations whatsoever.

The exception referred to above concerns a dog, who, after a 4 hours' artificial circulation, showed troubles of vision. At this moment, more than 6 months after the experiment, his sight is still diminished, but otherwise he is in a very good condition.

After these circulation experiments it seemed permitted to proceed with the aid of a thoracic surgeon to the complete operation, i.e. to open a heart cavity in dogs.

One has to realise, though, that the opening of the heart is a very serious procedure, during which the animal has to stand several interventions. First the catheters and canulas must be inserted, the blood sufficiently heparinised, and the artificial circulation with a mixture of the animal's own blood and that of a donor established. Then the animal has to be in deep narcosis and gets curare to facilitate work in the thorax. Further the thorax must be opened wide, and at last, after clamping eventually big vessels temporarily, the heart must be opened. The animal may endure each of these measures separately very well, but the accumulation of them may bring him in great danger.

The large thoracic wound can easily give rise to intra-thoracic afterbleeding in the heparinised animal, which may become fatal especially in dogs, as they have badly develloped mediastinal pleurae, compared with men.

Moreover, the nursing of dogs is rather difficult, much more difficult e.g. than of human patients. Thus altogether, the operative opening of a heartcavity is undoubtedly a serious procedure.

In the beginning of our experiments we did not realise this sufficiently and made the mistake to start at once with the complete operation. With the result that our first animals, although they had stood the opening of the heart very well and had all come in apparently good condition from the operation table, died 2 to 6 hours later.

Taught by this experience we proceeded step by step from now.

Artificial circulation in curarised dogs.

The animal is curarised so that he makes no respiratory movements.

Like the normal dog, the curarised animal stands the extra-corporeal circulation without harm and survives promptly.

Opening of the thorax in curarised dogs with artificial circulation.

In these experiments we payed special attention to the intrathoracic secondary hemorrhage.

As I pointed out already formerly (2, 3) the necessity to make use of an anti-coagulant is a weak point in artificial circulation procedures, during which big operations are performed. As soon as it is possible to construct the apparatus in such a way (e.g. by coating its surfaces or by using a special material as perhaps some plastics) that one is absolutely sure that no bloodclotting can occur, even after hours, we may leave out the anti-coagulant. But as long as such measures are not yet absolutely reliable, I think we had better make use of an anti-coagulant, as the smallest clot can cause unrestorable damage to the subject.

We used protamine-sulphate intra-venously as anti-heparine, but we have the impression that injection of protamine-sulphate is not quite harmless under the given circumstances. So in other experiments, we applied the protamine-sulphate locally on the innerside of the thorax-wound by means of spongostan soaked with protamine-sulphate. Moreover we gave transfusions with citrateblood. With both methods we had good results; the animals survive and recover in 1 of 2 days; we prefer, however, the local application of the protamine-sulphate.

One could also think of exsanguination after the operation, to eliminate the heparine, but we did not use this method.

Opening of a heart-cavity.

With this final operation we were also successfull now. The following is an abridged protocol of such an experiment.

Dog, male, 20 kg.
anaesthetic: morphine, kemithal.
heparine 110 mg.
After insertion of catheters and canulas curare (intercostrin): 10 mg.

16^{25}	:	*Artificial circulation started.*
16^{32}	:	Thorax is opened.
16^{35}	:	Incision of the pericard.
16^{36}	:	Pulmonary artery clamped.
16^{37}	:	Broad incision of the left auricle. The auricle is dry, the mitral valve is visible and accessible.
16^{48}—17^{11}	:	Closing of heart and thorax (drain).
17^{00}	:	2 mg prostigmine; 1 mg atropine.
17^{17}	:	*Artificial circulation stopped;* Catheters and canulas removed.
17^{30}	:	5 cc protamine-sulphate.
17^{40}	:	850 mg coramine; penicillin 100.000 U.

18^{00}—20^{00}: Transfusion of 400 cc citrate blood.
21^{00} : The animal has woken up from narcosis.
Second day: walks and drinks.
Third day: is very active, walks, drinks and eats.
Eighth day: sutures removed: dog behaves entirely normal.

Every time again it is amazing to see how, after clamping the pulmonary artery, the left auricle can be opened very widely without any blood flowing, so that one can see and has access to the mitral valves. If we open the right auricle both venae cavae are clamped, and in this case the venous blood of the coronary system has to be aspirated. This is easily done: the aspirated blood is collected in a sterile vessel and put in the reservoir of the system, so that it is not lost for the animal.

By the coronary vessels one can see very clear that the oxygen supply to the heartmuscle is good: one immediately sees the difference in colour between the veins and the arteries of the coronary system.

Although we reached our aim in so far that we have shown now that a dog can survive the whole procedure of the opening of an auricle and that it is thus possible with our method of artificial circulation to make the heart accessible to intracardial operations, we are continuing our experiments.

I am much indebted to my coworkers, especially to Dr VAN GOOR, who mostly directed the pre-operations, to Dr PUNT, who a.o. filmed our apparatus and experiments, and to the thorax-surgeon Dr BROM, who performed, with his assistents, the cardiac operations.

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BEIJERINCKIA, A NEW GENUS OF NITROGEN-FIXING BACTERIA OCCURRING IN TROPICAL SOILS

BY

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Almost half a century ago these Proceedings contained a paper, written by M. W. BEIJERINCK and entitled: „On oligonitrophilous bacteria". [3]. It was in this paper that the author published his discovery of a group of mobile microbes of unusual dimensions and of a very singular aspect. He decided to give them the name of *Azotobacter*, because they were evidently capable of assimilating atmospheric nitrogen, but so peculiar was their image, that BEIJERINCK hesitated between the names *Azotobacter* and *Parachromatium* for the new microbes; the latter name was intended to indicate a relationship with the genus *Chromatium*, well known among the *Thiorhodaceae*. This idea of a relationship remained active in BEIJERINCK's mind throughout his further work on *Azotobacter* and, as will appear further, it has received an unexpected justification quite recently by the work of WILSON. [17].

It is well nigh superfluous to mention that BEIJERINCK's important discovery has particularly stimulated research on microbial nitrogen-fixation all the world over.

Yet, research on the microbes, bringing about fixation of atmospheric nitrogen in the tropics, has been rather inadequate for more than one reason; especially in Indonesia. Moreover, the results obtained in this country have been rather conflicting.

In papers, dating from 5 to 8 years after BEIJERINCK's discovery of *Azotobacter*, one of his pupils, DE KRUYFF, concluded from investigations of Javanese soils that *Azotobacter chroococcum* „is very rare in tropical countries". [9]. The author mentions the presence, however, of several nitrogen-fixing bacteria, among which *Bacterium Krakataui* [8] and *Micrococcus* No. 1 [9]. These organisms have been described very superficially, but it is stated that they do not produce the well known wrinkled *Azotobacter*-pellicle on BEIJERINCK's classical mannitol-medium. This medium gradually turns into a thick, viscous mass: a striking phenomenon, also shown by the organisms, which form the object of the present paper. Of course, exact identification of DE KRUYFF's organisms is not possible with his scanty data.

Some years later the scarcity of *Azotobacter* in Javanese- and other soils was contested by GROENEWEGE [5], who was able to show the presence of typical *Azotobacter* in several kinds of soil from Java and

also from Hawai. Since then, *Azotobacter* has been found to be present in arable soils of suitable p_H, i.e. 6 or more, almost everywhere. Soil-deficiencies of some kind (for which see e.g. VAN NIEL [12]) may well inactivate *Azotobacter* in the soil and even prevent its development in BEIJERINCK's mannitol-medium, but their *presence* in these (rare) cases can as a rule be demonstrated by the addition of the necessary trace-elements to the medium mentioned.

BEIJERINCK's medium, viz. a solution made up of tap water, mannitol 2 % and K_2HPO_4 0.02 %, in thin layers, has been almost exclusively used in *Azotobacter*-studies, sometimes with some minor modifications (e.g. ASHBY's medium), until 1932. In that year WINOGRADSKY [18] published a new and very efficacious procedure for the isolation of nitrogen-fixing organisms, which consists in the distribution of a small quantity of sieved soil-particles on the surface of a nutrient silica gel-, or purified agar-plate. As will presently be seen, this method has led to an important deepening of our knowledge of nitrogen-fixing organisms.

Using sodium benzoate 0,5 % as a source of carbon, as proposed by WINOGRADSKY, some tentative experiments were made with the view of demonstrating the presence of *Azotobacter* in samples of arable soil from the Royal Botanic Garden in Buitenzorg. Without exception, typical *Azotobacter chroococcum* could be detected in such soils within a couple of days and several strains were isolated, differing in production and intensity of the well known black colour. An aberrant strain, not darkening on staying and producing a green fluorescent substance, soluble in the medium, was also isolated and provisionally labelled *Azotobacter vinelandii*.

Quite negative results, however, were obtained at the examination of forest-soils, these being either humus-rich soils from woody parts of the Buitenzorg garden (250 m. above sea-level), or soil from the virgin-forest in the neighbourhood of the Tjibodas mountain-garden (1400 m. above sea-level). The reaction of these soils was slightly acid (p_H 4.5—5.5) and, as could be anticipated, *Azotobacter* failed completely in these samples. This makes it probable that the principal reason of DE KRUYFF's failure to demonstrate *Azotobacter* in Javanese soil was an unfortunate choice of his soil-samples with regard to the p_H. It should be considered indeed, that the notion of p_H and the knowledge of its great influence on microbial life were still embryonic in the years 1906—1909, when the pioneer work of SÖRENSEN, MICHAELIS and CLARK was only just making a start (1909—1914—1915)!

However, when the benzoate-agar plates sprinkled with forest-soil, were re-examined after two or three weeks standing, very remarkable colonies had made their appearance. They are almost porcelain-white and rise high above the surface of the agar (some of them are almost spherical) and consist of microbes embedded in a tough elastic matter. Microscopic examination reveals the extremely uncommon appearance

of the microbes themselves: at first sight they resemble strongly refractive diplococci of about 1.5 μ in diameter, *lying some distance* (0.3—0.7 μ) *apart*. Closer examination shows these "cocci" to be united in pairs by an enveloping membrane, which often appears to be more or less constricted between them. The "cocci" are coloured yellow-brown by LUGOL's iodine-solution; methylene-blue and gentian-violet leave them colourless, but the intermediate space between two "cocci" is strongly stained; at the same time, a well defined capsular slime-mass is revealed in some strains.

Of course, it soon became clear that the organisms are in reality straight, or slightly curved, rods of about 3 μ by 1—1.5 μ, containing two polar globular bodies of lipoid character, which occupy the total breadth of the microbe and eventually cause a distinct local dilatation of the cell wall. The lipoid bodies are mostly of equal, but sometimes of different dimensions and may in older cultures become so voluminous as to completely distend the bacterial cells into large, misshapen and branched bodies.

The description has been given in such detail to make it clear that, though working on aerobic nitrogen-fixing organisms and in particular with *Azotobacter*, in no stage of the experiments with these newly unveiled organisms the slightest thought occurred to me that they might be a particular species of *Azotobacter*, or even an organism in some way related to this genus. In fact, I was quite puzzled!

I was aware, it is true, of the work of ALTSON [1] who, in 1936, had isolated "*Azotobacter*" from acid Malayan soils — a quarzite soil of p_H 4.5 and an alluvial soil of p_H 4.6 — *by means of the same sieved-soil distribution method of* WINOGRADSKY (on silica-gel-mannitol plates). But ALTSON states emphatically — and without giving further particulars on the morphology of his microbes — that „he studied the morphology of the organism during a period of about eighteen months" (l.c. p. 274) and that „microscopic examination showed that this (gelatinous growth) consisted of typical cells of *Azotobacter*" (l.c. p. 269). For this reason, ALTSON's paper was not given much attention at first. Yet, the manner in which this author found his organism, as well as its cultural and physiological behaviour, agree so closely with the corresponding properties of the organisms found in Buitenzorg-soils, that I am now quite convinced of the identity, or at least the very close parentage, of his — unnamed! — microbes and the organisms found here. Notwithstanding the fact that the latter most decidedly do not consist of typical *Azotobacter*-cells!

It will not be difficult now to imagine my astonishment in finding in the 6th edition of BERGEY's manual, which had just reached me, the description of *Azotobacter indicum* STARKEY and DE; *a description fitting in all details the organisms, which I had isolated* and which have been just described.

STARKEY and DE's original publication [16], containing two good microphotographic images and a subculture of the original strain of their *Azotobacter indicum* (kindly put at our disposal by KLUYVER from the collection at the Laboratory of Microbiology at the Technical University of Delft) sufficed to remove any doubt as to the identity of some of the Buitenzorg strains with *Azotobacter indicum* STARKEY and DE.

However, I am unable to understand, how ALTSON (whose earlier work is not mentioned by STARKEY and DE) as well as STARKEY and DE could include the organism in the genus *Azotobacter*, a genus showing such distinctive morphological and cultural characters. Neither has HOFER [6] in his investigation on the type of flagellation of several *Azotobacter* species, *among which Azotobacter indicum*, expressed any doubt on the systematic position of the latter organism.

Azotobacter indicum and its congeners are indeed different from the known Azotobacters in almost every respect: *morphologically* (small, single rods with hyaline protoplasm containing two (or sometimes three, in "buckled" rods) polar, strongly refractive lipoid-spherules; versus stout, short, mostly doublerods with distinctly differentiated (granulated) protoplasm in *Azotobacter sp.*); *culturally* (steadily increasing viscosity of the culture-solution by formation of a ropy to elastic slime; versus formation of a gradually thickening supernatant pellicle in *Azotobacter*-cultures) and, above all, *physiologically* (development in nutrient media with p_H 3.5—9, gradual formation of acid, which lowers the medium's initial p_H to about 3.5, versus growth only in media of $p_H > 6$ and no acid formation during growth of *Azotobacter*). These distinctive characters were, curiously enough, known to STARKEY and DE; they were established by these authors and could be confirmed on all points by our investigations.

In fact, the only feature these organisms have in common with typical *Azotobacter*, is their capacity of nitrogen-fixation under aerobic conditions!

It is therefore not surprising to find that the justness of incorporating *Azotobacter indicum* in the genus *Azotobacter* has already been called in question before.

After STARKEY's lecture on *Azotobacter indicum*, at the meeting held at New Brunswick, N.J. by the Third Commission of the International Society of Soil Science (Session of August 31, 1939), BURK put the following questions: „I would not ask if this organism just described is *Azotobacter*, but I would ask you, Dr STARKEY, this: „Is there any *doubt in your mind* that it is *Azotobacter?*" and „So I would ask, does any doubt lurk in your mind that this may not be *Azotobacter?*" [15].

From STARKEY's answer it becomes clear that it is „above all the capacity of using molecular nitrogen in the absence of available fixed nitrogen" (together with strict aerobiosis, non-sporulation, motility, etc.), which induced him to range the new organism under *Azotobacter*. Presumably ALTSON too has been primarily led by the same — or similar — arguments.

In matter of nitrogen-fixation, even with the restriction to aerobic nitrogen-fixation, it does not seem either advisable or justified to incorporate all known free living, aerobic, nitrogen-fixing organisms in one genus *Azotobacter*. Not only should such a physiological genus in that case also contain certain *Nostocaceae*, but the capacity of nitrogen-fixation has been shown in the past to be incident to several „common" microbes, belonging to very different classes and genera, such as *Bacterium lactis viscosum*, *Bact. radiobacter*, *Bact. prodigiosum*, *Bact. pyocyaneum*, *Bacillus danicus*, *Bac. malabarensis* etc. Nitrogen-fixation has even been claimed for some Saccharomycetes, such as *Torula bogoriensis rubra* DE KRUYFF [10], *Torula Wiesneri* ZIKES [19] and perhaps still other ones!

It is true that these assumptions have been regarded with some mistrust, but quite recently GEST and KAMEN have discovered an active nitrogenase-system in *Rhodospirillum rubrum*, and WILSON [17] very recently has succeeded in demonstrating nitrogen-fixation by all the species investigated of *Athiorhodaceae* and *Thiorhodaceae*; organisms not suspected hitherto of this faculty. This disclosure, by the way, involuntarily calls to mind BEIJERINCK's postulate of the parentage of *Azotobacter* and *Chromatium!*

But there are still other reasons, which induce to consider physiological genera with some circumspection:

Numerous are the instances encountered in the course of microbiological research, in which potencies, *which are not essential to the perpetuation of the species*, may not only become latent under certain conditions, but may actually be lost and vanish irreversibly. This may occur either by processes of adaptive selection within a population or by processes of — or equivalent to — mutation, spontaneous or induced, e.g., by radiation.

With special regard to *Azotobacter* e.g., it has been recognized that the development of these microbes is not essentially dependent on the ability to fix atmospheric nitrogen. This ability is not a *conditio sine qua non* for their multiplication and dispersion; they may thrive on bound inorganic nitrogen compounds as well. Consequently the nitrogenase-system can be dispensed with, without endangering the survival of the species, and it has been found indeed that *Azotobacter*-strains maintained in pure culture — i.e. under conditions artificial and very dissimilar to those prevailing in Nature — may ultimately lose the ability to assimilate free nitrogen (SCHRÖDER [13]).

On the other hand it seems not contrary to reason to presume that *Azotobacter*-clones, which are not — or no longer — able to fix atmospheric nitrogen might very well occur in nature; yet, as long as we only have at our disposal a method for the accumulation of *Azotobacter*, which is based on an elective action with respect to the ability of development in the absence of nitrogen, it is obvious that only those individuals will grow, multiply and finally appear, which are in possession of an active

nitrogenase. Nevertheless the presumed nitrogenase-less *Azotobacter*-forms should have just as much right to the generic name *Azotobacter* as their *confrères*, which have definitely lost the nitrogenase in the course of their seclusion in culture tubes, stuffed with artificial food.

A final remark on the question of physiological genera: Certainly it would be imprudent to generalize and to assert that the fixation of free nitrogen is a distinctive character of the genus *Nostoc*, but it would be absurd to split up this genus into two genera, one for the *Nostoc*-species, able to fix nitrogen and another for the *Nostoc*-species, which lack this ability.

Without wishing, at the present stage of the investigations, to put forward more than the merest suggestion toward a working-hypothesis, I should like to indicate some rather striking points of resemblance of *Azotobacter indicum* c.s. to species of the genus *Rhizobium* Frank. Members of this genus (especially from the *Medicago-Melilotus*-group) also show a decided tendency to acid formation in the course of their development and often produce an abundant slimy — and occasionally viscous — gum. Morphologically the rods of *Azotobacter indicum* c.s., often irregular and buckled, are reminding of the bacteroids of some Rhizobia, in which polar lipoid-bodies, equally coloured yellow-brown by dilute iodine-solution, are common (see e.g. the original drawings of Beijerinck [2] of the nodule-bacterium of *Vicia Faba*).

On the other hand, of course, the Rhizobia-*bacteroids* show no motility as is manifestated by *Azotobacter indicum*; they do not multiply as such, nor do they fix atmospheric nitrogen *in vitro*. It should, however, be kept in mind that the properties of *Rhizobium* sp. may have been profoundly modified by — and adapted to — their symbiotic mode of life. As well as parasitism, symbiosis may be the cause of the loss of several qualities originally present!

However this may be, in view of the very peculiar morphology of *Azotobacter indicum* and the related species, which have been found in tropical soils during these investigations, it appears quite justified to create a new genus for these organisms. It seems only natural to the present author to give this genus the name of Beijerinckia, in honour of one of the great pioniers in the field of general bacteriology, M. W. Beijerinck (16 March 1851—1 Jan. 1931) who, on New-Year's eve 1900, for the first time used the name *Azotobacter* for the aerobic nitrogen-fixing organisms he discovered.

The new genus may be defined as follows:

Beijerinckia *nov. gen.*

Straight, or slightly curved, or irregular, locally swollen to buckled, rods, characterized by the presence, at the extremities, of highly refractive spherical bodies, presumably consisting of lipoids. No endospores. Motility

present, at least in certain stages of the development. Flagellation peritrichous (HOFER, J. of Bact. **48**, 697—701 (1944)).

Decidedly aerobic; developing in media with p_H 3.5—9 and fixing atmospheric nitrogen in the absence of nitrogen compounds. Growth is accompanied by the formation of acid and, in some species, by the formation of large amounts of a tough to elastic slime. No surface pellicle is formed on liquid media. No development on peptone-broth-agar. Gram negative.

Found in soils of tropical countries.

Species typica: *Beijerinckia indica* (STARKEY et DE) Derx comb. nov.; syn. *Azotobacter indicum* STARKEY et DE, Soil Sci. **47**, 329—343 (1939), Science **89**, 267 (1939); *Azotobacter sp.* ALTSON Journ. of Agric. Sc. **26**, 268—280 (1936).

Distinctively characterized by the formation of leavan from saccharose and by the formation of a rusty-red to fulvous colour on media kept neutral, e.g. by the addition of $CaCO_3$.

In a separate paper some other species and forms of *Beijerinckia* will be described.

I take the occasion to thank Miss. E. J. KLERKX for the accurate execution of most of the experimental work concerned.

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FURTHER NOTES ON VARIATION IN LOXOTHYLACUS CARINATUS (KOSSM.)

BY

H. BOSCHMA

(Communicated at the meeting of January 28, 1950)

In a previous paper (BOSCHMA, 1949) some details were given of Rhizocephalan parasites of various crabs, which were regarded as belonging to one species, *Loxothylacus carinatus* (Kossm.). The peculiarities of the male genital organs, the colleteric glands and the cuticular excrescences of three of these specimens are dealt with here in more detail. This is done because the three specimens among each other show differences which might indicate that they are not really conspecific. When an extensive material of specimens occurring on one species of crab is available it stands to reason that the observed variation, if within reasonable limits, is individual, as, e.g., in the specimens of *Loxothylacus variabilis* on *Chlorodiella nigra* (Forsk.) (cf. BOSCHMA, 1940). But when the specimens are parasites of different crabs and show individual variation it is open to doubt whether they are conspecific or not. The characters of the three specimens (one from unknown host, one on an unidentified Portunid, and one on *Thalamita admeta* (Herbst)) therefore are described here in some detail to form a basis for a future decision whether or not they really belong to one species. The data concerning the localities, etc., are given in the 1949 paper cited above.

In the figures of parts of sections the excrescences of the external cuticle have been omitted.

In the specimen from unknown host, from Hongkong, the visceral mass is attached to the posterior part of the body in the region from which the stalk takes its origin (fig. 1*a*). Here the visceral mass has shifted slightly to the right side of the posterior region, but the place of attachment is not as distinctly separated from the region of the stalk as it is typical in the genus *Loxothylacus*. In this part of the body the vasa deferentia are narrow canals running along the posterior margin of the visceral mass. In a more dorsal region the vasa deferentia gradually become wider whilst their cavities obtain an irregular inner wall as a result of numerous ridges and other prominences (fig. 1*b*). Farther towards the dorsal region the left vas deferens again becomes narrower, and it passes into its testis which also remains narrow. The most dorsal part of the left testis is seen in fig. 1*d*; here its course chiefly

is in an anterior direction. The curvature of the left testis is evident in the region of fig. 1*c*. The right testis is much larger than the left, it has a much thicker wall and a much wider cavity. Fig. 1*c* shows the right testis in the region where it is connected with its vas deferens, a slightly more dorsal section is the one of fig. 1*d*, whilst the terminal part of the

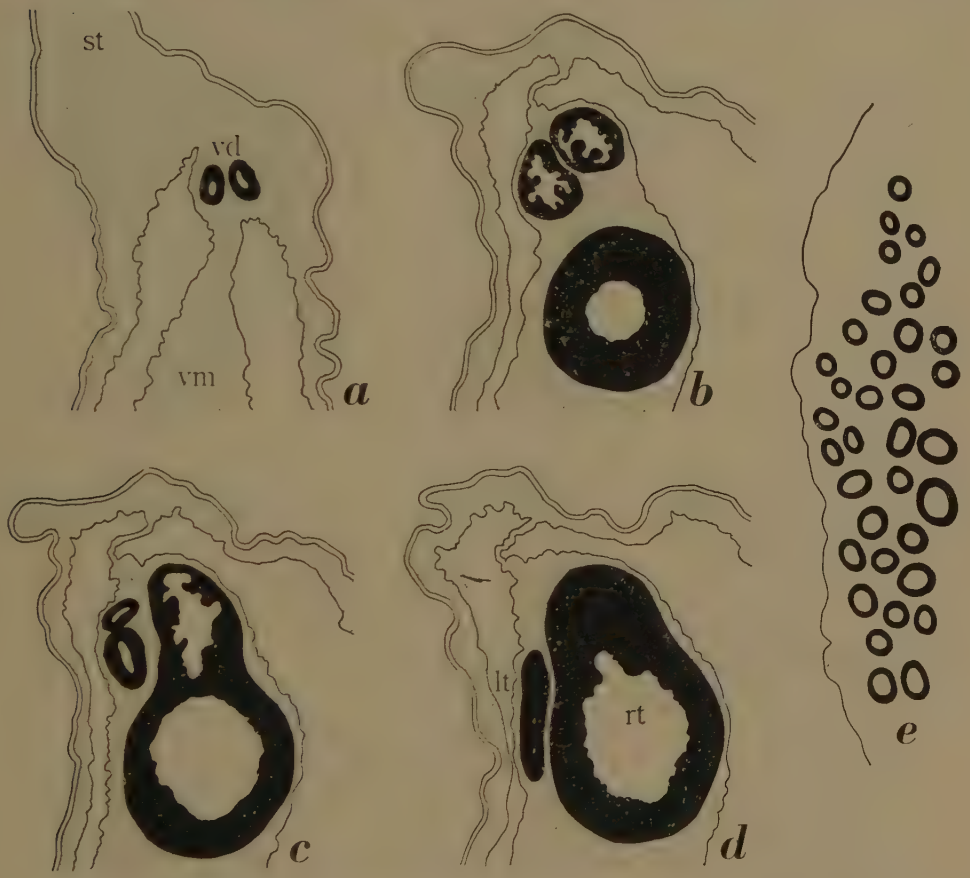

Fig. 1. *Loxothylacus carinatus*, specimen from unknown host. *a* — *d*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *e*, longitudinal section of one of the colleteric glands. *lt*, left testis; *rt*, right testis; *st*, stalk; *vd*, vasa deferentia; *vm*, visceral mass. *a* — *d*, × 30; *e*, × 107.

right testis, at the end of its curvature, is represented in fig. 1*b*. Here this terminal part is distinctly pointing in a ventral direction. In this specimen, therefore, the two male organs possess a similar, distinct curvature. Their size is so different that apparently only the right testis was functioning, the left is more or less rudimentary.

The colleteric glands occupy a not exactly central position in the lateral surfaces of the visceral mass, as they are slightly nearer to the anterior end than to the posterior extremity of the visceral mass. In a

longitudinal section of the most strongly divided part there are 35 canals (fig. 1*e*). In this specimen the canals of the colleteric glands do not possess chitin.

In the greater part of the mantle of the specimen from unknown host the external cuticle of the mantle is rather thin (9—12 μ), in some parts it is thicker (to about 21 μ). Its surface is crowdedly covered with excrescences consisting of a kind of chitin different from that of the

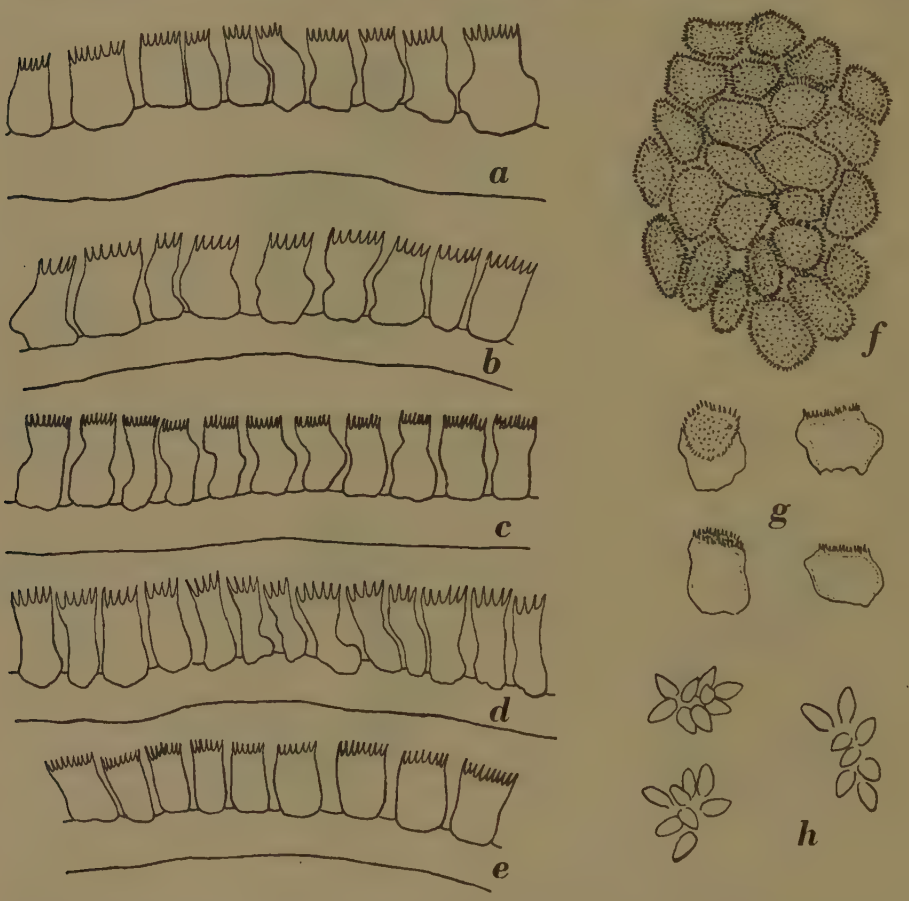

Fig. 2. *Loxothylacus carinatus*, specimen from unknown host. *a* — *e*, sections of the external cuticle; *f*, excrescences of the external cuticle in surface view; *g*, isolated excrescences; *h*, retinacula. × 530.

main layers by its being of a more hyaline composition and having very little affinity for stains. These excrescences (fig. 2*a*—*e*) are more or less cylindrical bodies which with comparatively broad bases are attached to the upper surface of the cuticle and at their terminal parts bear a variable number of spines. In this specimen the excrescences do not possess root-like expansions penetrating into the main layers of the cuticle. The excrescences are of rather variable shape and size (total

height 15—26 μ; transverse diameter 6—20 μ), and in some parts of the mantle their spines are larger than in other parts. In surface view it appears that the larger and most conspicuous spines are found on the upper margin of the excrescences, whilst smaller spines occur in great numbers on the more or less flat upper surfaces (fig. 2*f*). Sometimes the marginal parts of the excrescences appear to have a different structure from the more internal parts (fig. 2*g*), showing that the external chitinous matter is harder than the internal.

The internal cuticle of the mantle in this specimen bears numerous retinacula which are rather regularly distributed over its surface. The basal parts of these retinacula (well developed in other specimens) are not distinct here. There are only to be seen groups of spindles, each group consisting of 7 to 10 spindles, which have a length of 6 to 9 μ and do not seem to possess barbs (fig. 2 *h*).

The visceral mass of the specimen on the unidentified Portunid crab is attached to the mantle at the right side of the region of the stalk (fig. 3*a*). This situation of the visceral mass is intermediate between

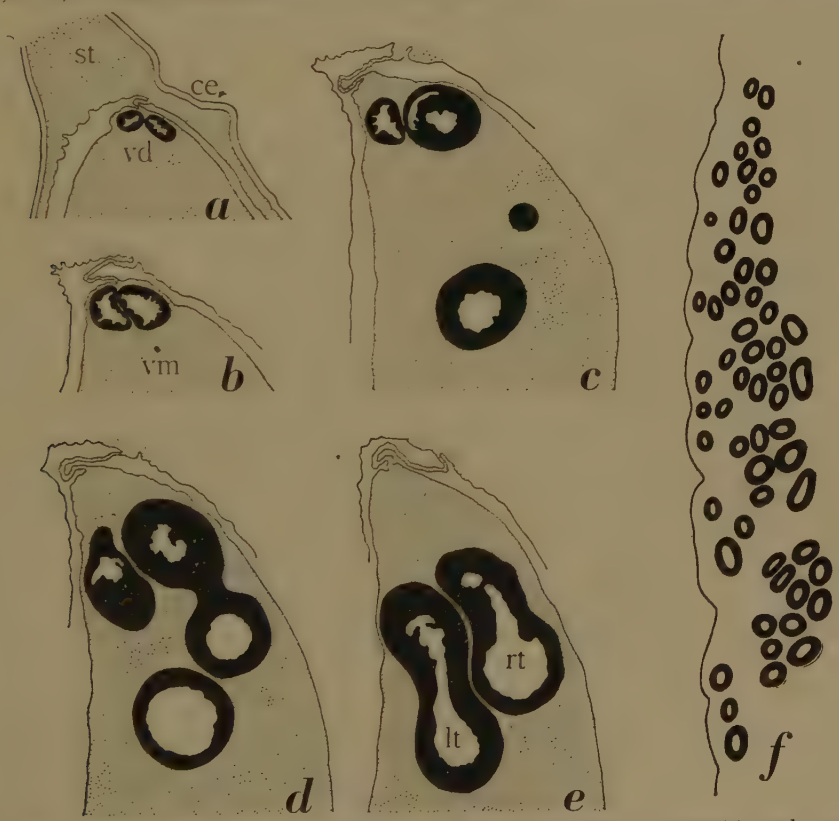

Fig. 3. *Loxothylacus carinatus*, specimen on unidentified Portunid crab. *a* — *e*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *f*, longitudinal section of one of the colleteric glands. *ce*, external cuticle; *lt*, left testis; *rt*, right testis; *st*, stalk; *vd*, vasa deferentia; *vm*, visceral mass. *a* — *e*, × 20; *f*, × 80.

that found typically in *Loxothylacus* and that occurring as a rule in *Heterosaccus* and *Drepanorchis*. In this region the vasa deferentia are narrow; they show already some ridges on their inner walls. In a more dorsal region the vasa deferentia become wider and their ridges become more prominent (fig. 3*b*). Gradually now the vasa deferentia pass into the testes, and from here the male organs show their curvature, which in this specimen is strongly developed. In a section of the terminal region of the testes it appears that these are found at a fairly large distance from the vasa deferentia (fig. 3*c*). In sections from a more dorsal region the transition of the vasa deferentia into the testes is visible (fig. 3*d*, *e*). The two male organs here have approximately the same shape and they are of about equal size. Especially in the dorsal region they are in close contact, but they remain completely separated.

The colleteric glands occupy the central region of each of the lateral surfaces of the visceral mass. They are more or less flattened and contain a fairly large number of canals; in a longitudinal section of the most strongly divided part (fig. 3*f*) there are 58 of these canals, which in the present specimen do not contain chitin.

In the specimen on the unidentified Portunid crab the external cuticle in various parts of the mantle has a thickness of 21 to 33 μ. Its surface is closely beset with excrescences consisting of a more or less hyaline kind of chitin, differing from that of the main layers. These excrescences are more or less cylindrical bodies that in some parts of the cuticle with their broad bases are attached to the surface layers (fig. 4*b*), whilst in the greater part of the cuticle they have well developed roots that are deeply imbedded in the main layers (fig. 4*a*, *c*, *d*). The size and the shape of the excrescences is subject to rather extensive variation. The total length of the excrescences (including the roots) is from 24 to 60 μ, the part extending above the surface of the cuticle is from 18 to 38 μ, their transverse diameter varies from 7 to 22 μ. The excrescences may be rather slender (fig. 4*d*) or broader and thicker (fig. 4*a*—*c*). At their extremities they bear a number of rather strong small spines. The more or less cylindrical parts extending above the cuticle often are rather contorted, as, e.g., in some of the isolated excrescences shown in fig. 4*f*. In surface view it appears that the spines chiefly are implanted on the margins of the excrescences (fig. 4*e*), the central part of the upper surface has a few smaller spines only.

On the internal cuticle of the mantle there are numerous retinacula, more or less regularly distributed over its surface. Each retinaculum (fig. 4*g*) has a well developed large basal part on the top of which there are 5 to 7 spindles. The latter have a length of 12 to 18 μ, they possess small barbs.

In the specimen on *Thalamita admeta* the visceral mass is attached to the mantle at some distance from the stalk (fig. 5*a*; the thickened part of the mantle in the upper part of the figure in a more ventral

section is connected with the region of the stalk). This section further shows the vasa deferentia, which here practically have no ridges on their inner walls, and the terminal, ventral part of the right testis. The region of transition of the right vas deferens into its testis is shown in

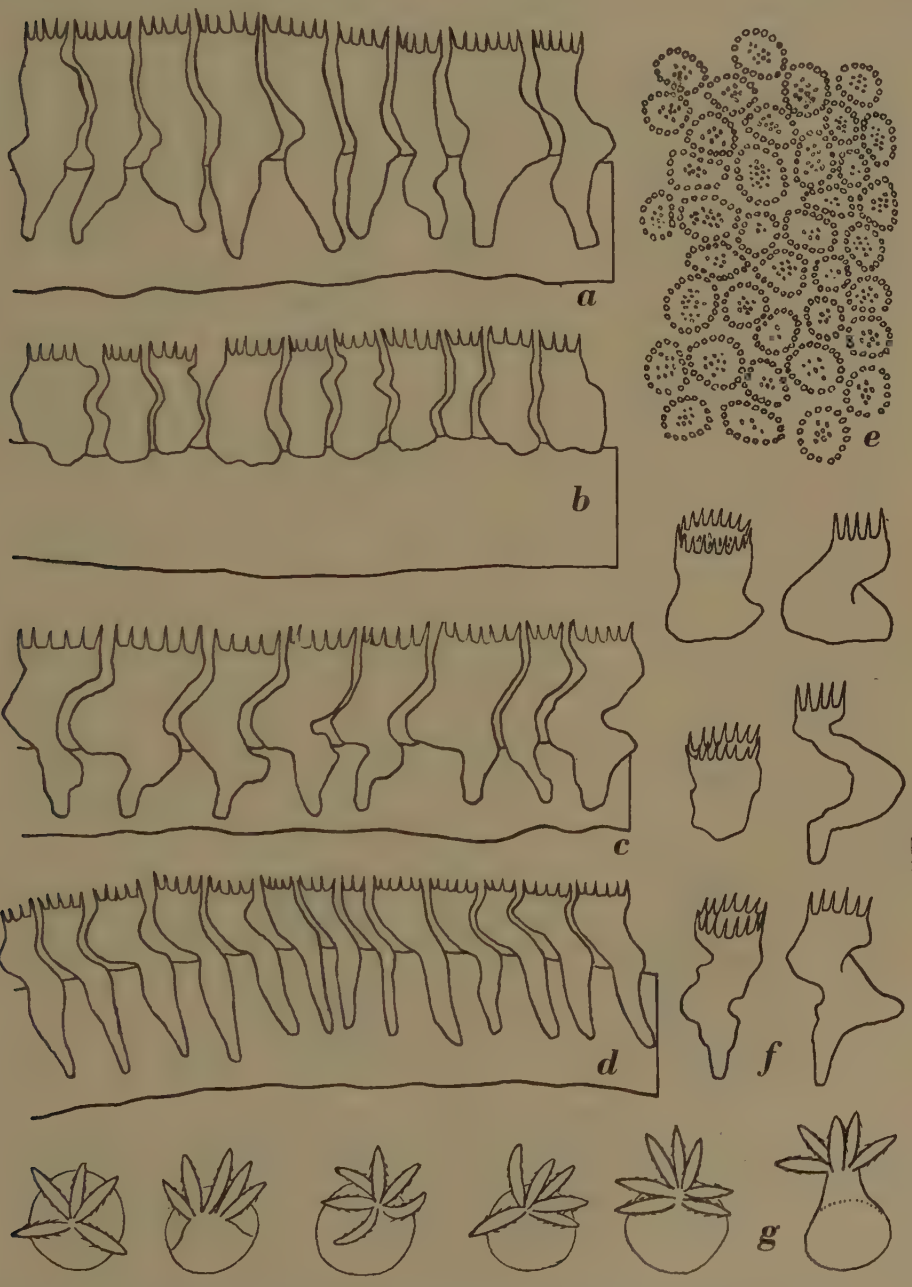

Fig. 4. *Loxothylacus carinatus*, specimen on unidentified Portunid crab. *a* — *d*, sections of various parts of the external cuticle; *e*, excrescences in surface view (basal parts of spines drawn only); *f*, isolated excrescences; *g*, retinacula. × 530.

fig. 5*b*, *c*; a more dorsal part of this testis is visible in fig. 5*d*. The right testis is of large size, it has a thick wall and a wide cavity. The left testis is much smaller, in its widest region (in its dorsal part, fig. 5*d*) it is not appreciably wider than the vasa deferentia. As the right testis the left shows a distinct curvature (fig. 5*c*), but it does not protrude

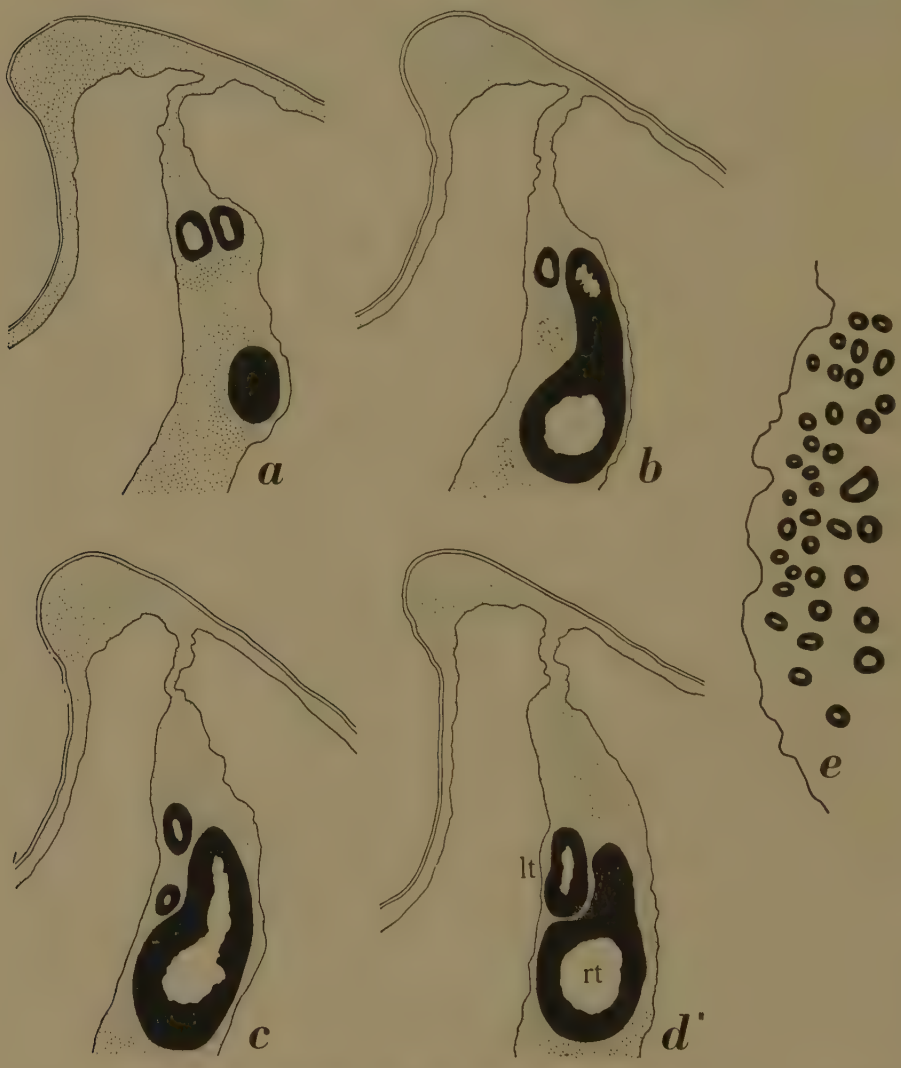

Fig. 5. *Loxothylacus carinatus*, specimen on *Thalamita admeta*. *a* — *d*, posterior parts of longitudinal sections, each following section from a more dorsal region than the preceding; *e*, longitudinal section of one of the colleteric glands. *lt*, left testis; *rt*, right testis. *a* — *d*, × 30; *e*, × 127.

as far in a ventral direction as the right. As in the specimen from unknown host the right testis of the parasite dealt with here apparently was fully functioning, the left being more or less rudimentary. The curvature of the male organs, especially that of the right, is rather wide.

The colleteric glands lie in the anterior half, at a very short distance from the centre of each lateral surface of the visceral mass. Their number of canals is not very large, in a longitudinal section of the most strongly divided region this number amounts to 36 (fig. 5*e*).

The external cuticle of the mantle of the specimen on *Thalamita admeta* is densely covered with excrescences which by their hyaline structure differ from the main layers of the cuticle. On the surface of the rather thin cuticle (thickness about 9 to 15 μ) there occur more or less cylindrical bodies that with broad bases are attached to the upper layers of the cuticle: at their free extremities they bear a number of

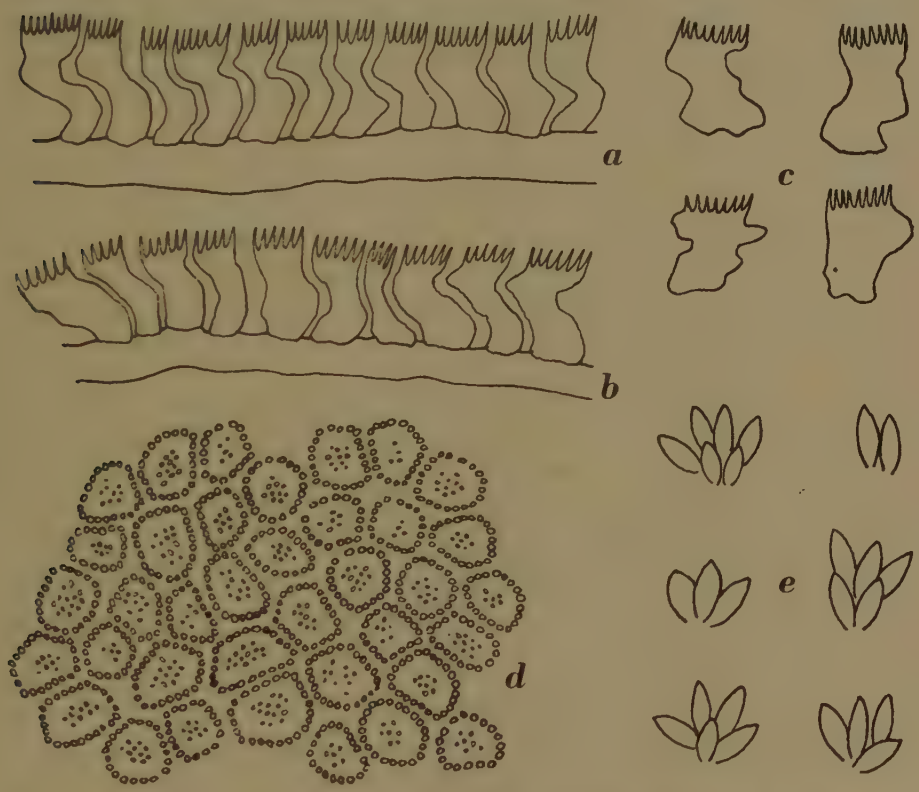

Fig. 6. *Loxothylacus carinatus*, specimen on *Thalamita admeta*. *a*, *b*, sections of the external cuticle; *c*, isolated excrescences; *d*, excrescences of the external cuticle in surface view (basal parts of spines drawn only); *e*, retinacula. × 530.

small spines (fig. 6*a*, *b*). The cylindrical bodies often have a more or less contorted shape: their total height varies from 22 to 33 μ, their transverse diameter from 7 to 23 μ. On their flat upper surfaces they show a few smaller spines within the outer row of stronger spines (fig. 6*d*). No excrescences were found here that penetrated with root-like expansions into the main layers of the cuticle.

The surface of the internal cuticle of the mantle bears numerous

retinacula in more or less regular distribution. No basal parts as they occur in most of the other specimens were to be found here; each retinaculum seems to consist of a group of spindles only (fig. 6*e*). The spindles are arranged in groups of 2 to 6 (as a rule 4 or 5), they have a length of 12 to 16 μ and do not show barbs.

In a previous paper (Boschma, 1949) attention was drawn to the similarity of the excrescences of *Sacculina cuspidata* to those of *Loxothylacus carinatus*. In this paper it was noted that the colleteric glands of *S. cuspidata* are found near the anterior extremity of the visceral mass. As in *L. carinatus* these glands as a rule have a more or less central position in the lateral surfaces of the visceral mass, this difference may serve as an additional character for distinguishing the two species. In fig. 7 longitudinal sections of the median part of the body are drawn of six specimens of *Loxothylacus carinatus* (fig. 7*a*—*d*, *g*, *h*) and of two

Fig. 7. Longitudinal sections of the visceral mass (with the colleteric glands, stippled) of various specimens of *Loxothylacus carinatus* (*a* — *d*, *g*, *h*) and *Sacculina cuspidata* (*e*, *f*). *a*, on *Thalamita prymna*; *b*, on *Neptunus hastatoides*; *c*, on *Lissocarcinus polybioides*; *d*, from unknown host; *e*, on *Pseudozius caystrus*; *f*, on *Chlorodopsis* spec.; *g*, on *Thalamita admeta*; *h*, on unidentified Portunid crab. *a* — *d*, × 8; *e*, *g*, × 13; *f*, × 20; *h*, × 7.

specimens of *Sacculina cuspidata* (fig. 7*e*, *f*). In the latter the colleteric glands are found in the vicinity of the anterior extremity of the visceral mass (in fig. 7*e* this extremity is pointing to one side), whilst in the specimens of *Loxothylacus carinatus* these glands occupy the central region of the visceral mass (fig. 7*a*, *c*, *h*) or have a slightly more anterior position (fig. 7*b*, *d*, *g*).

It is by no means certain that the parasites identified here as *Loxothylacus carinatus*, viz., the three specimens dealt with in a previous paper (Boschma, 1949) and the three specimens of which the characters are described in the present paper, are conspecific. The same uncertainty holds for the specimen on *Caphyra levis* mentioned in the cited paper; of the latter specimen the structure of the excrescences of the cuticle is known, but that of the male organs and of the colleteric glands is unknown.

In three specimens of *Loxothylacus carinatus* (on *Thalamita prymna*, on *Lissocarcinus polybioides*, and on the unidentified Portunid) the two male organs are of equal size; in the three other specimens (on *Neptunus hastatoides*, on unknown host, and on *Thalamita admeta*) the right testis is much larger than the left. This difference, however, not necessarily points to a specific distinction, as in other species of *Loxothylacus* a corresponding difference may occur as individual variation (e.g., in *L. variabilis*, cf. BOSCHMA, 1940).

The colleteric glands in the six specimens show a large degree of similarity. The number of canals is somewhat different, but these differences are not pointing to specific distinction.

The chief characters of the external and the internal cuticle of the mantle of the seven specimens may be summarized as follows:

Specimen on *Caphyra levis* (VAN KAMPEN & BOSCHMA, 1925, fig. 24): excrescences of the external cuticle with well developed roots; retinacula with basal part and 2 to 4 barbed spindles.

Specimen on *Thalamita prymna* (VAN KAMPEN & BOSCHMA, 1925, fig. 25): excrescences of the external cuticle with well developed roots; retinacula with basal part and 2 to 4 barbed spindles.

Specimen on *Neptunus hastatoides* (BOSCHMA, 1931, fig. 51*b*, *c*): excrescences of the external cuticle with small though distinct roots; retinacula with basal part and 4 to 6 barbed spindles.

Specimen on *Lissocarcinus polybioides* (BOSCHMA, 1931, fig. 51*a*): excrescences of the external cuticle without distinct roots; retinacula unknown.

Specimen from unknown host (fig. 2 in the present paper): excrescences of the external cuticle without roots; retinacula without basal part, consisting of 7 to 10 spindles without barbs.

Specimen on unidentified Portunid crab (fig. 4 in the present paper): excrescences of the external cuticle as a rule with well developed roots; retinacula with basal part and 5 to 7 barbed spindles.

Specimen on *Thalamita admeta* (fig. 6 in the present paper): excrescences of the external cuticle without roots: retinacula without basal part, consisting of 2 to 6 spindles without barbs.

Especially on account of the peculiar shape of the retinacula (without basal part and without barbs) the specimens on unknown host and on *Thalamita admeta* might be regarded as a distinct species, different from *Loxothylacus carinatus*. It is interesting that in these two specimens the right testis is much larger than the left. But in the third specimen showing this enormous difference in the size of the male organs (the specimen on *Neptunus hastatoides*) the retinacula have basal parts and possess barbs.

Though there are indications for specific distinctions the material up till now available is too scanty to justify a definite decision. For the present all the specimens dealt with above may be regarded as belonging

to the one species *Loxothylacus carinatus*. The details of the individual specimens described here may serve as a means to reach a better solution of the difficulties when more material becomes available.

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INFLUENCE OF LIGHT AND SUGAR ON GROWTH AND SALT INTAKE BY MAIZE

BY

O. M. VAN ANDEL, W. H. ARISZ AND R. J. HELDER

(Communicated at the meeting of January 28, 1950)

Introduction.

By various investigators it has been demonstrated that light may affect the absorption of ions.

SCHMIDT pointed out that this effect may be different for different ions. Thus he found with Sanchezia nobilis that for Ca, PO_4 and NO_3 the relation between the intensity of light and the absorption was represented by a minimum curve, for K and Mg by a saturation curve. With the same intake of water there was a much greater intake, especially of K, in light than in darkness. For this reason SCHMIDT presumed that K might play a part in photo-synthesis.

LUTTKUS and BÖTTICHER found that with maize in the dark there was exosmosis of phosphate and particularly of K. The K given off proved to come from the shoot. They came to the conclusion that potassium must be present in a labile binding with the protoplasm, which binding is broken up in the dark. Whether the binding was directly caused by the energy of light or that it was merely a question of the general state of nutrition they left undecided.

ALBERDA ascertained that with maize the influence of light on the intake of phosphate was dependent on the condition of the plant; he also, as did HOAGLAND and BROYER, made a distinction between "high-salt" plants, which during cultivation had been given the opportunity of taking in many salts, and the roots of which had a high salt and a low sugar percentage, and "low-salt" plants which were cultivated in a solution poor in salts, so that the roots contained little salt and much sugar. With "high-salt" plants, when placed in darkness, ALBERDA found a drop in the intake of phosphate and of the growth of the shoot; in most cases phosphate was even given off. "Low-salt" plants, however, showed an uptake for a considerable time even in darkness.

That the "high-salt" plants showed no growth in darkness cannot have been the result of decrease in the absorption of salt, for on tap-water there was growth in the light. So it could be assumed that the growth, i.e. the consumption of salts, determined the absorption and not vice versa.

As the only clear distinction between high- and low-salt plants is to be found in the percentage of carbo-hydrates it seemed possible that the

strong check on growth in the dark with high-salt plants was caused by lack of sugar.

WENT and BONNER succeeded in growing tomato-plants in the dark by administering sugar to the leaves. ALBERDA tried the same treatment with maize but obtained negative results, so that this point remains unsolved. He left open the possibility that some other difference between high and low salt plants played a part.

ALBERDA's experiments were continued by HELDER. Thereby it appeared that the intake is more sensitive to light and darkness than the growth of the shoot. With great intensity of light the intake considerably increases during the first few days. The growth of the shoot reaches a constant level already at an earlier period (apart from the periodicity also shown under continuous exposure to light and corresponding with the day and night rhythm). In the dark the intake immediately decreases: the growth of the shoot does not show a clear decrease before the second period (fig. 1).

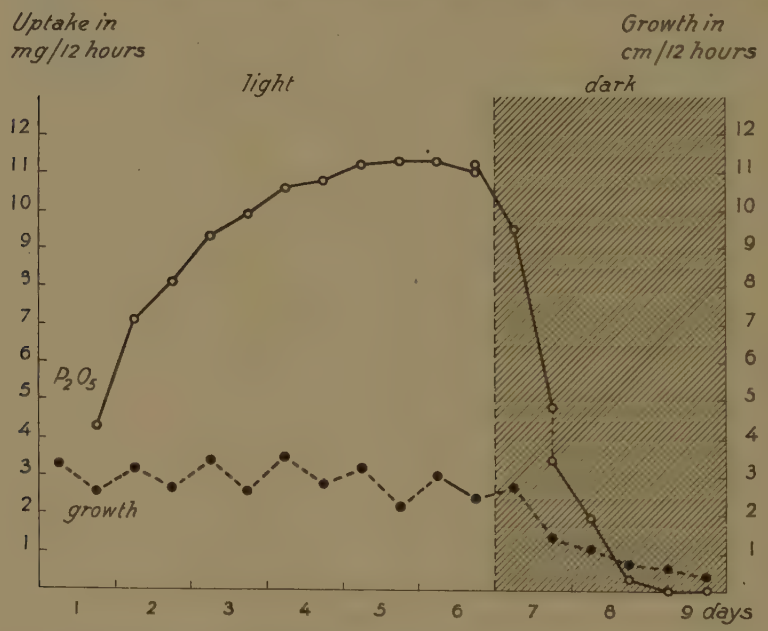

Fig. 1. Average phosphate uptake and growth of the shoot of high salt plants in light and darkness at 23° C. The experiment was started with 4 sets of plants, each consisting of 12 plants placed on 1 liter of a HOAGLAND solution, containing 16 mg P_2O_5. After 6, 7 and 8 days, a set was taken away each time for further analysis.

This makes it less probable that the intake is exclusively or chiefly determined by the growth of the shoot; the correlation between uptake and growth of the shoot may also be explained from a common factor, viz. the supply of sugar. In this connection it is of importance that PHILLIS and MASON (1940) found that the intake of Br by cotton plants within two hours of their being ringed is considerably lower than with plants

not ringed. From this they inferred that the intake is greatly dependent upon the supply of material from the shoot. The behaviour of low-salt plants is in agreement with this view. When placed on a nutrient solution of high concentration there is ample intake in light. At the same time, however, they will consume a good deal of their sugar; in darkness, therefore, the intake rapidly decreases (fig. 2, A). When they are put on a

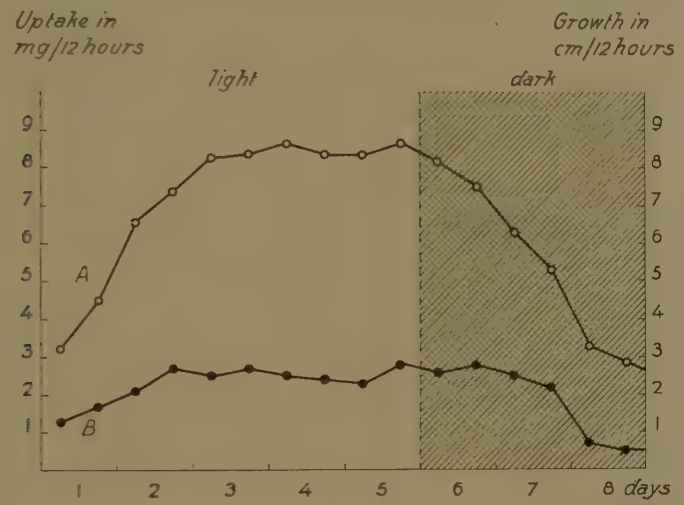

Fig. 2. Uptake of phosphate by low salt plants in light and darkness at 20° C. Average uptake by two sets of 12 plants. *A*: in 16 mg P_2O_5/l. *B*: in 4 mg P_2O_5/l.

solution of low concentration, however, then the intake and sugar consumption are both less; in the dark the intake may continue at the same rate for some time (fig. 2, B).

A strong indication for the correctness of the hypothesis was found in some orientating experiments whereby the shoot was placed in an atmosphere free of CO_2. With the low-salt as well as the high-salt plants, the intake decreases when, under continuous exposure to light, CO_2 is withdrawn from the atmosphere. The decrease may be just as great as in a dark period.

In the experiments described below, executed by Miss VAN ANDEL, the question as to in how far the influence of light on the intake and growth of the shoot is caused by the sugar formed during the photosynthesis is more closely examined. For this purpose the growth of the shoot and the intake of phosphate as well as potassium were determined, for high salt plants which at the same time are low sugar, on account of which deficiency of sugar may soon occur in the dark. Glucose was administered to the roots (cf. BURSTRÖM), whilst by a circulation apparatus the solution was kept as sterile as possible. The influence of sugar and light could, in this way, be separately examined.

Material.

Grains of maize were soaked in water for two days and then placed in the dark

to germinate at a temperature of 25° C. When the coleoptile had reached a length of 2—3 cm they were fastened two by two or separately to a disc of wood or bakelite, and placed on a glass vessel of 500 ml capacity. For the first week the vessel was filled with a solution of 0.00125 mol KNO_3, 0.00125 mol $Ca(NO_3)_2$, 0.0005 mol $MgSO_4$ 7 aq in tap-water with 1 ml of a saturated solution of iron tartrate per l. This solution was renewed once. In a week the nutrient solution was replaced by one containing 0.00025 mol KH_2PO_4 instead of iron tartrate. For series of eight plants the solution was renewed daily, for series of four every other day, so that high salt plants should be obtained.

When the plants had been on the nutrient solution from sixteen to twenty-one days they were used for the experiments. Two or three days previously the remainder of the seed was carefully removed by means of tweezers.

Method.

As a nutrient solution containing sugar had to be used for the experiments, a somewhat simplified form of an apparatus designed by WOODFORD and GREGORY for respiration-experiments was used in order to counteract the development of micro-organisms (fig. 3). The apparatus functioned as follows: When the root chamber *A*, to which the disc with the plants could be attached by a rubber ring and which was aerated through a sintered glass disc overflowed, the liquid flowed into the vessel *B* which was heated by an electric apparatus. From there the solution dripped through a condenser into the reservoir *C*. There was a difference from the apparatus of WOODFORD and GREGORY in that air was blown into *D* through a narrow tube by which the liquid was carried along upwards. From *E* the nutrient solution returned to the root chamber which thereby again overflowed, and so on. In about an hour the whole of the liquid circulated through the culture vessel. The rate of circulation can be regulated by tap *X*. At the end of the experiment the root chamber was emptied by means of a siphon and the liquid (plus minus 300 ml) was drawn off from the lower part of the apparatus through the tap *Z*. The whole apparatus was then rinsed a few times with distilled water until the original quantity of liquid was brought up to 1 l. When a solution containing sugar had been or was to be used, the apparatus was rinsed with alcohol. The experiments were carried out in a room of constant temperature, first at 20° C, later at 25° C.

For exposure to light, a lamp of 200 Watt in a reflector was used at a distance of 60 cm from the upper side of the culture vessel. Between the lamp and the plants was a Petri dish with water in order to protect the plants against the heat of the lamp. The nutrient solution was renewed every twelve hours. First HOAGLAND's solution was used 0.0025 mol KNO_3 0.0025 mol. $Ca(NO_3)_2$ 0.001 mol $MgSO_4$ 7 aq 0.0005 mol KH_2PO_4 dissolved in glass distilled water, p_H 5.6. With this nutrient solution the plants had at their disposal $\pm$ 10.8 mg P_2O_5 and $\pm$ 35.4 mg K. As sometimes however nearly all the phosphate or K was used up the concentration of the solution was increased in later experiments to 6/5 of the original value (12.9 mg P_2O_5 and 42.6 mg K). In certain cases it appeared that during boiling a precipitate might be formed. The solution, as such, did not show any precipitate during boiling. In the solution 10.93 mg P_2O_5 was present, 10.87 mg was re-found. Nor could any precipitate be shown when by means of a base the p_H was brought up to 7.0. With p_H 7.4 however, a distinct precipitate was evident. Of the 10.98 mg only 8.0 mg P_2O_5 was re-found. When a higher concentration was used, the possibility of a precipitate was even greater. As during the experiments, the p_H proved to undergo a considerable change and could rise to 6.8 or even higher, the solution was later modified as follows: 0.0025 mol KNO_3 0.0015 mol $Ca(NO_3)_2$ 0.001 mol $MgSO_4$ 7 aq 0.0005 mol KH_2PO_4 0.0005 mol $(NH_4)_2SO_4$ dissolved in glass distilled water (p_H 5.8). In this case the p_H moved to the acid side during the experiment.

The solution used by WOODFORD and GREGORY was also tried. It contained 0.001108 mol KNO_3 0.000408 mol $Ca(NO_3)_2$ 0.00033 mol $MgSO_4$ 7 aq 0.000602 mol KH_2PO_4 in glass distilled water (p_H 5.6).

The solution they used in full, half, quarter, and eighth concentration. They

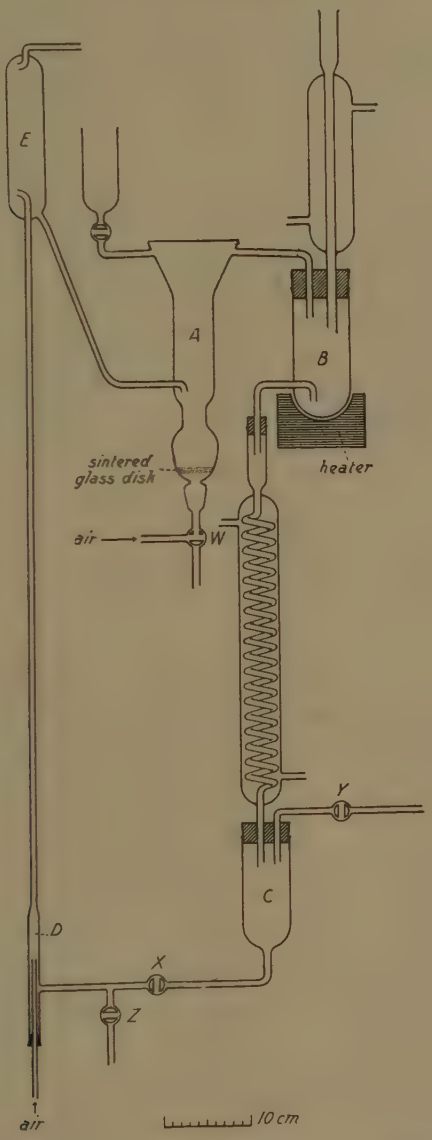

Fig. 3. Circulation apparatus according to WOODFORD and GREGORY (slightly altered).

made no mention whatever of the occurrence of a precipitate. It appeared that from the strongest concentration phosphate precipitated already with a p_H 6.6. Through this 0.7 mg P_2O_5 from the 12.6 mg P_2O_5 present was lost. So no more than HOAGLAND's was this solution useful for our experiments. For a nutrient containing sugar 2 % pure glucose was added to the solution. The liquid, before being used, was sterilised for twenty minutes at 100° C.

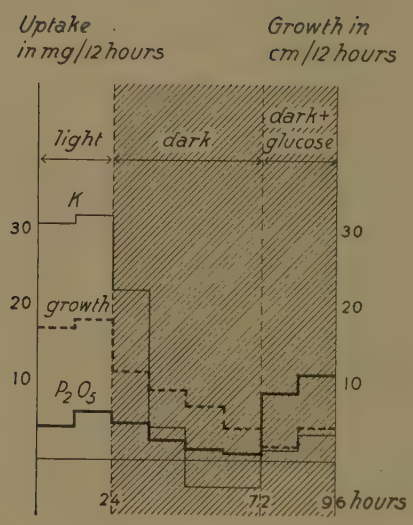

Fig. 4

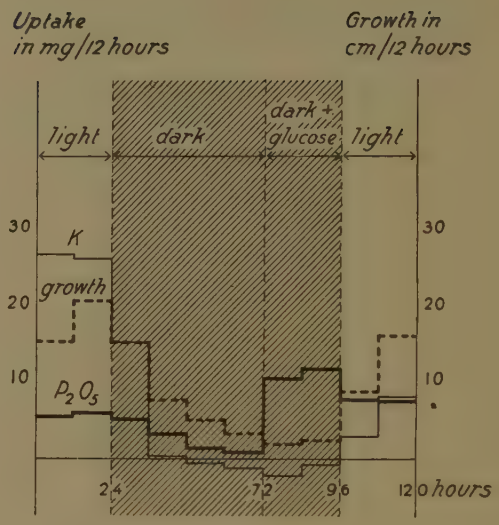

Fig. 5

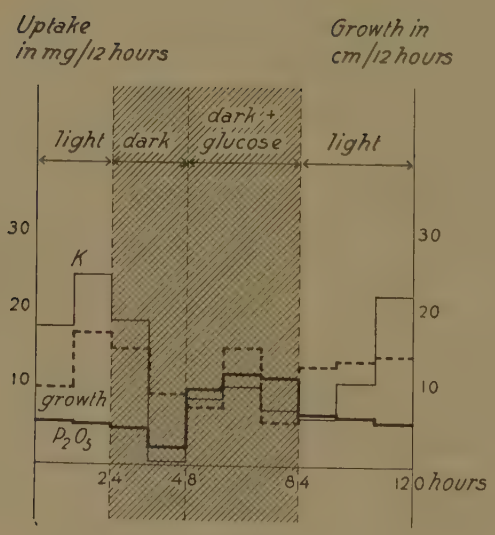

Fig. 6

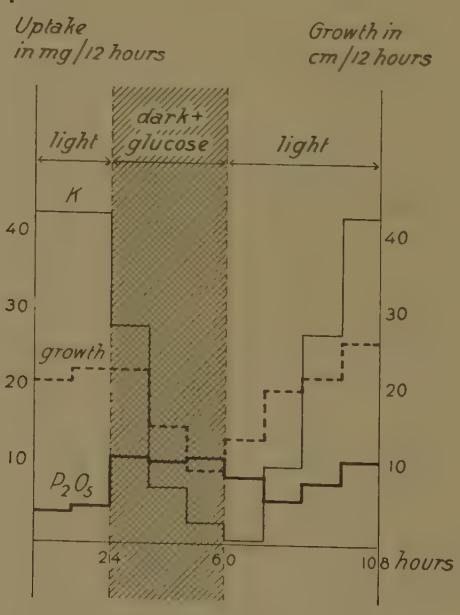

Fig. 7

Fig. 4—9. The influence of successive periods of light and darkness and the addition of glucose to the nutrient solution on the uptake of phosphate (—) and potassium (—) and on the growth of the shoot (— — —)

The phosphate percentage of the solution was determined according to the colorimetrical method of PARKER and FUDGE with a Cencophotelometer.

The K percentage was determined with a Perkin Elmer flamephotometer according to the method of the internal standard.

The growth of the shoot was measured in the manner described by ALBERDA. The length of the youngest leaf visible at the beginning of the experiment was measured every twelve hours; when the length had exceeded 40 cm a new leaf was measured. The figures obtained in this way have only a relative value.

Experiments.

I. Successive light — dark — dark and glucose (cf. fig. 4).

First the plants were exposed to light in the apparatus for 2 × twelve hours; they were then placed in darkness for 4 × twelve hours, after the lapse of which glucose was added to the nutrient solution for 2 × twelve hours in the dark. The intake of phosphate gradually decreased in darkness without dropping to nil. When glucose was added the intake rose to nearly twice the value which was found in light.

The intake of K greatly diminished in the dark; in the second half of the dark period, there was exosmosis of K. When sugar was added there was again a slight intake. The growth decreased in darkness without the coming to a complete standstill. In some cases, when sugar was added, decrease continued at first, after which the growth again increased a little.

The experiment was repeated three times at 20° as well as at 25°, with the same results.

I*a*. light — dark — dark and glucose — light (cf. fig. 5).

There was here, at the end, exposure to light for 2 × twelve hours.

The first part progressed entirely as in I. In the second period of light, in which no sugar was supplied, the intake of phosphate decreased, whereas the intake of K increased as well as the growth, which ultimately approached the value of the first light period. The same test was made with a shorter dark period (2 × twelve hours) after which glucose was added for 3 × twelve hours in the dark, and finally there was 3 × twelve hours exposure to light without glucose; but now also the K intake during the last light period rose to the same level as at the beginning (cf. fig. 6).

II. Successive light — dark and glucose — light (cf. fig. 7).

After the first period of light (2 × twelve hours) glucose was immediately added to the solution in the dark, and subsequently there was exposure to light without glucose for 4 × twelve hours. The intake of phosphate immediately increased again in the dark as soon as sugar was present in the nutrient solution; it also decreased at first in the second light period through lack of sugar and then once more gradually increased. The intake of K greatly diminished in darkness, it is true, but had not reached nil in 36 hours. Under the subsequent exposure to light, the decrease at first continued, then in the light, the intake gradually rose to the original value.

In darkness the growth remained for the first twelve hours the same as in light, then diminished, and, when exposed to light, immediately began to increase.

III. Successive light without glucose — light and glucose (cf. fig. 8). Under continuous exposure to light, glucose was alternately added to

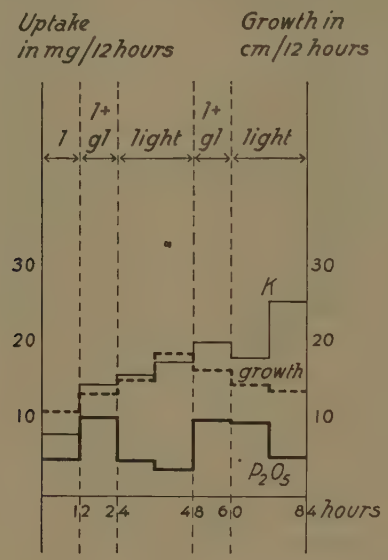

Fig. 8

and withheld from the nutrient solution. Also in this case the intake of phosphate immediately increased when glucose was present and decreased sometimes sharply, sometimes gradually, when this was absent. Neither the intake of K nor the growth showed a similar course, at first both gradually increased; but after the lapse of 48 hours the growth began somewhat to decrease whereas the intake of K was still increasing.

III*a*. Successive light without glucose — light and glucose — light without glucose (cf. fig. 9). In some other tests glucose was added to the nutrient solution for 3 × twelve hours after a light period of 4 × twelve hours, after which there was again exposure to light for twelve hours. The whole test took place under exposure to light. In the last period of light and glucose the nutrient solution was renewed every six hours. During the first light period growth and intake had not yet reached a constant level, on account of which the effect of the sugar was less clear in this case. The intake of phosphate, however, showed, when glucose was added, a conspicuously greater increase than before and decreased immediately when placed on solution without sugar, so that, also with experiments in light, the influence of the addition of glucose is unmistakable. The intake of K continued to increase during the whole experiment;

the intake was high in the period during which sugar was added, but also somewhat irregular.

After the addition of glucose the growth had a tendency to diminish

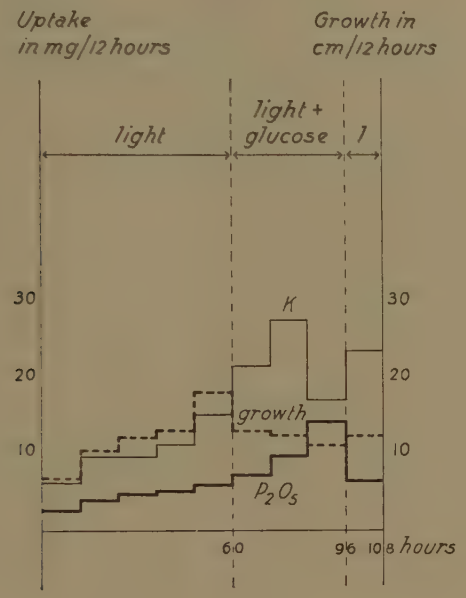

Fig. 9

and then remained fairly constantly on this level. In the subsequent period of light the growth increased in the one case and remained constant in the other.

Discussion.

When maize plants are placed in the dark, the intake of phosphate decreases (cf. ALBERDA 1949). This decrease is mostly a gradual one; exosmosis of phosphate was not found in these experiments. It appears that the intake of phosphate is strongly enhanced when glucose can be taken up by the roots. This is the case when the plants have first been in the dark for some time as well as when there has been a previous period of light. Also with plants continuously exposed to light, glucose enhances the uptake. When the plant after a dark period with glucose is once more placed in the light, but without glucose, the uptake of phosphate diminishes. From this it follows that the sugar content of the root is, under normal circumstances, too low to obtain a maximal intake of phosphate. When in darkness sugar is added only a slight correlation is found between the growth of the shoot and the absorption of phosphate. After a previous intake in light, the intake of phosphate sometimes rises to even double the amount, whilst the growth of the shoot is much smaller than in light without glucose.

From these data it becomes understandable that the intake of phos-

phate in darkness after a period of light cannot but diminish rapidly since the supply of sugar diminishes and then becomes a limiting factor. In how far the decrease of the growth of the shoot in fig. 1 can also be the result of the smaller supply of sugar in darkness cannot be concluded, since we do not know whether the sugar administered to the roots goes to the shoot. In this connection it must be pointed out that SPOEHR did not succeed in cultivating albino-maize on a nutrient solution containing sucrose, but that he did succeed in doing so when the sugar was administered to the leaves. This points to the fact that a considerable part of the glucose taken up from the nutrient solution remains in the roots and is not conveyed to the shoot. So it is conceivable that the growth of the shoot is checked in the dark by the lack of carbo-hydrates, but this decrease might also be due to a deficiency of other substances. As a matter of fact it cannot be said with certainty if the correlation between the uptake and the growth of the shoot in darkness may be explained by the common factor sugar. It remains possible as well that the growth of the shoot affects the uptake.

As the growth of the root was not measured we do not know if there is a correlation between the uptake and this growth. It is, however, probable. For the phosphate taken up may, as far as is known, be disposed of in three ways. In the first place it may be fixed in the cortical cells of the root where it is taken up. Secondly, it may be conveyed to the growing parts of the root which work as a sink (WALKLEY, WILLIAMS). Thirdly it is given off to the vessels in which it is conveyed to the shoot. With high-salt plants the second process, especially, may be considerably intensified by the addition of glucose. The influence of addition of sugar on the third process seems to be rather weak (cf. WENT e.a.).

The growth of the shoot as well as the uptake of potassium prove to re-act only slightly upon the addition of glucose. The uptake of potassium, as well as the growth of the shoot are to a great extent dependent upon the exposure to light. It is conspicuous that often growth and uptake of potassium are correlated. This points to a fixation of K by growth and a regulation of the uptake by consumption.

In what manner light affects the uptake of potassium cannot with certainty be concluded. LUTTKUS and BÖTTICHER imagined that in darkness, the binding of potassium to the protoplasm was broken up and the free K ions are removed. It is noticeable that on the change from darkness to light, the uptake of potassium does not reach its full strength immediately. Mostly this does not happen before the third or fourth period of twelve hours. Also the drop in the uptake of potassium on changing from light to darkness takes place more gradually. This points to the fact that the influence of light on the potassium content of the shoot is more of an indirect nature. The correlation between the exosmosis of potassium and the decrease of the growth indicates that the light not directly influences the binding of K to the protoplasm and that in darkness

the fixation of K in the shoot decreases when the processes of synthesis, which fix the potassium in the growing parts, come to a stop. Then there arises in the shoot a surplus of potassium which, via the root, is conveyed to the external solution. The potassium then behaves in principle in the same way as phosphate, which, under certain circumstances, is also given off to the medium by the roots. The quantity, however, is considerably greater for potassium. This may be due to the fact that the binding of the potassium to the protoplasm in light (LUTTKUS and BÖTTICHER) is broken up in darkness, but it is also possible to explain it from the greater mobility of the potassium in the plant, whilst the phosphate is soon organically bound. It is a known fact the potassium is given off again from older cells and is used by younger parts during their growth. If, after the processes of synthesis have come to a stop on account of the plant being placed in the dark, this exosmosis of potassium continues, the surplus of potassium in the shoot will increase and will lead to a removal of potassium to the roots. It is, however, probable, that this removal is not, as is supposed by LUTTKUS and BÖTTICHER, a diffusion process, but, as was pointed out by COLLANDER, MASON and ALBERDA, takes place along the phloem vessels to the root. As a result the concentration in the root increases and the uptake from the medium diminishes or changes into exosmosis.

In fig. 10 a sketch is given of the influence of light and sugar on the uptake and consumption of K and phosphate. The salt taken up is used in the root for accumulation and growth, whilst, in addition, salt is given

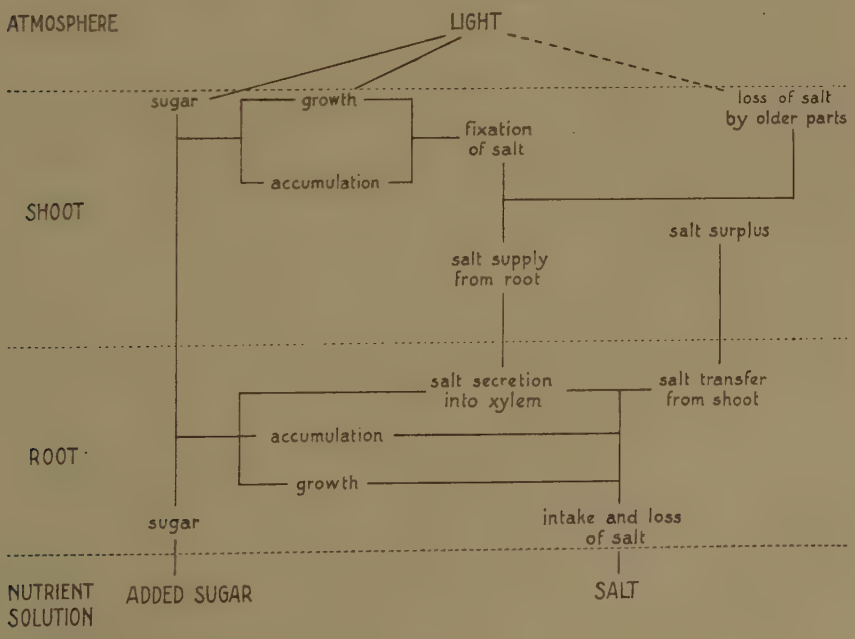

Fig. 10

off to the xylem and carried to the shoot. In the shoot the salt is withdrawn from the sap in the xylemvessels and used for growth and accumulation whilst a surplus is conveyed back to the root. We have drawn a dotted line between light and loss of salt by older parts. This indicates that it is uncertain whether the exosmosis of potassium by older cells is influenced by the light. In the left part of the figure it is indicated that these processes are all more or less dependent upon sugar, which is either formed in the light during the photo-synthesis or which is taken up from the nutrient solution.

Summary.

With a somewhat modified form of the circulation apparatus of WOODFORD and GREGORY it was possible with maize plants to elucidate the influence of sugar which was added to the nutrient solution, on the uptake of potassium and phosphate and on the growth of the shoot. The development of micro-organisms in the nutrient solution in twelve hours is very slight. In order to prevent the phosphate from precipitating through boiling, it is necessary to take care that the p_H does not rise higher than 7.

With high salt plants the uptake of phosphate is increased in light as well as in darkness by the addition of glucose. So the fact that the uptake of phosphate after a period of light decreases in darkness may be explained by the reduced supply of sugar to the roots (PHILLIS and MASON 1940). It may be that along with this, the diminished growth of the shoot exercises influence (ALBERDA 1948). The sugar added may further the uptake of phosphate in different ways; The fixation of P in synthetic processes e.g. in root growth seems to be an important process.

The uptake of the K and the growth of the shoot are both dependent upon the exposure to light, but are not furthered to any considerable extent by the addition of sugar to the nutrient solution. As we do not know with certainty that the sugar added goes also to the shoot, this does not prove that sugar has no influence on the growth of the shoot.

Exosmosis of phosphate in darkness was not found; on the other hand there was a very distinct exosmosis of K, especially when the dark period lasted for some days, and the growth had greatly decreased. When in darkness the growth continues, some K may be taken up. The exosmosis of the K can be explained by assuming that the transport from the shoot is greater than the supply to it. By the decrease of the growth, the consumption of K becomes small, while the older parts continue to give off K. In light the growth is promoted; the fixation of K can now become greater again than the quantity liberated from the older parts; there results an uptake from the medium. So the exosmosis of K need not be due to a sensitiveness to light of the binding of K to the plasma.

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ON EASTON's PERIOD OF 89 YEARS

BY

S. W. VISSER

(Communicated by Prof. F. A. VENING MEINESZ at the meeting of Jan. 28, 1950)

In 1928 C. EASTON published his monumental collection of West European winter data. [1])

This publication was proceeded, already in 1917, by considerations concerning a period of 89 years, deduced from 13 temperature waves of this duration between A.D. 760 and 1916, in these Proceedings. [2]) I arrived at this same period by quite another way and have tested my views by means of EASTON's figures. I give my results in this paper.

Remarkable meteorological extremes, especially of the winter temperature, such as we are experiencing nowadays, appear to have occurred in historical times repeatedly. Instrumental observations during $2\frac{1}{2}$ centuries [3]) reveal the occurrence of time intervals during which extremes accumulate, alternating with intervals during which excessive values are rare.

The investigation showed that, quite in the same manner as nowadays, such accumulations of extreme winters occur when the 11-years sun spot period obtained exceptionally high maxima.

CLAYTON [4]) has remarked that the sunspot activity is characterized by a sequence of 4 strong 11-years periods followed by three faint ones and he supposed therefore a long period of seven 11-years periods or 77,7 years. This regularity, however, is nowadays no longer present and my investigations led me to surmising that we have to do with a long period of eight 11-yearlies, exactly EASTON's period. EASTON himself had seen this connection between his period and the 11-years solar period. He had divided his period in 4 parts, each of $22\frac{1}{4}$ years, suiting two sunspot periods, each part with its own characteristic temperature averages.

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[3]) A. LABRIJN. Het klimaat van Nederland gedurende de laatste twee en een halve eeuw. Med. en Verh. Kon. Ned. Met. Inst. De Bilt **49**, (1945).

[4]) H. H. CLAYTON. Solar cycles. SMITHS. Misc. Coll. Washington, **106**, nr. 22 (1947).

In order to investigate the accumulations of extremes I have characterized them by means of the standard deviation [5]).

Intervals with large standard deviations must alternate with intervals with small ones. The results arrived at were hopeful, but the interval with instrumental observations was too short. To test the surmized period it is necessary to investigate EASTON's data. His character figures have by no means the same evidence as those of thermometer observations. Yet, EASTON having spent much care on his character figures, it seemed to be the wrong way to neglect his work because they do not suit modern claims. Moreover, is it not the task of the statistician to draw the best conclusions from insufficient data?

My research started with the year A.D. 1205, the first year of EASTON's uninterrupted series, containing up to 1917 eight periods of 89 years.

Yearly overlapping 11-years average values of EASTON's character figures were calculated. Each average was inscribed at the central year (the sixth) of the group of 11. The deviations of the observed values and these averages were calculated and from these deviations, again for overlapping 11-years groups standard deviations were determined. These standard deviations are not wholly identical with those of theoretical statistics, because the averages are changing continually, but confusion is impossible.

The results are represented in the figure by 8 curves, each of 89 years. The starting year of each period is indicated at the left hand side of each curve. The beginning of the 9th period (1917—1949), calculated by LABRIJN [6]) for De Bilt, has been added. Moreover the average range of the 89-years period for 1205—1916 has been represented in the figure (curve m). The average value of the standard deviation for the whole interval (15,5) has been drawn as a straight line in each curve.

The average curve (m) shows a fairly regular shape. It may divided without objection in 4 equal parts, each of two 11-years periods.

The first and the third parts are characterized by high standard deviations, the second and the fourth parts by low ones. The limits between high and low values coincide fairly well with the limits between the four parts; only at the end of the first part a small irregularity is present. The amplitude of the first half of the whole period is distinctly larger than that of the second half.

Though these features in each of the eight curves are not as clearly developed as in the average curve, yet they show obviously common traits, especially since A.D. 1561.

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[5]) S. W. VISSER. Les taches solaires et la météorologie, III Les hivers extrêmes. Ann. de Géoph. Paris, **5**, 74—82 (1949).

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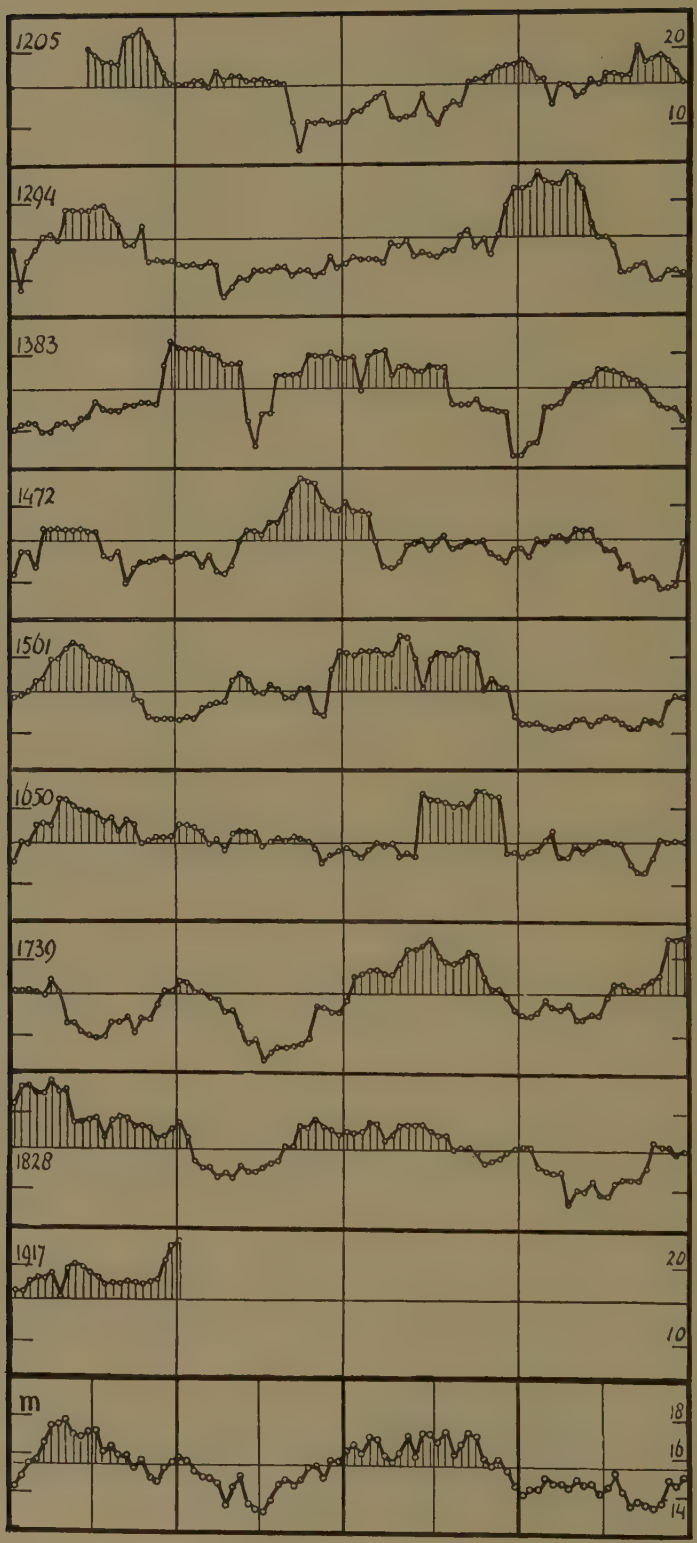
[6]) A. LABRIJN. Ts K.N.A.G. **64**, 307 (1947). The years 1948 and 1949 have been added with the aid of data, received from the Roy. Neth. Met. Inst. at De Bilt.


1205
20
10
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Probably the tops may advance or retard in the same manner as f.i. the meteorological seasons may deviate from their average epochs.

The strongest argument for the physical reality of the period of 89 years is its concordancy with the 11-years sunspot period.

The first year of the long period was since the beginning of the 18th century a sunspot maximum year: 1739, 1829, 1917. We may surely accept that the same has been the case before and so we can fix the sunspot maxima since 1205.

For the years with instrumental observations a high correlation has been revealed between the standard deviations of the winter temperature and the deviations of the long solar period (for the years 1831—1937 I found for Prague a correlation coefficient + 0,74, for De Bilt + 0,82).

Now finding periodically shaped standard deviations of winter temperatures since A.D. 1205, pointing to periodical accumulations of extreme winters, it is allowed to ascribe these accumulations to a solar-activity period with a duration of 89 years or eight 11-years periods.

Concerning the occurrence of extreme winters, the 11-years maxima behave differently according to their situation in the 89-years period; they coincide resp. with normal, high, normal and low standard deviations (see the *m*-curve in which the 8 maxima are indicated by vertical lines). The same applies for the sunspot minima; here, however, the succession of standard deviations is high, high, low, low. This conclusion is important; two succeeding 11-years sunspot minima may occur during high intensity of the long solar period. The consequence is extreme winters coinciding with sunspot minima and this means a solar influence on atmospheric phenomena independent of the common 11-years period. Recently BURGAUD [7]) has arrived at a similar conclusion when investigating solar influences on disturbances of terrestrial magnetism.

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[7]) M. BURGAUD S. J. Eruptions chromosphériques, crochets et perturbations magnétiques à début brusque. Ann. de Géoph. Paris, 3, 264—281 (1947).

SOME METRICAL THEOREMS OF DIOPHANTINE APPROXIMATION. IV

BY

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(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of Nov. 26, 1949)

1. *Introduction.*

This paper is in some ways a continuation of III and uses the same notation as far as possible. It is, however, completely self-contained.

Throughout this paper let

$$f_1(\theta),\ f_2(\theta),\ \ldots,\ f_n(\theta),\ \ldots$$

be a sequence of functions with positive monotonely non-decreasing continuous derivatives all defined in the range [1])

$$0 \leqslant \theta \leqslant 1.$$

We shall denote by $\{x\}$ the fractional part of x. For $0 \leqslant \alpha \leqslant \beta \leqslant 1$ we define $F_N(\alpha,\ \beta : \theta)$ to be the number of $n \leqslant N$ such that

$$\alpha \leqslant \{f_n(\theta)\} < \beta. \tag{1}$$

Further we put

$$R_N(\alpha,\ \beta : \theta) = F_N(\alpha,\ \beta : \theta) - N(\beta - \alpha)$$

so that $R_N(\alpha, \beta : \theta)$ is the excess of the number of solutions of (1) over the number to be expected at random.

Finally we put

$$\mathfrak{R}_N(\theta) = \operatorname*{Max}_{\alpha,\beta} |R_N(\alpha,\ \beta : \theta)|$$

so that $\mathfrak{R}_N(\theta) = ND(N)$ in KOKSMA's notation.

In III of this series I proved, and ERDÖS-KOKSMA [1] proved it independently and practically simultaneously by a different method, that if $f'_m(\theta) - f'_n(\theta)$ is monotonic for all m, n and if

$$|f'_m(\theta) - f'_n(\theta)| \geqslant K > 0 \tag{2}$$

for some K and all m, $n \neq m$, θ then for almost all θ

$$\mathfrak{R}_N(\theta) = O(N^{1/4} \log^{(5/4)+\varepsilon} N). \tag{3}$$

[1]) The extension of the results proved to a more general interval $a \leqslant \theta \leqslant b$ instead of $0 \leqslant \theta \leqslant 1$ is trivial.

In this paper I show that the estimate (3) can be improved if the derivatives $f_n'(\theta)$ increase fast enough.

We denote by $A_0, A_1, A_2, \ldots$ positive absolute constants and by A an absolute positive constant, not necessarily the same in all contexts.

Theorem. *Suppose there is a positive function $\varphi(n)$ of the positive integer variable n such that*

$$\text{(i)} \quad f_n'(\theta) \geqslant e^{\varphi(n)} f_{n-1}'(\theta) \ , \ f_1'(\theta) \geqslant 1$$

for all θ and $n > 1$.

$$\text{(ii)} \quad \log n \log\log n \geqslant \varphi(n) \geqslant c > 0$$

when n is large enough, where c is some constant independent of n.

$$\text{(iii)} \quad \varphi(n) \text{ and } \log n \log\log n\, \varphi^{-1}(n)$$

are monotonic non-decreasing when n is large enough.

Then for almost all θ there is an $N_0 = N_0(\theta)$ such that

$$\text{(4)} \qquad \mathfrak{R}_N(\theta) \leqslant A_0 N^{1/2} \log^{1/2} N \log\log N\, \varphi^{-1/2}(N) \qquad (N > N_0)$$

where A_0 is some absolute constant.

We note the two extreme cases.

$\varphi(n) = \log n \log\log n$. Then (4) becomes $\mathfrak{R}_N(\theta) \leqslant A_0 N^{1/2} \log\log^{1/2} N$. This is best possible apart from the constant A_0. Indeed it is an immediate consequence of the statistical "law of the iterated logarithm" when μ_n is any strictly increasing sequence of integers and $f_n(\theta) = 2^{\mu_n}\theta$ that

$$\limsup_N \frac{\mathfrak{R}_N(\theta)}{N^{1/2} \log\log^{1/2} N} \geqslant \limsup_N \frac{|R_N(0, 1/2 : \theta)|}{N^{1/2} \log\log^{1/2} N} = 2^{-1/2}$$

for almost all θ, however quickly μ_n tends to infinity. [cf. KHINTCHINE [3]].

$\varphi(n) = c = \text{constant}$. This is the "lacunary" case and then (4) becomes $\mathfrak{R}_N(\theta) \leqslant A_0 c^{-1/2} N^{1/2} \log^{1/2} N \log\log N$. This estimate is stronger than one obtained by ERDÖS-KOKSMA [2], who, moreover, required a condition on $f_n''(\theta)$.

2. *Notation.*

We use the following symbols:—

$[x]$: the greatest integer not greater than x.

$||x||$: the difference between x and the nearest integer, i.e.

$$||x|| = \operatorname*{Min}_{n=0, \pm 1, \pm 2, \ldots} |n - x|.$$

$\{x\}$: the fractional part of x. Thus $x = [x] + \{x\}$.

Define a function $r(\alpha, \beta : x)$ of the variables α, β, x by

$$r(\alpha, \beta : x) = 1 - \{\beta - \alpha\} \text{ if } \{x - \alpha\} < \{\beta - \alpha\}$$
$$= -\{\beta - \alpha\} \text{ otherwise.}$$

It is easily verified that

$$R_N(\alpha, \beta : \theta) = \sum_{n \leqslant N} r(\alpha, \beta : f_n(\theta)) \tag{5}$$

when $0 \leqslant \alpha \leqslant \beta \leqslant 1$ (i.e. whenever the symbol $R_N(\alpha, \beta : \theta)$ has been defined). We shall use the equation (5) to define $R_N(\alpha, \beta : \theta)$ for other values [1]) of α, β. More generally we define

$${}_{N_1}R_{N_2}(\alpha, \beta : \theta) = \sum_{N_1 < n \leqslant N_2} r(\alpha, \beta : f_n(\theta)).$$

We note also

Lemma 1. *For each integer $n \geqslant 1$ the function $r = r(\alpha, \beta : x)$ satisfies*

$$r^n = U_n r + V_n \gamma$$

where $\gamma = ||\beta - \alpha||$ and U_n, V_n are independent of x. Further

$$|U_n| \leqslant 1, \quad |V_n| \leqslant 1, \quad V_1 = 0.$$

For $r(\alpha, \beta : x)$, when α, β are fixed, takes only the values $1 - \gamma$ and $-\gamma$. The identity then follows with

$$U_n = (1-\gamma)^n - (-\gamma)^n$$
$$V_n = (1-\gamma)\, U_{n-1}$$

and the rest is trivial.

3. *An estimation lemma.*

Our proof depends essentially on the following lemma:—

Lemma 2. *Suppose $g_1, \ldots, g_l$ are any l of the functions $f_1, f_2, \ldots$ (l any integer). Write $r(x)$ for $r(\alpha, \beta : x)$ where $\gamma = ||\beta - \alpha||$ and α, β are any numbers. Then*

$$\left| \int_0^1 r(g_1(\theta))\, r(g_2(\theta)) \ldots r(g_l(\theta))\, d\theta \right|$$
$$\leqslant 2\gamma \left\{ \frac{2l-1}{g_l'(0)} + 2 \int_0^1 \frac{g_1'(\theta) + \ldots + g_{l-1}'(\theta)}{g_l'(\theta)}\, d\theta \right\}.$$

The case $l = 2$ is practically the special case $h_m(x) = h_n(x) = r(x)$, $\varphi_m = \varphi_n = \gamma$ of lemma 2 in paper II. The proof for general l runs similarly.

Write

$$r^*(x) = \int_0^x r(\xi)\, d\xi$$

so that

$$r^*(1) = \int_0^1 r(x)\, dx = 0, \qquad |r^*(x)| \leqslant \gamma.$$

---

[1]) This extension of the meaning of $R_N(\alpha, \beta : \theta)$ does not affect the definition of $\Re_N(\theta)$ as

$$R_N(\alpha, \beta : \theta) = R_N(\{\alpha\}, \{\beta\} : \theta) = -R_N(\{\beta\}, \{\alpha\} : \theta)$$

and at least one of the last two expressions has a meaning in the original sense.

We first show that if $g(\theta)$ is one of the $g_j(\theta)$

$$|\int_a^b r(g(\theta))\,d\theta| \leqslant \frac{2\gamma}{g'(a)} \tag{6}$$

whenever $0 \leqslant a \leqslant b \leqslant 1$. Indeed

$$\int_a^b r\ (g(\theta))\,d\theta = \int_{\theta=a}^{b} \frac{r(g(\gamma))}{g'(\gamma)}\,d(g(\theta)) = \left[\frac{r^*(g(\gamma))}{g'(\gamma)}\right]_{\theta=a}^{b} - \int_{\theta=a}^{b} r(g(\theta))\,d\left(\frac{1}{g'(\gamma)}\right)$$

and hence

$$|\int_a^b r(g(\theta))\,d\theta| \leqslant \frac{\gamma}{g'(a)} + \frac{\gamma}{g'(b)} + \int_{\theta=a}^{b} \gamma'\,|d\left(\frac{1}{g'(\theta)}\right)| \leqslant \frac{2\gamma}{g'(a)}$$

by the monotonicity of $g'(\theta)$.

Now let $\lambda_i^{(j)}$ $(i = 1, 2, \ldots, k_j)$ be the set of values of θ for which *either* $\{g_j(\theta)\} = \alpha$ *or* $\{g_j(\theta)\} = \beta$ $(j = 1, \ldots, l-1)$ i.e. the set of points of discontinuity of $r(g_j(\theta))$. Further let $\mu_1, \ldots, \mu_m$ be the numbers $0, 1, \lambda_i^{(j)}$ $(j = 1, \ldots, l-1)$ arranged in order of magnitude. Then the product

$$r(g_1(\theta))\,r(g_2(\theta)) \ldots r(g_{l-1}(\theta))$$

is constant and numerically less than 1 in each interval $\mu_n < \theta < \mu_{n+1}$. Hence

$$\left.\begin{aligned} |\int_0^1 r(g_1(\theta)) \ldots r(g_l(\theta))\,d\theta| &\leqslant \sum_{n=1}^{m-1} |\int_{\mu_n}^{\mu_{n+1}} r(g_l(\theta))\,d\theta| \leqslant 2\gamma \sum_{n=1}^{m-1} \frac{1}{g_l'(\mu_n)} \text{ (by (6))} \\ &\leqslant 2\gamma \left\{\frac{2l-1}{g_l'(0)} + \sum_{j=1}^{l-1} \sum_{i=3}^{k_j} \frac{1}{g_l'(\lambda_i^{(j)})}\right\}. \end{aligned}\right\} \tag{7}$$

But

$$\int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} g_j'(\theta)\,d\theta = \int_{\theta=\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} dg_j(\theta) = 1 \quad (3 \leqslant i \leqslant k_j)$$

and hence

$$\sum_{i=3}^{k_j} \frac{1}{g_l'(\lambda_i^{(j)})} = \sum_{i=3}^{k_j} \int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} \frac{g_j'(\theta)}{g_l'(\lambda_i^{(j)})}\,d\theta \leqslant \sum_{i=3}^{k_j} \int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} \frac{g_j'(\theta)}{g_l'(\theta)}\,d\theta \leqslant 2\int_0^1 \frac{g_j'(\theta)}{g_l'(\theta)}\,d\theta. \tag{8}$$

The lemma follows on substituting (8) in (7).

We note also the [1])

Corollary. *Suppose* $g_1, \ldots, g_M$ *are any* M *of the functions* $f_1, f_2, \ldots$ *where* $M > l$

Then

$$\left.\begin{aligned} &\sum_{j_1 < j_2 < \ldots < j_l} |\int_0^1 r(g_{j_1})\,r(g_{j_2}) \ldots r(g_{j_l})\,d\theta| \\ &\leqslant 2\gamma \left\{(2l-1) \sum_{j=1}^{M} \frac{1}{g_j'(0)} + 2 \sum_{i<j\leqslant M} j^{l-2} \int_0^1 \frac{g_i'(\theta)}{g_j'(\theta)}\,d\theta\right\}. \end{aligned}\right\} \tag{9}$$

[1]) We suppress the argument θ except when its absence might cause ambiguity.

This follows directly by summation. We note that the right hand side of (9) is an increasing function of l (if $g_1, \ldots, g_M$ remain fixed).

4. *The principal lemmas.* The kernel of the proof lies in the next two lemmas.

Lemma 3. *Let α, β be any two numbers and let $\gamma = ||\beta - \alpha||$. Write $r(x)$ for $r(\alpha, \beta: x)$. Suppose that $g_1, \ldots, g_M$ are any M of the functions f_n and that*

$$M\gamma = M\,||\beta - \alpha|| \geqslant s$$

where s is a positive integer. Put

$$\mathfrak{r}(\theta) = \sum_{j=1}^{M} r(g_j(\theta))$$

and suppose

$$\mathfrak{S}\,(\text{say}) = (4s-1)\sum_{j=1}^{M} \frac{1}{g_j'(0)} + 2\sum_{i<j\leqslant M} j^{2s-2} \int_0^1 \frac{g_i'(\theta)}{g_j'(\theta)}\,d\theta \leqslant M.$$

Then

$$\int_0^1 \mathfrak{r}^{2s}(\theta)\,d\theta \leqslant 2 \cdot \frac{(2s)^{2s+2}}{s!} \cdot (M\gamma)^s.$$

The proof depends on setting up an identity of the type

$$\mathfrak{r}^{2s}(\theta) = D_0 + \sum_{l=1}^{2s} D_l \sum_{j_1<\ldots<j_l} r(g_{j_1})\, r(g_{j_2}) \ldots r(g_{j_l}) \tag{10}$$

by expanding and applying lemma 1, where $D_0, D_1, \ldots, D_{2s}$ are independent of θ and satisfy certain inequalities. The lemma will follow from the corollary to lemma 2, on integration.

In the first place we have an identity of the type

$$\mathfrak{r}^{2s}(\theta) = \sum_{\substack{n \\ a_1,\ldots,a_n}} B(a_1, \ldots, a_n) \sum_{j_1<\ldots<j_n} r^{a_1}(g_{j_1}) \ldots r^{a_n}(g_{j_n})$$

where

(i) the first sum is over all sets of positive integers $n, a_1, \ldots, a_n$ with

$$\sum_{\nu=1}^{n} a_\nu = 2s$$

(ii) the numbers $B(a_1, \ldots, a_n)$ are non-negative integers and

$$\sum_{\substack{n \\ a_1,\ldots,a_n}} B(a_1, \ldots, a_n) \leqslant \underbrace{(1+1\ldots+1)^{2s}}_{2s \text{ summands}} \leqslant (2s)^{2s}. \tag{11}$$

Further, $B(a_1, \ldots, a_n)$ is unchanged by permutation of $a_1, \ldots, a_n$.

Substituting from lemma 1, we obtain [1])

$$(12)\quad r^{2s}(\theta) = \sum_{\substack{l,m \\ b_1,\ldots,b_l,\, c_1,\ldots,c_m}} B(b_1,\ldots,b_l,c_1,\ldots,c_m)\, U_{b_1}\ldots U_{b_l}\, V_{c_1}\ldots V_{c_m} \sum_{\substack{j_1<\ldots<j_l \\ k_1<\ldots<k_m \\ (j_1,\ldots,j_l,k_1,\ldots,k_m)_{\neq}}} \gamma^m r(g_{j_1})\ldots r(g_{j_l}),$$

where $l,\ m,\ b_1,\ldots,b_l,\ c_1,\ldots,c_m$ are any set of numbers such that

$$\sum_{\lambda=1}^{l} b_\lambda + \sum_{\mu=1}^{m} c_\mu = 2s;\quad b_\lambda > 0,\ c_\mu > 0;\ l,\ m \geqslant 0.$$

Since $V_1 = 0$, we may assume that

$$c_\mu \geqslant 2 \quad (\mu = 1,\ldots,m)$$

and hence

$$m \leqslant s;\quad m \leqslant s-1 \quad \text{if } l > 0.$$

We now deduce the identity (10) where $D_0,\ldots,D_{2s}$ are independent of θ (but may depend on M, γ as well as s) and satisfy the inequalities

$$|D_0| \leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s \quad ; \quad |D_l| \leqslant \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1} \quad (l \geqslant 1).$$

Indeed if D_0 is the sum of the terms in (12) independent of the r's, we have

$$\begin{aligned}
|D_0| &= \Big|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1,\ldots,c_m)\, V_{c_1}\ldots V_{c_m} \sum_{k_1<\ldots<k_m\leqslant M} \gamma^m\Big| \\
&\leqslant \Big|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1,\ldots,c_m)\frac{(M\gamma)^m}{m!}\Big| \text{ since } |V_c| \leqslant 1, \\
&\leqslant \Big|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1,\ldots,c_m)\Big|\frac{(M\gamma)^s}{s!} \text{ since } m \leqslant s,\ M\gamma \geqslant s, \\
&\leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s \text{ by (11).}
\end{aligned}$$

Similarly the coefficient of a term $r(g_{j_1})\ldots r(g_{j_l})$ is

$$D_l = \sum_{\substack{m \\ b_1,\ldots,b_l \\ c_1,\ldots,c_m}} B(b_1,\ldots,b_l,c_1,\ldots,c_m)\, U_{b_1}\ldots U_{b_l} V_{c_1}\ldots V_{c_m} \sum_{\substack{k_1<\ldots<k_m\leqslant M \\ (j_1,\ldots,j_l,k_1,\ldots,k_m)_{\neq}}} \gamma^m$$

which clearly depends only on l (and M, γ, s) and not on the individual numbers $j_1,\ldots,j_l$. Further, almost as for D_0,

$$\begin{aligned}
|D_l| &\leqslant \Big|\sum_{\substack{m \\ b_1,\ldots,b_l \\ c_1,\ldots,c_m}} B(b_1,\ldots,b_l,c_1,\ldots,c_m)\frac{(M\gamma)^m}{m!}\Big| \\
&\leqslant \Big|\sum_{\substack{n \\ a_1,\ldots,a_n}} B(a_1,\ldots,a_n)\Big|\frac{(M\gamma)^{s-1}}{(s-1)!} \text{ since } M\gamma \geqslant s,\ m \leqslant s-1 \\
&\leqslant \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}.
\end{aligned}$$

[1]) The symbol $(x, y, \ldots, t)_{\neq}$ means that the numbers $x, y, \ldots, t$ are unequal in pairs.

Now integrate (10) and we obtain

$$\int_0^1 \mathfrak{r}^{2s}(\theta)\,d\theta \leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s + \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}\sum_{l=1}^{2s}\Big|\sum_{j_1<\ldots<j_l}\int_0^1 r(g_{j_1})\ldots r(g_{j_l})\,d\theta\Big|$$

$$\leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s + \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}\cdot 2s\cdot 2\gamma\mathfrak{S}$$

since, by the corollary to lemma 2, each of the terms of the outer summation is not greater than $2\mathfrak{S}$. Since $\mathfrak{S} \leqslant M$ by hypothesis, this proves the lemma.

Lemma 4. *Suppose $N > 100$ and suppose there is a positive integer $s \geqslant 4$ such that*

$$\text{(13)}\qquad \begin{cases} \mathfrak{S}_{a,M} = (4s-1)\sum_{j=1}^{M}\dfrac{1}{f'_{a+j}(0)} + 2\sum_{i<j\leqslant M} j^{2s-2}\int_0^1 \dfrac{f'_{a+i}(\theta)}{f'_{a+j}(\theta)}\,d\theta \\ \qquad\qquad \leqslant M \end{cases}$$

for all positive integers a, M with $a + M \leqslant N$, $M \geqslant N^{1/2}$. Then there is an absolute constant A_1 such that

$$\max_{n\leqslant N}\mathfrak{R}_N(\theta) \leqslant A_1 s^{1/2} N^{1/2}\log^{p/s} N$$

except, possibly, in a set E of θ of measure

$$\text{(14)}\qquad |E| \leqslant 4\log^{-2p} N,$$

where p is any positive number.

Choose U, V integers such that

$$2^{2U} \leqslant s\,N < 2^{2U+2} \quad , \quad 2^{2V+2} \leqslant s^{-1}N < 2^{2V+4}.$$

We shall show first that there is a set E for which (14) holds and

$$\text{(15)}\qquad \max_{\substack{u=0,1,\ldots,[N_2^{-U}]\\ v=0,1,\ldots,2^V-1}} |R_{u2^U}(0, v2^{-V}:\theta)| \leqslant A s^{1/2} N^{1/2}\log^{p/s} N$$

except, possibly, when $\theta \in E$.

Let a, y, b, z be any four integers with

$$y \geqslant U \;;\; 0 \leqslant a2^y < (a+1)2^y \leqslant N$$
$$0 < z \leqslant V \;;\; 0 \leqslant b2^{-z} < (b+1)2^{-z} \leqslant 1.$$

Then

$$\mathfrak{r}(\theta) = \mathfrak{r}_{a,y;b,z}(\theta) = {}_{a2^y}R_{(a+1)2^y}(b2^{-z}, \overline{b+1}\,2^{-z}:\theta)$$
$$= \sum_{j=1}^{2^y} r(f_{a2^y+j}(\theta)),$$

where $r(x) = r(b2^{-z}, \overline{b+1}\,2^{-z}:x)$ satisfies the conditions of lemma 3 by (13), and so

$$\int_0^1 \mathfrak{r}^{2s}(\theta) \leqslant 2\cdot\frac{(2s)^{2s+2}}{s!}2^{s(y-z)}.$$

Hence

$$|\mathrm{r}_{a,y;\,b,z}(\theta)| \leqslant A s^{1/2} \cdot \log^{p/s} N \cdot N^{1/4}\, 2^{1/4(y-z)}$$

except, possibility, in a set $E_{a,y;\,b,z}$ of θ of measure

$$|E_{a,y;\,b,z}| \leqslant (\log N)^{-2p} \cdot N^{-s/2}\, 2^{s/2(y-z)}.$$

We shall take for the E of the theorem

$$E = \cup\, E_{a,y;\,b,z}.$$

Then certainly

$$|E| \leqslant \sum |E_{a,y;\,b,z}| \leqslant (\log N)^{-2p} \Big(\sum_{a,y} N^{-s/2}\, 2^{sy/2}\Big) \Big(\sum_{b,z} 2^{-(sz/2)}\Big).$$

Now for any given value of z there are 2^z values of b and hence

$$\sum_{b,z} 2^{-sz/2} = \sum_{z} 2^{-\left(\frac{s-2}{2}\right)z} \leqslant 2 \qquad (s \geqslant 4).$$

Similarly for any value of y there are at most $[N\, 2^{-y}]$ values of a and hence

$$\sum_{a,y} N^{-s/2}\, 2^{sy/2} \leqslant \sum_{2^y \leqslant N} (2^y N^{-1})^{1-(s/2)} \leqslant 2 \qquad (s \geqslant 4).$$

Hence, finally,

$$|E| \leqslant 4\, (\log N)^{-2p}.$$

For the rest of the proof of this lemma we suppose $\theta \in E$ and suppress the argument θ. Now consider a general

$$R_{u\,2^U}(0, v\, 2^{-V}).$$

By expressing u and v in the binary scale and making use of the basic identities

$${}_{N_1}R_{N_3} = {}_{N_1}R_{N_2} + {}_{N_2}R_{N_3}\ ;\ R(\alpha, \beta) = R(\alpha, \gamma) + R(\gamma, \beta)$$

we have

$$R_{u\,2^U}(0, v\, 2^{-V}) = \sum{}^{*}\, \mathrm{r}_{a,y;\,b,z}$$

where the * indicates a sum depending on u and v in which each pair of values y, z occurs at most once. Hence

$$|R_{u\,2^U}(0, v\, 2^{-V})| \leqslant A s^{1/2} \log^{p/s} N \sum_{\substack{2^y \leqslant N \\ z > 0}} N^{1/4}\, 2^{1/4(y-z)} \leqslant A s^{1/2} \log^{p/s} N \cdot N^{1/2}$$

as required.

We now complete the proof of the lemma. If $0 \leqslant n \leqslant N$ we can find a $u\, 2^U$ such that $u\, 2^U \leqslant N \leqslant (u+1)\, 2^U \leqslant u\, 2^U + s^{1/2} N^{1/2}$. Hence from (15) and the trivial inequalities $|R_{a+b}| \leqslant |R_a| + |{}_aR_{a+b}| \leqslant |R_a| + b$ we have

$$|R_n(0, v\, 2^{-V})| \leqslant A s^{1/2} N^{1/2} \log^{p/s} N + s^{1/2} N^{1/2} \leqslant A s^{1/2} N^{1/2} \log^{p/s} N.$$

Next, if v, w are any two integers in the range $0 \leqslant v, w < 2^V$ we have

$$|R_n(w\,2^{-V},\ v\,2^{-V})| \leqslant |R_n(0, w\,2^{-V})| + |R_n(0, v\,2^{-V})| \leqslant As^{1/2} N^{1/2} \log^{p/s} N.$$

In particular

$$|R_n(\delta_1, \delta_1 + 2^{-V})| \leqslant As^{1/2} N^{1/2} \log^{p/s} N$$

if δ_1 is of the form $v\,2^{-V}$. Suppose now $\{\delta - \delta_1\} < 2^{-V}$. Then, by definition,

$$\begin{aligned} R_n(\delta_1, \delta) &= F_n(\delta_1, \delta) - n\,\{\delta - \delta_1\} \\ R_n(\delta_1, \delta_1 + 2^{-V}) &= F_n(\delta_1, \delta_1 + 2^{-V}) - n\,2^{-V} \\ 0 &\leqslant F_n^1(\delta_1, \delta) \leqslant F_n(\delta_1, \delta_1 + 2^{-V}), \end{aligned}$$

where

$$0 \leqslant n\,\{\delta - \delta_1\} \leqslant n\,2^{-V} \leqslant N\,2^{-V} \leqslant 4s^{1/2} N^{1/2}.$$

Hence

$$|R_n(\delta_1, \delta)| \leqslant |R_n(\delta_1, \delta_1 + 2^{-V})| + 4\,s^{1/2} N^{1/2} \leqslant As^{1/2} N^{1/2} \log^{p/s} N.$$

Finally, if α, β are any numbers we can find α_1, β_1 of the form $v\,2^{-V}$ such that $\{\alpha - \alpha_1\} < 2^{-V}$, $\{\beta - \beta_1\} < 2^{-V}$. Then

$$|R_n(\alpha, \beta)| \leqslant |R_n(\alpha_1, \alpha)| + |R_n(\beta_1, \beta)| + |R_n(\alpha_1, \beta_1)| \leqslant As^{1/2} N^{1/2} \log^{p/s} N$$

for all n, α, β. Since

$$\mathfrak{R}_n = \operatorname*{Max}_{\alpha, \beta} |R_n(\alpha, \beta)|$$

this proves the lemma.

5. *An elementary lemma.* Before going to the proof of the theorem we give an almost trivial lemma. We again suppress the argument θ.

Lemma 5. *Let the sequence of functions*

$$f_1, \ldots, f_N \tag{16}$$

be decomposed into a number (say t) of distinct subsequences:

$$\begin{cases} f_1^{(1)}, & f_2^{(1)}, \ldots, f_{N_1}^{(1)} \\ \vdots & \\ f_1^{(t)}, & f_2^{(t)}, \ldots, f_{N_t}^{(t)} \end{cases} \tag{17}$$

so that every element in (16) *occurs just once in* (17) *and vice-versa. Suppose also that the elements of any row of* (17) *occur in the same order in* (16). *Then*

$$\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_n(\theta) \leqslant \operatorname*{Max}_{n \leqslant N_1} \mathfrak{R}_n^{(1)}(\theta) + \ldots + \operatorname*{Max}_{n \leqslant N_t} \mathfrak{R}_n^{(t)}(\theta)$$

where the upper affixes in the $\mathfrak{R}_n^{(\tau)}$ refer to the sequence $f_n^{(\tau)}$.

For we have

$$R_n(\alpha,\beta)=\sum_{\nu=1}^{n} r(\alpha,\beta:f_\nu)=\sum_{\tau=1}^{t}\sum_{\nu_\tau=1}^{n_\tau} r(\alpha,\beta:f_{\nu_\tau}^{(\tau)})=\sum_{\tau=1}^{t} R_{n_\tau}^{(\tau)}(\alpha,\beta)$$

for some integers n_τ. Hence

$$\operatorname*{Max}_{n\leqslant N}\mathfrak{R}_n=\operatorname*{Max}_{\substack{\alpha,\beta\\ n\leqslant N}}|R_n(\alpha,\beta)|\leqslant\sum_{\tau=1}^{t}\operatorname*{Max}_{\substack{\alpha,\beta\\ n_\tau\leqslant N_\tau}}|R_{n_\tau}^{(\tau)}(\alpha,\beta)|=\sum_{\tau=1}^{t}\operatorname*{Max}_{n_\tau\leqslant N_\tau}\mathfrak{R}_{n_\tau}^{(\tau)}.$$

6. *Proof of the theorem.*

We first adapt lemma 4.

Lemma 6. *Suppose the conditions of the theorem are satisfied. Then there is an absolute constant A_2 and a constant C_0 depending only on the function $\varphi(n)$ such that for any given $N>C_0$ the inequality*

$$\operatorname*{Max}_{n\leqslant N}\mathfrak{R}_N(\theta)\leqslant A_2\,n^{1/2}\log^{1/2}N\log\log N\,\varphi^{-1/2}(N)$$

holds except, possibly, in a set E_N of θ of measure

$$|E_N|\leqslant\log^{-2}N.$$

We shall denote by C a constant depending only on the function $\varphi(n)$, not necessarily the same in different contexts (so c of (ii) of the theorem is a C). Put

$$p=3 \tag{18}$$

and let $s=s_N$ and $t=t_N$ be the integers

$$s=[\log\log N]+1,\quad t=\left[\frac{6s\log N}{\varphi(N)}\right]+1. \tag{19}$$

We may assume that N is so large that

$$s\geqslant 4,\quad t\leqslant\log^2 N \tag{20}$$

by condition (ii) of the theorem.

The proof depends on decomposing the sequence $f_1,\ldots,f_N$ into t subsequences, in the sense of lemma 5, and then applying lemma 4 to the subsequences.

Consider the t subsequences

$$f_1^{(\tau)},\ldots,f_{N_\tau}^{(\tau)}\quad(\tau=1,\ldots,t)$$

where

$$f_n^{(\tau)}=f_{(n-1)t+\tau},\quad|N_\tau-(N/t)|\leqslant 1.$$

We now estimate the sum $\mathfrak{S}_{a,M}^{(\tau)}$ $(M\geqslant N_\tau^{1/2})$ of lemma 4 for each of the subsequences $f_n^{(\tau)}$:

$$\begin{aligned}\mathfrak{S}_{a,M}^{(\tau)}&=(4s-1)\sum_{j=1}^{M}\frac{1}{f_{a+j}^{(\tau)}(0)}+2\sum_{i<j\leqslant M}j^{2s-2}\int_0^1\frac{f_{a+i}^{(\tau)}(\theta)}{f_{a+j}^{(\tau)}(\theta)}\,d\theta\\&=(4s-1)\,\sigma_1+2\,\sigma_2\ \text{(say)}.\end{aligned}$$

Now by (i) and (ii) of the theorem,

$$\sum_{n=1}^{\infty} \frac{1}{f_n''^{(\theta)}} \leqslant 1 + \sum_{n=2}^{\infty} e^{-\varphi(2)\ldots-\varphi(n)} < C < \infty$$

and *a fortiori*

$$\sigma_1 < C.$$

Also, if $0 < i < j$,

$$\frac{f_{a+i}'^{(\tau)}(\theta)}{f_{a+j}'^{(\tau)}(\theta)} = \frac{f'^{(\theta)}_{(a+i-1)\,t+\tau}}{f'^{(\theta)}_{(a+j-1)\,t+\tau}} \leqslant e^{-\varphi\overline{(a+j-1}\,t+\tau)-\ldots-\varphi(\overline{a+j-2}t+\tau+1)}$$

$$\leqslant e^{-t\varphi(j)}$$

since $\varphi(n)$ is non-decreasing and $\overline{a+j-2}\,t+\tau+1 = (a+j-2)\,(t-1) + (\tau-1)+a+j \geqslant j$. Hence

$$\sigma_2 \leqslant \sum_{j \leqslant M} j^{2s}\, e^{-t\varphi(j)}$$

$$\leqslant \sum_{\log j \leqslant \sqrt{\log N}} j^{2s} + \sum_{\substack{\log j > \sqrt{\log N} \\ j \leqslant M}} e^{2s\log j - t\varphi(j)}$$

$$= \sigma_3 + \sigma_4 \text{ (say).}$$

But trivially $\sigma_3 < N^{1/3}$ if N is large enough. Further

$$t\,\varphi(j) \geqslant t\,\varphi(N)\log^{-1} N \log\log^{-1} N \log j \log\log j \geqslant 6 \log j \log\log j$$

by the monotonicity of $\log n \log\log n\, \varphi^{-1}(n)$ and hence

$$\sigma_4 \leqslant \sum_{\log j \geqslant \sqrt{\log N}} e^{2s\log j - 6\log j \log\log j}$$

$$\leqslant \sum e^{-\log j \log\log j} < A < \infty$$

since $2s = 2\,[\log\log N] + 2 < 5 \log\log j$ if $\log j \geqslant \log N$. Combining these inequalities we deduce

$$(21) \qquad \mathfrak{S}_{a,M}^{(\tau)} \leqslant 4s\sigma_1 + 2\sigma_3 + 2\sigma_4 < C3 + 2N^{1/3} + A < \left(\frac{N}{t} - 1\right)^{1/3} \leqslant N_\tau^{1/3}$$

for N greater than a C, the third inequality in (21) being a trivial deduction from (19). Hence $\mathfrak{S}_{a,M}^{(\tau)} \leqslant M$ for all $M \geqslant N_\tau^{1/3}$ and, by lemma 4 applied to $f_n^{(\tau)}$ it follows that

$$\operatorname*{Max}_{n \leqslant N_\tau} \Re_n^{(\tau)}(\theta) \leqslant A s^{1/3} N_\tau^{1/3} \log^{p/3} N_\tau$$

except, possibly, in a set $E_N^{(\tau)}$ (say) of measure

$$|E_N^{(\tau)}| \leqslant 4 \log^{-2p} N_\tau.$$

Write

$$E_N = \bigcup_\tau E_N^{(\tau)}.$$

Then in the first place

$$|E_N| \leqslant 4t \log^{-2p}\left(\frac{N}{t} - 1\right)$$

$$\leqslant \log^{-2} N$$

by (18) and (20) if N is greater than a C. Also, provided $\theta \in \overline{\in} E_N$ we have

$$\begin{aligned}\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_N(\theta) &\leqslant \sum_{\tau=1}^{l} \operatorname*{Max}_{n_\tau \leqslant N_\tau} \mathfrak{R}^{(\tau)}_{n_\tau}(\theta) \\ &\leqslant A\, t\, s^{1/2} \left(\frac{N}{t}+1\right)^{1/2} \log^{p/2} N \\ &\leqslant A\, t^{1/2}\, s^{1/2}\, N^{1/2} \log^{p/2} N\end{aligned}$$

since each $N_\tau \leqslant (N/t)+1 \leqslant N$. Hence using the values (18), (19) we have

$$\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_n(\theta) \leqslant A\, N^{1/2} \log^{1/2} N \log\log N\, \varphi^{-1/2}(N)$$

which proves the lemma.

It is now an easy matter to prove the theorem. The proof follows familiar lines.

We shall first prove that for almost all θ there is a $T_0(\theta)$ such that

$$\operatorname*{Max}_{N \leqslant 2^T} \mathfrak{R}_N(\theta) \leqslant A_2\, 2^{1/2 T} \log^{1/2} 2^T \log\log 2^T\, \varphi^{-1/2}(2^T)$$

for all $T \geqslant T_0(\theta)$ where A_2 is the A_2 of lemma 6. For if E_N is the E_N of lemma 6, that lemma shows that

$$\sum_T |E_{2^T}|$$

is convergent. Hence for almost all θ there is a $T_0(\theta)$ such that

$$\theta \in \overline{\in} E_{2^T} \quad \text{all} \quad (T \geqslant T_0).$$

Now put $N_0(\theta) = 2^{T_0(\theta)}$. Then for all $N \geqslant N_0(\theta)$ there is a $T \geqslant T_0(\theta)$ such that $N \leqslant 2^T \leqslant 2N$. But then

$$\mathfrak{R}_N(\theta) < \operatorname*{Max}_{N \leqslant 2^T} \mathfrak{R}_N(\theta) < A\, (2N)^{1/2} \cdot \log^{1/2} 2N \cdot \log\log 2N\, \varphi^{-1/2}(N)$$

since $\varphi(n)$ is non-decreasing, and finally

$$\mathfrak{R}_N(\theta) \leqslant A\, N^{1/2} \log^{1/2} N \log\log N\, \varphi^{-1/2}(N)$$

for some A. This proves the theorem.

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THE ASYMPTOTIC EXPANSION OF THE CONFLUENT HYPERGEOMETRIC FUNCTION $M_{\omega/2,0}(2\omega)$

BY

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§ 1. Elsewhere [1]) I have found that $M_{\omega/2,0}(2\omega)$ is for large positive ω asymptotic equal to

$$2^{1/3}\,3^{-1/6}\,\pi^{-1}\,\omega^{1/6}\{\Gamma(1/3)\cos(\omega/2-1/6)\,\pi\sum_0 a_p\,\omega^{-2p}+$$

$$+\,2.3^{1/3}\,\Gamma(2/3)\cos(\omega/2+1/6)\sum_0 b_p\,\omega^{-2p-4/3}\}.$$

Here

$$a_0=1\ ;\ b_0=-{}^{11}/_{280}\ ;\ a_1=-{}^{13}/_{900}\ ;\ b_1={}^{4}/_{7}+{}^{1}/_{9}+{}^{2}/_{13}-{}^{7}/_{32}-{}^{377}/_{625}.$$

The object of this paper is to deduce a numerical upper bound of the remainder in this formula.

§ 2. We start with the fundamental formula

$$M_{\omega/2,0}(2\omega)=(2\omega)^{1/3}\,1/2\pi i\int_{-\infty}^{(1^+)} e^{\omega z}\,(z-1/z+1)^{\omega/2}\,(z^2-1)^{-1/3}\,dz.$$

We put $\arg(z-1)=0$ and $\arg(z+1)=0$ for $z>1$. We may write

$$(2\omega)^{-1/3}\,M_{\omega/2,0}(2\omega)=-1/\pi\,Re\{\exp(\omega\pi i/2)\int_a^{-\infty} e^{\omega\varphi(z)}\,(1-z^2)^{-1/3}\,dz\}.$$

Here a denotes an arbitrary number >1 and the path of integration lies above the real axis; $\varphi(z)=z+{}^{1}/_{2}\ln 1-z/1+z$.

According to the method of steepest descent we define the path of integration by $Im\,\varphi(z)=0$, thus if $z=x+iy$,

$$\begin{cases} x^2=2y\,\mathrm{cotg}\,2y+1-y^2 & 0<y<\pi/2 \\ y=0 & 0<x<1. \end{cases}$$

Therefore the contour consists of a segment S of the real axis from $x=1$ to $x=0$ and moreover of a curve L in the second quadrant with asymptote $y=\pi/2$ and with the tangent $y=x\,\mathrm{tg}\,2\pi/3$ at the origin. The

[1]) The use of confluent hypergeometric functions in mathematical physics and the solution of an eigenvalue problem. Appl. Sci. Res. (A) 1950.

function $\varphi(z)$ increases on S monotonously from $-\infty$ to 0 and decreases on L monotonously from 0 to $-\infty$.

By the transformation $\zeta = \{-\varphi(z)\}^{1/3}$, where $\arg\zeta = 0$ for z on S and $\arg\zeta = 2\pi/3$ for z on L, we find for $\pi(2\omega)^{-1/2} M_{\omega/2,0}(2\omega)$ the expression

$$Re\, e^{\omega\pi i/2}\left\{\int_0^{\infty} e^{-\omega\zeta^3}\frac{d\zeta}{d\zeta/dz\,(1-z^2)^{1/2}} + \int_0^{\infty\exp 2\pi i/3} e^{-\omega\zeta^3}\frac{d\zeta}{d\zeta/dz\,(1-z^2)^{1/2}}\right\}.$$

By applying BURMANN's theorem we obtain

$$\pi(2\omega)^{-1/2} M_{\omega/2,0}(2\omega) = Re\, e^{\omega\pi i/2}\{\sum_0^{n-1} m_k (A_k - B_k) + (C_n - D_n)\},$$

where

$$(1)\quad \begin{cases} A_k = \int_0^{\infty} e^{-\omega\zeta^3}\zeta^k\, d\zeta = 1/3\,\Gamma(k+1/3)\,\omega^{-(k+1/3)}; \\ B_k = \int_0^{\infty\exp 2\pi i/3} e^{-\omega\zeta^3}\zeta^k\, d\zeta = 1/3\,\Gamma(k+1/3)\,\omega^{-(k+1/3)\pi i}\, e^{(2k+2/3)\pi i}; \\ m_k = 1/2\pi i \int^{(0+)} (1-t^2)^{-1/2}\,(-t-1/2\ln 1-t/1+t)^{-(k+1/3)}\, dt; \end{cases}$$

$$(2)\quad \begin{cases} C_n = \int_0^{\infty} e^{-\omega\zeta^3}\zeta^n R_n(\zeta)\, d\zeta \quad ; \quad D_n = \int_0^{\infty\exp 2\pi i/3} e^{-\omega\zeta^3}\zeta^n R_n(\zeta)\, d\zeta; \\ R_n(\zeta) = 1/2\pi i \int \frac{dt}{(1-t^2)^{1/2}(-t-1/2\ln 1-t/1+t)^{n/3}\{(-t-1/2\ln 1-t/1+t)^{1/3}-\zeta\}}; \end{cases}$$

The contour encloses all the values of t for which $(-t-1/2 \ln 1-t/1+t)^{1/3} = \zeta$. It is easily seen that $m_k = 0$ for k odd so that in virtue of the vanishing of the factor $\sin(k+1/3)\pi$ for $k \equiv 2 \pmod 3$ we find

$$M_{\omega/2,0}(2\omega) = \frac{2^{1/2}\omega^{1/6}\cos(\omega/2-1/6)\pi}{3^{1/2}\pi}\sum_0^{[n-1/6]} m_{6p}\,\Gamma(2p+1/3)\,\omega^{-2p} +$$

$$+ \frac{2^{1/2}\omega^{-1/6}\cos(\omega/2+1/6)\pi}{3^{1/2}\pi}\sum_0^{[n-5/6]} m_{6p+4}\,\Gamma(2p+1+2/3)\,\omega^{-2p-1} + U_n,$$

where the remainder U_n is equal to

$$(3)\qquad U_n = (2\omega)^{1/3}/\pi\,\{C_n\cos(\omega\pi/2) - Re\,(D_n\, e^{\omega\pi i/2})\}.$$

In the paper mentioned in note [1]), in which I have derived the same asymptotic expansion, I have found

$$m_0 = 3^{1/3} \quad ; \quad m_4 = -11.3^{1/3}/280 \quad ; \quad m_6 = -13.3^{1/3}/400;$$
$$m_{10} = (4/7 + 1/9 + 2/13 - 7/32 - 377/625)\,3^{11/3}/40.$$

§ 3. In this section I deduce for k even an upper bound for the absolute value of m_k, viz.

$$(4)\qquad |m_k| < s\,(2/\pi)^{\frac{k+4}{3}}\left\{0{\cdot}64 + \frac{3.2^{-\frac{k+1}{6}}}{k+1}\right\},$$

where

$$s = 1 \quad \text{if} \quad k \equiv 0 \text{ or } 4 \pmod 6$$
$$s = 2 \quad \text{if} \quad k \equiv 2 \pmod 6.$$

In (2) I choose as path of integration two contours resp. around $+1$ and -1 and beginning and also ending resp. at $+\infty$ and $-\infty$. In this manner I obtain

$$m_k = (1/\pi)\, Re\, (\exp\,(k+1/3)\,\pi i - 1) \int_1^\infty (t^2-1)^{-1/2} \left(\theta + \frac{\pi i}{2}\right)^{-(k+1/3)} dt,$$

where $\theta = t + {}^1/_2 \ln t - 1/t + 1$, and therefore we obtain

$$|m_k| \leqq s/\pi \int_1^\infty (t^2-1)^{-1/2} \left(\theta^2 + \frac{\pi^2}{4}\right)^{-(k+1/6)} dt.$$

By using

$$\theta^2 + (\pi^2/4) \geqq (\pi^2/4) \qquad \text{for} \quad 1 < t < a$$
$$\theta^2 + (\pi^2/4) \geqq (t-b)^2 + (\pi^2/4) \qquad \text{for} \quad t > a,$$

where a is an arbitrary number $> {}^5/_4$ and $b = {}^1/_2 \ln a - 1/a + 1$, we find

$$(5) \quad \begin{cases} |m_k| < s/\pi\, (2/\pi)^{k+1/3} \{\ln (a + \sqrt{a^2-1}) + \\ + \int_c^\infty (u^2+1)^{-(k+1/6)} \{u^2 + (4/\pi)\, bu - (4/\pi^2)\, (b^2-1)\}^{-1/2}\, du\}, \end{cases}$$

where

$$c = 2/\pi\, (a-b) = 2/\pi\, (a - {}^1/_2 \ln a + 1/a - 1).$$

Here

$$u^2 + (4/\pi)\, bu - 4/\pi^2\, (b^2-1) > u^2.$$

In fact we have to prove the inequality

$$2\, b\, (a-b) > b^2 - 1,$$

which is equivalent to

$$3\, b^2 - 2\, ab - 1 < 0,$$

that is

$$b < \frac{a + \sqrt{a^2+3}}{3};$$

this inequality follows from

$$a + \sqrt{a^2+3} - {}^3/_2 \ln (a + 1/a - 1) > 0$$

which is obvious since the left hand side represents a monotonously increasing function which is positive at $a = {}^5/_4$.

Consequently the integral occurring in (5) is less than

$$\int_c^\infty u^{-1}(u^2+1)^{-(k+1/6)}\,du = {}^1/_2 \int_{1+c^2}^\infty v^{-(k+7/6)}/1-(1/v)\,dv = {}^1/_2 \sum_0^\infty \int_{1+c^2}^\infty v^{-((k+7/6)+j)}\,dv$$

$$= {}^1/_2 \sum_0^\infty \frac{1}{(1+c^2)^{(k+7/6)+j}\,((k+1/6)+j)} \leqq$$

$$\leqq \frac{3}{k+1} \sum_0^\infty \frac{1}{(1+c^2)^{(k+1/6)+j}} = \frac{3}{(k+1)\,c^2\,(1+c^2)^{k-5/6}}.$$

We define a by the equation $c=1$. In that case $a<1{,}94$ and $\ln(a+\sqrt{a^2-1})<1{,}28$. This establishes the proof of formula (4).

§ 4. The object of the following sections is to deduce an upper bound for the expression $R_n(\zeta)$ occurring in (2). We choose the same path of integration as in the preceding section, namely two contours resp. around $+1$ and -1, and beginning and also ending resp. at $+\infty$ and $-\infty$. Thus we find for $2\pi R_n(\zeta)$ the sum

$$\int_1^\infty \frac{\exp(n-1/3)\,dt}{(t^2-1)^{1/2}(\vartheta+(\pi i/2))^{n/3}\{(\vartheta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}\}} + \int_1^\infty \frac{\exp-(n+1/3)\,\pi i\,dt}{(t^2-1)^{1/2}(\vartheta-(\pi i/2))^{n/3}\{(\vartheta-(\pi i/2))^{1/3}-\zeta e^{\pi i/3}\}}$$

$$-\int_1^\infty \frac{dt}{(t^2-1)^{1/2}(\vartheta+(\pi i/2))^{n/3}\{(\vartheta+(\pi i/2))^{1/3}-\zeta\}} - \int_1^\infty \frac{dt}{(t^2-1)^{1/2}(\vartheta-(\pi i/2))^{n/3}\{(\vartheta-(\pi i/2))^{1/3}-\zeta\}}$$

where $\vartheta = t+{}^1/_2\ln(t-1/t+1)$.

This is in absolute value at most

$$\int_1^\infty \frac{1}{(t^2-1)^{1/2}(\vartheta+(\pi i/2))^{n/3}} \left| \frac{\exp(n+1/3)\,\pi i}{(\vartheta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}} - \frac{1}{(\vartheta+(\pi i/2))^{1/3}-\zeta} \right| dt +$$

$$+\int_1^\infty \frac{1}{(t^2-1)^{1/2}(\vartheta-(\pi i/2))^{n/3}} \left| \frac{\exp-(n+1/3)\,\pi i}{(\vartheta-(\pi i/2))^{1/3}-\zeta e^{-\pi i/3}} - \frac{1}{(\vartheta-(\pi i/2))^{1/3}-\zeta} \right| dt.$$

Hence

$$(6)\qquad |R_n(\zeta)| \leqq (1/\pi)\int_1^\infty \frac{T\,dt}{(t^2-1)^{1/2}(\vartheta^2+(\pi^2/4))^{n/6}\,|(\vartheta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\;|(\vartheta+(\pi i/2))^{1/3}-\zeta|},$$

where

$$T=(\vartheta^2+(\pi^2/4))^{1/6}+3^{1/2}|\zeta| \quad \text{if} \quad n\equiv 4 \pmod 6$$

$$T=(\vartheta^2+(\pi^2/4))^{1/6} \quad \text{if} \quad n\equiv 0 \pmod 6.$$

We consider only values of n which are either $\equiv 0 \pmod 6$ or $\equiv 4 \pmod 6$.

§ 5. In this section I shall prove the inequality

$$(7)\quad |(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\,|(\theta+(\pi i/2))^{1/3}-\zeta| \geqq \frac{(\pi/2)^{2/3}}{1+h_1\zeta} \text{ if } \arg\zeta=0,$$

where

$$h_1=(2+\sqrt{3})\,(2/\pi)^{1/3}=3{\cdot}2105;$$

and

$$(8)\quad |(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\,|(\theta+(\pi i/2))^{1/3}-\zeta| \geqq |\theta+(\pi i/2)|^{2/3} \text{ if } \arg\zeta=(2\pi/3).$$

For $t \geqq 1$ the function θ runs though all real values, so that the point $(\theta+(\pi i/2)^{1/3}=X+i\,Y$ lies on the curve F defined by

$$Im\,(X+i\,Y)^3=\pi/2 \quad \text{i.e.} \quad 3\,X^2Y-Y^3=\pi/2.$$

This curve has the symmetry axis $Y=X\,3^{-1/2}$ and two asymptotes $Y=0$ and $Y=X\,3^{1/2}$. Now we distinguish two cases.

1. Put $\arg\zeta=0$. Choosing a new rectangular coordinate system the x-axis of which coincides with the symmetry axis of F, we find for the equation of F

$$x^3-3\,xy^2=\pi/2$$

that is

$$x=(\pi/2)^{1/3}\,\lambda \qquad y=3^{-1/3}\,(\pi/2)^{1/3}\,\lambda^{-1/3}\,(\lambda^3-1)^{1/3}.$$

With respect to this system the coordinates of the point $\zeta e^{\pi i/3}$ are $(^1/_2\,\zeta\sqrt{3},\ ^1/_2\,\zeta)$ and those of ζ are $(^1/_2\,\zeta\sqrt{3},\ -^1/_2\,\zeta)$. By putting $\zeta=2.3^{-1/2}\,(\pi/2)^{1/3}\mu$, we obtain

$$|(\theta+(\pi i)2))^{1/3}-\zeta e^{\pi i/3}|^2\,|(\theta+(\pi i/2))^{1/3}-\zeta|^2=$$
$$=\frac{(\pi/2)^{4/3}}{9\,\lambda^2}\{9\,\lambda^2\,(\lambda-\mu)^4+6\,\lambda\,(\lambda-\mu)^2\,(\lambda^3+\lambda\mu^2-1)+(\lambda^3-\lambda\mu^2-1)^2\}.$$

The substitution

$$v=4\,\lambda^2\,(\lambda-\mu) \quad w=4\,\lambda\,(\lambda-\mu)^2$$

transforms the right hand side into

$$\frac{(\pi/2)^{4/3}}{9\,\lambda^2}\,(v^2-vw+w^2-v-w+1).$$

In virtue of $\lambda \geqq 1$ and $\mu \geqq 0$ we obtain $v^2 \geqq 4\,w \geqq 0$ and $v\,(v-w) \geqq 0$. In the next section we shall prove that these inequalities imply

$$(9)\qquad v^2-vw+w^2-v-w+1 \geqq \frac{9\,v^2}{\{f\,(v-w)+(4\,vw)^{1/3}\}^2},$$

where $f=2+4.3^{-1/3}=4.309$.

Hence the left hand side of (7) is at most equal to $(\pi/2)^{2/3}\,(1+h_1\zeta)^{-1}$ where h_1 has the above indicated value.

2. Put $\arg \zeta = 2\pi/3$. The square of the left hand side of (8) is equal to

$$(X^2 + Y^2 + 2|\zeta| X + |\zeta|^2)\ (X^2 + Y^2 + |\zeta| X - |\zeta| Y\sqrt{3} + |\zeta|^2) =$$
$$= (X^2 + Y^2)^2 + |\zeta|\ (X^2 + Y^2 + |\zeta|^2)\ (X\sqrt{3} - Y)\sqrt{3} +$$
$$+ 2|\zeta|^2\ (2X^2 - XY\sqrt{3} + Y^2) + |\zeta|^4 \geqq (X^2 + Y^2)^2,$$

since each point of F satisfies the inequality $Y \leqq X\sqrt{3}$. This proves inequality (8).

§ 6. In this section the inequality (9) will be proved. We distinguish four cases, namely

1. $v \leqq 0$; 2. $v \geqq 0$ $w \geqq 4$; 3. $v \geqq 0$, $0 \leqq w \leqq (2 - \sqrt{3})^2$
4. $v \geqq 0$, $(2 - \sqrt{3})^2 \leqq w \leqq 4$.

1. The left hand side of (9) is $\geqq {}^3/_4$ and the right hand side of (9) is $\leqq {}^9/_{16}$.
2. Writing $v = w + \delta$ $\delta \geqq 0$ we have

$$(v^2 - vw + w^2 - v - w + 1)^{1/2} = \{(w-1)^2 + \delta\ (w-1) + \delta^2\}^{1/2} \geqq w - 1\ ;$$

$$\frac{3v}{f(v-w) + (4vw)^{1/2}} \leqq \frac{3(w+\delta)}{f\delta + (4w^2)^{1/2}} \leqq \frac{3w^{1/2}}{2^{2/3}}$$

(9) follows from

$$w - 1 \geqq 3.2^{-2/3} w^{1/2},$$

which is obvious.

3. For $0 \leqq w \leqq (2 - \sqrt{3})^2$ we have

$$(v^2 - vw + w^2 - v - w + 1)^{1/2} = \{(v - {}^1/_2\, w - {}^1/_2)^2 + {}^3/_4\ (1-w)^2\}^{1/2} \geqq$$
$$\geqq (\sqrt{3}/2)\ (1-w) \geqq 3\ (2 - \sqrt{3});$$

and, writing $v = 2\varepsilon w^{1/2}$ $\varepsilon \geqq 1$

$$\frac{3v}{f(v-w) + (4vw)^{1/2}} = \frac{6\varepsilon}{f(2\varepsilon - w^{1/2}) + 2\varepsilon^{1/2}} \leqq \frac{6\varepsilon}{f(2\varepsilon - 2 + \sqrt{3}) + 2}$$

(9) follows from

$$f \geqq \frac{\{2\varepsilon - 2(2-\sqrt{3})\}(2+\sqrt{3})}{2\varepsilon - (2-\sqrt{3})} \geqq 2 + \sqrt{3}.$$

4. Writing again $v = 2\varepsilon w^{1/2}$, we have

$$v^2 - vw + w^2 - v - w + 1 = (w - w^{1/2} + 1)^2 + 2\ (\varepsilon - 1)\ w^{1/2}\ (-w + 2w^{1/2} + 2\varepsilon w^{1/2} - 1)$$
$$\geqq (w - w^{1/2} + 1)^2.$$

As before (9) follows from

$$w - w^{1/2} + 1 \geqq \frac{6\varepsilon}{f(2\varepsilon - w^{1/2}) + 2},$$

which is equivalent to

$$(2 - w^{1/2})\ (f\,w - f\,w^{1/2} - 2\,w^{1/2} + f - 2) + 2\ (\varepsilon - 1)\ (f\,w - f\,w^{1/2} + f - 3) \geqq 0.$$

The last inequality is obvious since the second term is $\geqq 0$ already for $f > 4$ and the first term vanishes for the first time in the w interval for $f = 2 + 4.3^{-1/2}$. It is impossible to find a better constant in (9), since the combination $v = -2 + 2\sqrt{3}$, $w = 4 - 2\sqrt{3}$ for which $v^2 = 4w$ transforms (9) into an equality.

§ 7. The results of the last sections will be applied to the remainders R_n and U_n occurring in (2) and (3). Substituting (7) and (8) in (6) we obtain

$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h\zeta)\cdot(1/\pi)\int_1^\infty (t^2-1)^{-1/2}(\theta^2 + (\pi^2/4))^{-(n/6)}\, T\, dt$$

if $\arg \zeta = 0$;

$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}\cdot(1/\pi)\int_1^\infty (t^2-1)^{-1/2}(\theta^2 + (\pi^2/4))^{-(n+2/6)}\, T\, dt$$

if $\arg \zeta = (2\pi/3)$.

From § 3 it follows

$$(10)\quad 1/\pi \int_1^\infty (t^2-1)^{-1/2}(\theta^2 + (\pi^2/4)^{-(k/6)}\, dt \leqq (2/\pi)^{k/3+1}\left\{0\cdot 64 + \frac{3.2^{-k/6}}{k}\right\}$$

valid for all positive integer values of k.

Denoting the left hand side of (10) by e_k we get

$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_1\zeta)(1 + h_2\zeta)\, e_{n-1} \quad (\arg\zeta = 0;\ n \equiv 4 \pmod 6),$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_1\zeta)\, e_{n-1} \quad (\arg\zeta = 0;\ n \equiv 0 \pmod 6);$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_2\zeta)\, e_{n+1} \quad (\arg\zeta = 2\pi/3;\ n \equiv 4 \pmod 6);$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}\, e_{n+1} \quad (\arg\zeta = 2\pi/3;\ n \equiv 0 \pmod 6),$$

where

$$h_1 = (2 + \sqrt{3})(2/\pi)^{1/3} = 3{\cdot}2105 \qquad \text{and} \qquad h_2 = \sqrt{3}\,(2/\pi)^{1/3} = 1{\cdot}4900.$$

Hence we find for U_n the upper bound

$$|U_n| \leqq (2^{7/6}\omega^{1/2}/\pi^{5/3})\left\{|\cos \omega\pi/2| \int_0^\infty e^{-\omega\zeta^3}(1 + h_1\zeta)\,(1 + h_2\zeta)\,\zeta^n\, d\zeta\cdot e_{n-1} + \right.$$
$$\left. + \int_0^\infty e^{-\omega\zeta^3}(1 + h_2\zeta)\,\zeta^n\, d\zeta\cdot e_{n+1}\right\}.$$

if $n \equiv 4 \pmod 6$ and

$$|U_n| \leqq (2^{7/6}\omega^{1/3}/\pi^{5/3})\left\{|\cos \omega\pi/2| \int_0^\infty e^{-\omega\zeta^3}(1 + h_1\zeta)\,\zeta^n\, d\zeta\cdot e_{n-1} + \right.$$
$$\left. + \int_0^\infty e^{-\omega\zeta^3}\,\zeta^n\, d\zeta\cdot e_{n+1}\right\},$$

if $n \equiv 0 \pmod 6$.

Or finally

$$(11a)\quad |U_n| \leqq \frac{2^{7/6}\omega^{1/2}}{3\pi^{5/3}}\left\{|\cos \omega\pi/2|\,(\gamma_0 + c_1\gamma_1 + c_2\gamma_2)\, e_{n-1} + (\gamma_0 + c_3\gamma_1)\, e_{n+1}\right\},$$

if $n \equiv 4 \pmod 6$, and

11b) $$|U_n| \leqq \frac{2^{7/6}\,\omega^{1/3}}{3\,\pi^{4/3}} \{|\cos \omega\pi/2|\,(\gamma_0 + c_4\gamma_1)\,e_{n-1} + \gamma_0\,e_{n+1}\},$$

if $n \equiv 0 \pmod 6$, where

$$\gamma_j = \Gamma\left(\frac{n+j+1}{3}\right)\omega^{-\frac{n+j+1}{3}} \qquad (j = 0, 1, 2),$$

$$c_1 = 2\,(1+\sqrt{3})\,(2/\pi)^{1/3} = 4{\cdot}7005; \quad c_2 = (3+2\sqrt{3})\,(2/\pi)^{2/3} = 4{\cdot}7837;$$

$$c_3 = \sqrt{3}\,(2/\pi)^{2/3} = 1{\cdot}4900; \quad c_4 = (2+\sqrt{3})\,(2/\pi)^{1/3} = 3{\cdot}2105.$$

To show the practible applicability of (4) and (11) a numerical example will be given:

$$M_{4,0}(8) = 1{\cdot}4494 - 0{\cdot}0136 - 0{\cdot}0013 + U_{10} = 1{\cdot}4345 + U_{10}.$$

(11a) gives

$$|U_{10}| \leqq 0{\cdot}0009$$

Further we have

$$m_6 = -0{\cdot}047 \text{ and } e_6 = 0{\cdot}184.$$

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COMPARISON OF *LEPIDOCYCLINA ZEIJLMANSI* TAN FROM BORNEO WITH *LEPIDOCYCLINA BIRMANICA* RAO FROM BURMAH

BY

M. G. RUTTEN

(Communicated by Prof. H. A. BROUWER at the meeting of January 28, 1950)

Abstract: The foraminiferal species *Lepidocyclina zeijlmansi* and *L. birmanica* are studied. They are found to be good, distinct, though closely related, species.

In 1936 TAN described a new species of *Lepidocyclina* from the Eocene of northern central Borneo. This form, called *L. zeijlmansi*, was remarkable in being the first representative of the subgenus *Polylepidina* in the East Indies, and in being the first instance of *Lepidocyclina* occurring in the Eocene of these regions.

In 1942 RAO also described a new form, *L.* (*Polylepidina*) *birmanica*, from the Upper Eocene of the Yaw Stage in Burmah. As RAO was not aware of TAN's publication, and only compared his form with species from the New World, a comparison of both species mentioned was indicated.

Both the "Hoofd van de Dienst van de Mijnbouw" at Bandoeng and the Director of the Geological Survey of India were so kind to put relevant material at my disposal, for which I wish to express my sincerest gratitude.

The Borneo material consists of rock specimens, and the fossils can only be studied in oblique sections. In addition however to the exellent figures of TAN's publication, this proved to be quite sufficient. From Calcutta I received a sample of detached tests of *L. birmanica*. The Yaw Stage material is not so well preserved as that from Borneo, which will be evident to anyone, comparing the figures of TAN and RAO. I have nothing to add to the description of *L. zeijlmansi*, but wish to make two remarks on *L. birmanica*. These regard the pillars and the structure of the embryonal apparatus of the megalospheric form.

RAO describes from the axial section of *L. birmanica*: "lateral chambers in regular tiers between narrow pillars, which appear at the surface as granules". The lateral chambers sometimes are arranged in regular tiers, but often they are not. Moreover pillars are absent in my specimen. At low magnification pillars seem to be present locally, but in reality this impression is formed by the slight overlap of floors and roofs of alternating chambers in adjacent tiers (fig. 2). Pillars may of course be present in other specimen but they are not so prevalent as is suggested in the original description.

As to the embryonal apparatus, this is described by RAO as consisting of four chambers, arranged in an intricate pattern, with the first chamber the largest. This is not a normal type of embryonal apparatus for *Polylepidina*, and it seems probable that nepionic chambers have been included in the description (cf. TAN, 1939). Horizontal sections are difficult to study, as the tests are partly recrystallised, and as the aequatorial layer, together with the embryonal chambers, is very low. Best results were obtained by studying a rather thick section under the binocular microscope. Fig. 1 gives a camera lucida drawing of the centre of an aequatorial section, made in this way. The embryonal apparatus

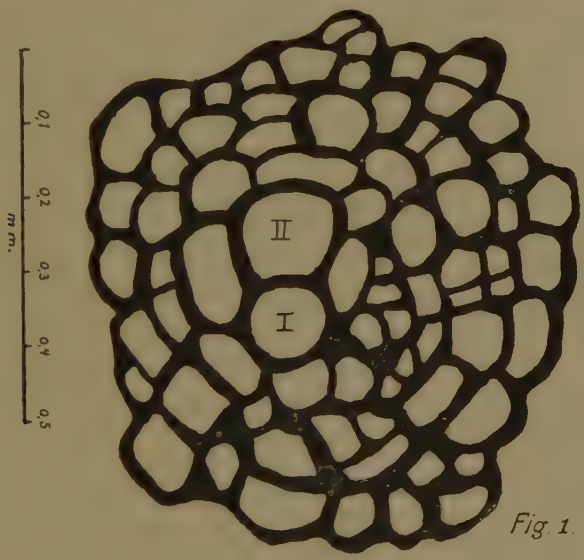

Fig. 1.

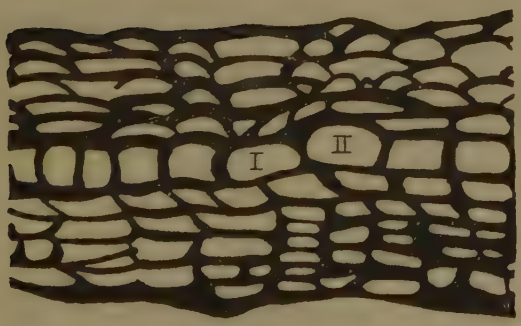

Fig 2.

Figs. 1, 2. *Lepidocyclina* (*Polylepidina*) *birmanica* RAO, × 50. I, II = first and second embryonal chamber. Geol. Inst. Amst., coll. nr. Re 7127, Re. 7129.

Fig. 1. Horizontal section of central part of aequatorial layer, showing the two-chambered nucleoconch, with adauxillary chambers and nepionic spirals.

Fig. 2. Vertical section of central part of test, showing lateral chambers without pillars and low aequatorial chambers and nucleoconch.

consists of two chambers only. These are of small size, and of nephrolepidine type. There are two adauxillary chambers, from which four short nepionic spirals start, completely encircling the embryonal chambers. As one of the adauxillary chambers is slightly larger than the other, this type of embryonal apparatus comes under TAN's subdivision of: "Forms with an asymmetric quadriserial nepiont" (TAN, 1939, p. 62). *L. zeijlmansi* belongs to the same group.

In comparing the two species, we find:

L. zeijlmansi:	*L. birmanica:*
with umbo	without umbo
rather robust form. diameter-thickness = 1 : 3 to 1 : 4.	mucht thinner, waferlike. diameter-thickness = 1 : 12.
strong pillars in umbonal region	pillars absent or rare, always thin
Moreover the aequatorial chambers in *L. zeijlmansi* are arranged somewhat more regularly than in *L. birmanica* (compare TAN, 1936, fig. 10 and RAO, 1942, fig. 1).	

These differences are enough to separate the two forms specifically.

Although there is thus no conspecificity in the genus *Lepidocyclina* between the Eocene of Borneo and of Burmah, the two forms are very narrowly related. As stated above they belong to the same group within the subgenus *Polylepidina*, characterised by an asymmetrical quadriserial nepiont. This is the highest evolutionary step attained by this subgenus, showing the shortest nepionic spirals, or in other words, the shortest span between embryonal and adult stages. *Polylepidina* as a whole being a subgenus of short duration and rapid evolution, the different pylogenetic stages within the subgenus are of necessity closely related.

The conclusion as to the narrow relationship of a *Polylepidina* from Burmah to another species from Borneo is in accordance with older views, lastly expressed by BEETS (1949), as to the faunal resemblance of Burmah and the East Indies during the Eocene.

Geological Institute, University of Amsterdam, Dec. 1949.

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SOME REMARKS ON PROGNATHISM WITH SPECIAL REFERENCE TO THE SKULL OF *PONGO PYGMAEUS* (HOPPIUS)

BY

A. M. HUSSON S.C.J.

(Communicated by Prof. C. J. VAN DER KLAAUW at the meeting of Jan. 28, 1950)

One of the most striking differences between the human skull and the skull of other mammals is the protrusion of the upper jaw which determines the prognathism of the skull. This difference has been studied from the beginning of comparative craniological investigations. With the papers of DAUBENTON (1764) and CAMPER (1782; 1791) begins an important series of research on prognathism. However, the correct method for the determination of the degree of prognathism in the various skull types is still a matter of discussion. IHERING (1872), RIVET (1909) and LÜTHY (1912, pp. 1—13) give a historical and critical summary of the methods in measuring prognathism. One of the best papers on this subject is certainly that by LISSAUER (1872), who studied the prognathism on median sagittal sections of the skull. TOPINARD (1872, 1873, 1885) divided prognathism into some categories. BÜCHNER (1912) drew attention to a new method for measuring prognathism. BOLK (1922, p. 417; 1923, p. 373) introduced the term pseudoprognathism for the skulls of several apes. CAMERON (1928, 1929) called attention to the facial triangle (with the landmarks basion, nasion and alveolar point) and refuted the commonly used gnathic index of FLOWER (1879). Recently WEIDENREICH (1943, pp. 101—102; 139—140) joined in the discussion. He argued to make use of the angle at the nasion; it must be remarked, however, that this angle has been used previously. Many other authors studied the problem of the prognathism of the human skull, but in my opinion the papers mentioned give a sufficient idea of the difficulties which form the background of this part of craniological research.

In recent years Prof. C. J. VAN DER KLAAUW studied the architecture of the vertebrate skull. At his suggestion I studied some features of the skull of *Pongo pygmaeus* (HOPP.). As the result of this study I arrived at the conclusion that it is necessary to introduce two new indices that provide a better indication of prognathism for a comparison of the skulls of the various mammalian groups.

VAN DER KLAAUW explicitly stressed the individuality of components of the skull. In his opinion (1948, p. 1) "the skull may be regarded as a complex of relatively separate functional components, sometimes detached, sometimes united to a morphological whole, but even in this

case to a certain extent independent in size, relative position and grouping". With regard to prognathism I may point to the paper of BOLK (1922, 1923), who arrived at the conclusion that the cerebral position of the nasal cavity and the degree of prognathism in man and apes are not correlated. In my study on the protrusion of the upper jaw, the starting point was also this relative individuality of the facial skull which strongly influences the architecture of the skull as a whole. Within the facial skull, the upper jaw again shows to possess a relative "individuality", such as it has been expressed by VAN DER KLAAUW (1945, p. 20). However, the present note has a descriptive character, because the anatomical, functional and ecological factors which influence the architectonic structure of the skull and the upper jaw are left out of consideration.

It is obvious that in the first place it is necessary to analyse critically the methods commonly used in craniometry and if necessary to find new ones. Also it is worth while to note WEIDENREICH's (1943, p. 102) opinion: "The investigator must adopt the method he considers best for his purpose; he should not hesitate to disregard standard measurements and apply new ones".

When studying the literature on prognathism we are struck by a variety of opinions mainly referring to the manner by which the degree of prognathism may be expressed. At the same time we see that the concept of prognathism of PRICHARD (1843; *vide, e.g.*, LÜTHY, 1912, p. 1) has been extended, but in a way which certainly cannot be considered as an improvement for comparative investigations on skulls.

In my opinion, therefore, it is more correct to distinguish two concepts of prognathism which I propose to term: prognathism *in sensu stricto* and prognathism *in sensu lato*.

By prognathism *in sensu stricto* is meant the protrusion of the upper jaw in respect to a certain plane or a certain line in skulls in median sagittal section.

The meaning of prognathism *in sensu lato* may be illustrated by the words of WEIDENREICH (1943, p. 102): "The facial triangle with the angle at the nasion as indicator of prognathism.... is in every respect far superior to the profile angles generally accepted". This infers that this investigator considers the angle at the basion to be constant in the different human races. Moreover, in WEIDENREICH's concept, the degree of prognathism depends partly on the position of the basion. Therefore, the angle at the nasion will only be useful for comparative investigations if the angle at the basion does not show too great differences in different groups of mammals. In my opinion, however, the comparative investigation of the skull cannot start from such a presumption. *A priori* such a statement is not probable in view of the variability of the base of the skull with the component parts basi-

sphenoid, praesphenoid and ethmoid. Moreover, the more backward position of the basion in the mammalian skull involves too the variation of the angle at the basion in respect to this angle in the human skull.

The gnathic index of FLOWER too is only of value (although merely of relative value, because the facial height is left out of consideration) if the mentioned presumption on the angle at the basion is accepted. I want to define the idea of prognathism in the sense of FLOWER and WEIDENREICH as prognathism *in sensu lato*.

Concerning the angle at the nasion the following remark must be made. Both CAMERON (1928) and WEIDENREICH (1943, p. 101) give the impression that with the angle at the nasion a new craniometric angle has been introduced, the value of which had not been realized till that time. We cannot deny that WEIDENREICH (1943, p. 140) has brought something new by introducing the terms hyperprognathous, prognathous, *etc.*, instead of using only the size of the angle at the nasion. Besides, he gives the parallelism of these terms with the classification of degree of prognathism given by LÜTHY (1912, p. 39) who used the facial profile angle. The two investigators seem to be ignorant of the fact that the Netherlands scientist KOSTER (1860, p. 284) already regarded the use of the facial triangle and of the angle at the nasion as suitable for indicating the degree of prognathism. KOSTER made this statement in a review of J. VAN DER HOEVEN's paper "Catalogus craniorum diversarum gentium". KOSTER, however, did not develop this method by further investigations on the subject. As, *e.g.*, may be inferred from remarks by RIVET (1909, p. 37, note 5) and LÜTHY (1912, pp. 10, 66), it may be useful to give here a translation of KOSTER's statements:

"Perhaps it may be possible to determine the degree in which the face is protruded as follows. When with a 'compas d'épaisseur' we measure the distance from the centre of the anterior margin of the foramen magnum to the nasion, from the latter to the alveolar arch between the two median incisors, and finally from that point to the centre of the anterior margin of the foramen magnum, we have the three sides of a triangle, situated in the median sagittal section of the head, and the angle between the two first mentioned lines will depend on the upper jaw being more or less protruded. The angle becomes larger when the distance between the centre of the anterior margin of the foramen magnum and the centre of the alveolar arch increases and *vice versa*. Once the three sides have been measured in this manner, we immediately find the required angle by some trigonometric formula. When we call the angle A and the opposite side a we obtain:

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)\,(s-c)}{bc}}."$$

In this formula a, b, c are the sides of the triangle and $s = \frac{a+b+c}{2}$.

As mentioned above the protrusion of the upper jaw in respect to a horizontal plane or line determines the prognathism *in sensu stricto*. The situation of the basion has nothing to do with this. Opinions differ

about the way to express prognathism by means of angle, ratio or formula. The idea that the skull possesses a natural position is the cause of all these different opinions. The horizontal plane should indicate this natural position and this plane is also considered to be a fixed component around which the skull elements are placed. Everybody agrees, however, that the angle between the horizontal line and the line nasion-alveolar point indicates the exact degree of prognathism. But the question is how to determine the horizontal plane or line. SOLLAS (1933, p. 389) framed the problem in distinct words: "Throughout the history of craniology we may trace a constant endeavour to find some fixed line, *i.e.* a line which by maintaining a constant position in human skulls might provide us with a secure basis for comparison. The result of this endeavour would appear to be the discovery that the only constant feature of the skull is its variability".

In practice the Frankfort horizontal line has proved to be the most suitable for human skulls, but it is a matter of discussion whether the ear-eye-line can be used for other mammalian skulls. GIRARD (1947) and DE BEER (1947) did find a physiological horizontal line that comes up to the requirements. It is almost impracticable, however, for a great number of skulls. And then, anyhow, is there any sense in talking about the most natural position of the skull or head?

In connexion with the present methods of research it seems justified to drop the matter of the horizontal line of the skull altogether. In my opinion it is quite justified to express the degree of prognathism into the elements of the facial triangle considered as a mathematical figure. The facial triangle determined by the landmarks nasion, basion and alveolar point as early as the year 1860 has been used by KOSTER. RIVET (1909) and CAMERON (1928 *b*) in this triangle regarded the angle at the alveolar point as an exact measure for the degree of prognathism (*in sensu stricto*). But for explaining my own views on this triangle in respect to the problem of prognathism I want to make a few remarks about the facial triangle itself.

For the determination of the landmarks nasion, basion and alveolar point I consulted BUXTON & MORANT (1933). The dynamic development of the skull with its constant demolition and formation of bone leads to irregularities which at the same time are influenced by the attachment or action of muscle groups. The basion especially may cause difficulties in obtaining an exact determination as indicated by several investigators. Though PEARSON (1925) drew attention to the confusion caused by the names prosthion and alveolar point, his proposal for improvement of the definitions has not yet been generally accepted. It is, however, a more important question whether the cranial base: nasion-basion, has to be considered as a border line between facial and cerebral skull. It is a fact that the two landmarks nasion and basion offer great advantages for craniometrical investigations, because they can be determined easy

and accurately. It sometimes appears that the line nasion-basion is not ideal. The fact itself that the basion is located in the cerebral skull and the nasion undoubtedly belongs to the facial skull, indicates that there is something amiss with this border line. Moreover, this line does not indicate anything about the anatomical and morphological structure of the skull base. The differences of the angles between basisphenoid, praesphenoid and ethmoid therefore are not evident. The same difficulty, however, arises for the two other sides of the facial triangle.

The nasion is an extremely variable landmark. It depends on the purpose of the investigation of the skull whether the nasomaxillofrontal, the supraorbital, the glabella or the supraglabellar should be used as landmarks. But it does not seem advisable to use different landmarks in one investigation, when an identical phenomenon is to be characterized. KEIL (1933, tab. 73) uses the line alveolar point-nasomaxillofrontal to determine the profile angle, but for determination of the gnathic index ("Kiefer-Index") he uses the basion-supraglabellar line. Although he speaks of "Kiefer-Index" he gives an inverse ratio of the gnathic index of FLOWER. From KEIL's tab. 93 we may therefore conclude that his skulls of *Pongo* were orthognathous (when the classic definition is meant). Such a procedure certainly is not conducive for a comparison of the results of other investigators.

SELENKA (1898, pp. 50—51, figs. 56*a*—74) already pointed out the great variability of the nasal bones in *Pongo*. GAUL (1933, p. 367) also came to this conclusion, namely that the nasion varies too much for a reliable landmark. Research of my own confirmed these observations. In my research on the prognathism in the skull of *Pongo pygmaeus* I used the supraglabellar as a landmark. This choice of the supraglabellar has its advantages and its disadvantages like the other available landmarks. After much consideration I concluded that the two landmarks basion and supraglabellar are independent of the development of the facial skull. In this manner the difficulties in connexion with the ossification between the orbits in adult skulls have been eliminated. Therefore with this method juvenile and adult skulls can be compared more correctly.

As mentioned above, in the facial triangle the size of the angle at the alveolar point is a correct measurement for the degree of prognathism *in sensu stricto*. The smaller this angle the more the upper jaw is protruding or the more prognathous the skull. But one angle of the triangle does not mathematically determine the triangle and therefore the position of the nasion, supraglabellar, *etc.* and the basion are not determined by the angle at the alveolar point. Will also the prognathism *in sensu lato* be expressed, then a second angle is needed. I therefore propose the following method.

Figs. 1 and 2 represent the median sagittal section of a skull of

Pongo pygmaeus. Ba, alv.p. and sg are the landmarks basion, alveolar point and supraglabellar. Line *h* has been dropped perpendicularly from *sg* to the profile length basion-alveolar point. This perpendicular divides the profile length into two parts *p* and *q*. The length of *q* gives also the

Fig. 1

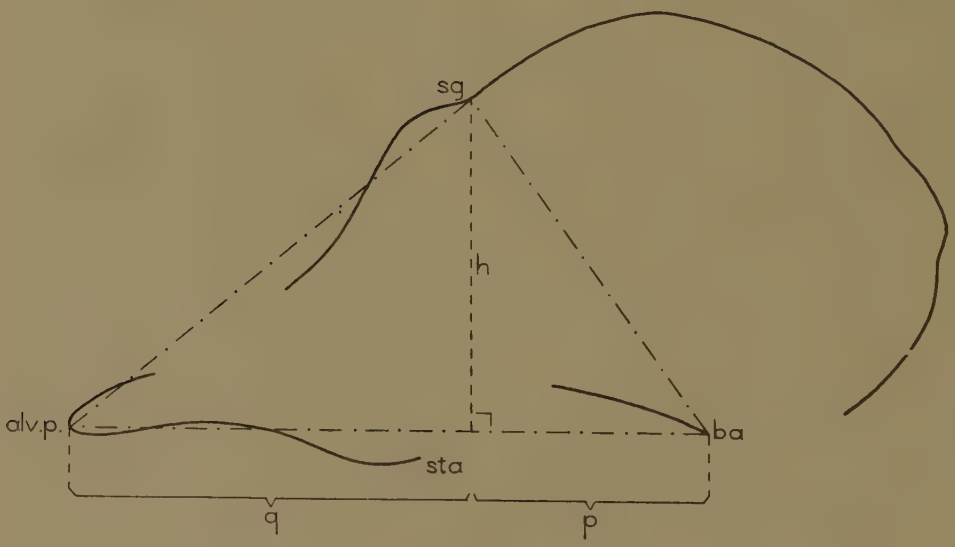

Fig. 2

absolute degree of prognathism *in sensu stricto*. By computing the ratio $\frac{100 \times q}{h}$ we have the relative degree of the prognathism *in sensu stricto*. This index has the same value as the cotangens of the angle at the alveolar point. It is obvious that the factor 100 does not at all alter the size of this angle. When the ratio values lie

a. between 0 and 57.735, the angle will be between 90° and 60°;
b. between 57.735 and 100, the angle stands between 60° and 45°;
c. between 100 and 173.205, the angle varies between 45° and 30°;
d. between 173.205 and ∞, then the angle decreases from 30° to 0°.

The length of p gives the absolute size of the position of the basion in respect to the projection of the supraglabellar on the profile length. The value of the index $\frac{100 \times p}{h}$ gives the cotangens of the angle at the basion. This index determines the position of the cerebral skull in respect to the other components of the skull. By using the same denominator h in these two new indices, we see how p and q are related, and how alveolar point, basion and supraglabellar (or another landmark in the median sagittal section of the skull) are located in respect to each other.

In this way the prognathism *in sensu stricto* and *in sensu lato* may be correctly determined and this method will be useful for comparative investigations of the skull.

Mr. W. Duk (Zoologisch Laboratorium, Leiden) constructed a new craniometrical instrument that enables to measure immediately the absolute length of p, q and h.

TABLE I

Measurements of h, p and q in mm, and the values for the two new indices in the female skulls of *Pongo pygmaeus* (Hopp.)

No. Hooijer	h	p	q	$I = \frac{100 \times p}{h}$	$I = \frac{100 \times q}{h}$
16	61	53	42	86.89	68.85
19	63	60	40	95.24	63.49
24	64	55	52	85.94	81.25
27	61	53	58	86.89	95.08
28	64	49	62	76.56	96.88
30	68	51	75	75.00	110.29
31	69	74	64	107.25	92.75
60	80	55	94	68.75	117.50
61	67.5	61	79	90.37	117.04
62	67	65.5	72.5	97.76	108.21
63	67	65	72	97.01	107.46
65	65	69	75	106.15	115.38
66	70.5	57	81	80.85	114.89
68	68	55	86	80.88	126.47
69	78.5	62	91	78.98	115.92
71	64	66	65	103.13	101.56
72	67.5	50	80	74.07	118.52
74	61.5	54	75	87.80	121.95
75	77.5	65	97	83.87	125.16
76	73	50	90	68.49	123.29
77	71	56	85	78.87	119.72
78	74	59	82	79.73	110.81
80	64	80	72	125.00	112.50

I want to thank Prof. H. Boschma (Leiden Museum), Prof. L. F. de Beaufort (Zoological Museum at Amsterdam) and Prof. J. E. W. Ihle (Zoological Laboratory at Amsterdam) who placed the specimens mentioned by Hooijer (1948, pp. 179—181) at my disposal.

Tables I and II give the measurements of h, p, q, as well as the values for the two new ratios in the examined skulls of *Pongo pygmaeus*. To arrive at a conclusion on the changes that take place in the skull during

TABLE II

Measurements of h, p and q in mm, and the values for the two new indices in the male skulls of *Pongo pygmaeus* (Hopf.)

No Hooijer	h	p	q	$I = \frac{100 \times p}{h}$	$I = \frac{100 \times q}{h}$
4	54	48	32	88.89	59.26
5	57	52	33	91.23	57.89
8	57	46	31	80.70	54.39
10	53	50	36	94.34	67.92
15	60	52	39	86.67	65.00
33	75	62	87	82.67	116.00
35	72	60	75	83.33	104.17
36	74	75	106	101.35	143.24
37	78	60	123	76.92	157.69
39	76	70	105	92.11	138.16
40	80	69	105	86.25	131.25
42	83	68	149	81.93	179.52
44	77	70	101	90.91	131.17
45	82	73	122	89.02	148.78
46	86	69	106	80.23	123.26
47	86	65	105	75.58	122.09
49	77	65	116	84.42	150.65
50	75	73	104	97.33	138.67
51	88	68	111	77.27	126.14
52	74	61	97	82.43	131.08
53	71	67	104	94.37	146.48
54	87	50	100	57.47	114.94
56	88	56	112	63.64	127.27
58	75	60	85	80.00	113.33
59	74	55	85	74.32	114.86

postnatal growth and development, it becomes necessary to use a statistical method. I am indebted to Mr. H. R. van der Vaart, who drew my attention to a new statistical method described by Mann & Whitney (1947), as well as for his assistance in computing the statistic U as used by these authors. This method is an extension of a test devised by Wilcoxon (1945). For the application of this method it is not necessary to assume the normality of the populations studied. The usual method of Student (see *e.g.*, Simpson & Roe, 1939, pp. 129-221) for testing the difference of the means of two samples needs this

assumption of normality. In this respect, the new method is to be preferred.

As only a small number of infantile and juvenile skulls could be examined it was decided to set these two together against the adult skulls. It is generally understood that *Pongo pygmaeus* does not form geographical races (SCHULTZ, 1941, pp. 79, 108; HARPER, 1945, p. 164). In my opinion, however, the examined skulls do not at all form an ideal series, because the skulls belong to specimens from several localities in the islands Borneo and Sumatra. For a comparative investigation of the skull or a research in growth and development of the skull the first condition is that specimens of the same population or of one locality are examined.

The statistic U and the deviation of the computed U-values were computed as indicated by MANN & WHITNEY. The probability of a deviation from the mean being greater in absolute value than the deviation found, or equal to it, was:

a. for the prognathism index $\frac{100 \times q}{h}$ of juvenile males and juvenile females equal to 0.26 (= 26 per cent);
b. for the prognathism index of juvenile females and adult females equal to 0.00045 (= 0.045 per cent);
c. for the prognathism index of juvenile males and adult males equal to 0.00032 (= 0.032 per cent);
d. for the prognathism index of adult females and adult males equal to 0.00031 (= 0.031 per cent).

Thus in this series of the skulls of *Pongo* the juvenile males and juvenile females also do not show a significant difference in the degree of prognathism *in sensu stricto*. The other mentioned groups of skulls, however, do not have the same degree of protrusion of the upper jaw. When we arrange these data in order of the increase of prognathism then we get the following picture for this series:

$$\text{juv.}\ \begin{matrix}\♂\♂\\ \♀\♀\end{matrix}\ \text{skulls} < \text{adult}\ \♀\♀\ \text{skulls} < \text{adult}\ \♂\♂\ \text{skulls}.$$

The research into the change of the angle at the basion showed an entirely different result. The probability of a deviation from the mean being greater in absolute value than the deviation found, or equal to it, was:

a. for the index $\frac{100 \times p}{h}$ of juvenile females and juvenile males equal to 1 (= 100 per cent);
b. for the index of juvenile females and adult females equal to 0.857 (= 85.7 per cent);

c. for the index of juvenile males and adult males equal to 0.4472 (= 44.72 per cent);
d. for the index of adult females and adult males equal to 0.5486 (= 54.86 per cent).

On account of these data we cannot conclude to a significant difference in the size of the angle at the basion in this series. During growth and development of the skulls the value of the angle at the basion remains fairly constant. In respect to the high percentage it is doubtful whether a larger series of the skulls of *Pongo* will lead to a significant difference in the size of the angle at the basion. The remark may be made that in this case the angle at the nasion determines also the degree of prognathism for the skull of *Pongo pygmaeus*. This supposition cannot be denied, but it does not diminish the value of my distinction in prognathism *in sensu stricto* and *in sensu lato*, because for every mammalian skull this supposition must be proved.

At the same time it is evident that the area where facial and cerebral skull meet, should be considered as fixed in the skull of *Pongo pygmaeus*. This is in agreement with KEIL's (1933, p. 218) conclusion, though he worked on altogether different lines: "Je weiter die untersuchten Entfernungen im Gebiete des Gehirnschädels liegen, desto weniger werden sie vom Gesichtswachstum erfasst".

In the literature on this subject I found only very few data referring to the prognathism of a series of the skulls of *Pongo*. More detailed data are given by HRDLIČKA (1906, p. 547) who used the gnathic index of FLOWER; by KEIL (1933, tab. 73 and 93) mentioned before, and by PITTARD & SEYLAN (1937, p. 108) who computed the index of FLOWER.

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DE F_5-ZAADGENERATIE VAN 1936 NA KRUISINGEN VAN TWEE ZUIVERE LIJNEN VAN PHASEOLUS VULGARIS. I

BY

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(Communicated by Prof. J. BOEKE at the meeting of Dec. 17, 1949)

Deze mededeling [1]) over de F_5-zaadgeneratie van 1936 sluit aan bij een publicatie in Genetica (1948) over de F_4-generatie 1935. De F_5-zaadgeneratie van 1936 is gegroeid uit uitgangsbonen van F_4-1935. Bovendien zijn ook in 1937 nog bonen van F_4-1935 uitgezaaid. Zij leverden dus F_5-1937. Al deze bonenopbrengsten zijn terug te voeren tot kruisingen van 1932. We beschikken voor ons onderzoek over metingen en wegingen van 175 bonenopbrengsten van F_5-1936 en bovendien nog van 43 bonenopbrengsten van F_5-1937. Ze zijn opgenomen in een tabel, die de uitgangsbonen van F_4-1935 en de gemiddelden van de bonenopbrengsten van F_5-1936 (en voor een klein aantal die van F_5-1937) bevat.

We berekenden de erfelijkheid als correlatie van de uitgangsbonen van F_4-1935 en de gemiddelden van de bonenopbrengsten van F_5-1936 (tab. 1). De positieve correlatie-coëfficienten, berekend met behulp van de formule van BRAVAIS zijn vrij groot, iets groter dan die van F_3-1934 en F_4-1935 en dan die van F_2-1933 en F_3-1934; de laatste zijn onregelmatig en vaak onbetekenend (tab. 2 en 3; ze behoren bij mijn publicatie in Genetica, 1947, doch zijn daar niet gepubliceerd). Dit groter worden van de correlatie-coëfficienten wijst op een toeneming van de erfelijkheid (raszuiverheid) in het materiaal. De correlatie-coëfficient van 2 op elkaar volgende generaties is in de zuivere lijnen 0 of onbetekenend (1947, Proceed. Vol. 50, 798). Hier is geen tegenstrijdigheid. Als we een boon van een bonenopbrengst van b.v. F_5-1936, die homozygoot zou zijn voor de onderzochte eigenschappen, konden uitkiezen en als we van haar bonenopbrengst alle bonen zouden voortkweken en aldus een groot materiaal er van verzamelen, dan zou blijken, dat de correlatie-coëfficient van de 2 op elkaar volgende generaties van dit materiaal ook 0 of onbetekenend is. Er is daar nl. volkomen regressie.

Zoals we, in de op elkaar volgende generaties, bonen om uit te zaaien, kozen, verrichtten we geen eigenlijke selectie-proeven. Onze keuze was niet ingesteld op selectie in op elkaar volgende generaties. We kozen in de op elkaar volgende jaren verschillende bonen van de laatste generatie uit, dus bonen met zeer grote of

[1]) Zie voor inleidende opmerkingen deze Proceedings, 52, 76 (1949).

kleine afmetingen, of met verschillend gewicht, ook wel middelmatig grote bonen, ook bonen, waarvan de ene afmeting groot, een andere klein was. We werden geleid door bijzonderheden van de bonen van de generatie, die aan de beurt was, om te worden voortgekweekt. We zijn niet uitgegaan van b.v. bonen met grote afmetingen en zijn niet, in de volgende generaties van deze bonen, blijven selecteren op grootte. Van iedere latere bonenopbrengst kennen we weliswaar de volledige ascendentie.

De grootste lengte van de bonen van F_5-1936 is $l = 18.1$ mm (dan $= 18.1$, 18.0, 17.9, 17.9 e.v.), de kleinste lengte is $l = 8.7$ mm (dan $= 8.8$, 8.9, 9.1, 9.2 e.v.). De variatie-breedte voor de lengte is dus 8.7—18.1 mm. Van de bonen van de I-lijn van 1936 is de variatie-breedte $= 8.5$—19.2 mm (dan $= 18.1$); van de bonen van de II-lijn is ze $= 8.0$ (dan 9.7)—13.3 mm. Van de I- en de II-lijn tezamen is dus de variatie-breedte 8.0 (dan 8.5)—19.2 (dan 18.1) mm. De variatie-breedte van de bonen van F_5-1936 verschilt dus niet of weinig van die van de bonen van de I- en van de II-lijn van 1936 samen.

De grootste gemiddelde lengte van de bonenopbrengsten van F_5-1936 is 16.2 (dan 16.0, dan 15.8, dan 15.7) mm; de grootste gemidd. lengte van bonenopbrengsten van de I-lijn van 1936 is 15.5 mm. De grootste gemidd. breedte van de bonenopbrengsten van F_5-1936 is 10.0 mm (dan volgt 9.9 en 9.6 mm); de grootste gemidd. breedte van de bonenopbrengsten van de I-lijn van 1936 is 9.5 mm. Er zijn dus bonenopbrengsten van F_5-1936 met een grotere gemiddelde lengte en breedte dan we bij bonenopbrengsten van de I-lijn van 1936 aantreffen. De kleinste gemidd. lengte van bonenopbrengsten van F_5-1936 is $l_m = 10.6$ mm de kleinste gemidd. breedte is $b_m = 7.5$ en 7.6 mm. De kleinste gemidd. lengte van bonenopbrengsten van de II-lijn van 1936 is $l_m = 10.1$ en 10.3 mm, de kleinste gemidd. breedte is $b_m = 7.5$ en 7.7 mm. Er zijn dus geen bonenopbrengsten van F_5-1936 met kleinere gemidd. lengte en breedte dan bij de bonenopbrengsten van de II-lijn van 1936.

De variabiliteit van de bonen van F_5-1936, zoals uitgedrukt door standaard-variatie en variatie-coëfficient is niet groter dan van bonen van de I-lijn en de II-lijn (tab. 4).

We hebben het materiaal ingedeeld volgens de 8 classen van ons tetrahybride schema (1947, deze Proceed. Vol. 50, p. 798) en gaven van iedere bonenopbrengst een korte beschrijving. Voor zover van deze bonenopbrengsten in 1937 bonen zijn voortgekweekt, is ook deze in 1937 gegroeide F_6-zaadgeneratie 1937 in de beschrijvingen opgenomen. Van dit uitgebreide manuscript geven we hier een overzicht. We willen vooral in het licht stellen, in hoeverre in de latere generaties, na de kruising, erfelijke variaties gevormd worden. Dit laat zich het scherpst onderzoeken voor bonenopbrengsten van cl 2 en van cl 7, omdat deze vergeleken kunnen worden met bonenopbrengsten van de I- en de II-lijn.

Cl. 1a. Form. L_1L_2B *Th, 25 gevallen* (tab. 7). De uitgangsbonen voor deze bonenopbrengsten hebben een zeer grote lengte (1 groter dan 15.5 mm),

een grote breedte (b = 8.6 mm en groter) en een grote dikte (th = 6.6 mm en groter). In één geval (pl. 783) is de formule van de gemiddelden ook L_1L_2B Th, cl 1a; in 7 gevallen is ze L_1l_2B Th, cl 1b.

Pl. 783 (tab. 5d en 6d). De uitgangsboon is van pl. 297, F_4-1935 (Fig. 1, K 33).

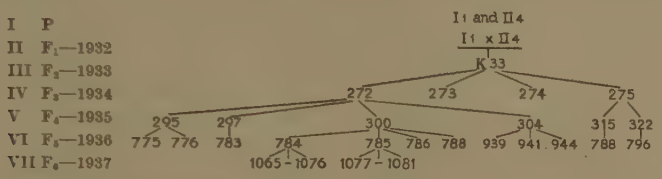

Fig. 1. Some beanyields, descendants of K 33, F_2-seedgeneration 1933.

De 1ste en 3de, laatste, boon van de peul van de uitgangsboon hebben ook een zeer grote lengte. Van de gemiddelden van de bonenopbrengst van pl. 297 is de gemiddelde lengte belangrijk kleiner dan van de meeste bonenopbrengsten van de I-lijn van 1935. Alle 3 gemiddelde indices zijn hoger dan van bonenopbrengsten van de I-lijn van 1935. Volgens de classificatie van de bonenopbrengst van pl. 297 zijn er zeer veel bonen in cl 1a en cl 1b, in cl 2b en is er een enkele boon in andere classen. Van één boon in cl 5 (1 B Th) is de lengte even onder de grenswaarde; van die van cl 6 (1 B th) heeft ze de grenswaarde. De ene boon in cl 7 is de laatste boon in de rij van de peul. Van de 3 bonen in cl 8 behoren er twee tot een peul met nog al onderling verschillende bonen. Het gaat bij deze enkele bonen waarschijnlijk om kleine genotypische verschillen. Ook de verschillen van de bonen van cl 1 en cl 2 zijn niet zeer groot. Dit blijkt uit de vergelijking der indices.

We hebben bij de bonenopbrengst van pl. 297 te doen met een gelijkmatig phaenotype met de form. L B Th (cl 1), waarbij de dikte der bonen niet veel verschilt van die van bonen met de form. L B th (cl 2). Van de uitgangsboon van pl. 297 voor pl. 783 is de dikte groter dan bij bonen van de I-lijn van 1935 voorkomt.

Volgens de classificatie behoort slechts één boon van de bonenopbrengst van pl. 783 niet tot cl 1. Deze boon is de laatste boon in de rij van de peul; ze behoort tot cl 2, form. L B th, doch de dikte is niet zeer klein (th = 6.3 mm).

Van de zeer grote gemiddelden van de bonenopbrengst van pl. 783 is de gemiddelde lengte groter dan de grootste gemiddelde lengte van vergelijkbonenopbrengsten van de I-lijn van 1936 (pl. 81, l_m = 15.5 mm). We hebben hier een voorbeeld van transgressieve variabiliteit. De grootste gemiddelde breedte van vergelijkbonenopbrengsten van de I-lijn van 1936 (pl. 31, b_m = 9.7 mm, n = 16; dan volgt pl. 25, b_m = 9.3 mm, n = 28) is duidelijk kleiner dan die van pl. 783. De gemiddelde dikte komt voor een vergelijking niet in aanmerking; immers, de formule van de bonen van de I-lijn is L B th. Door de verschillende dikte van de bonen van cl 1 en cl 2 is ook het gemiddelde gewicht van de bonenopbrengst van pl. 783 groter dan het gemidd. gewicht van bonenopbrengsten van de I-lijn.

In *7 van de 25 gevallen*, waar de formule van de uitgangsboon

L_1L_2 B Th, cl 1*a*, is, die van de gemiddelden der bonenopbrengsten L_1l_2 B th, cl 1b.

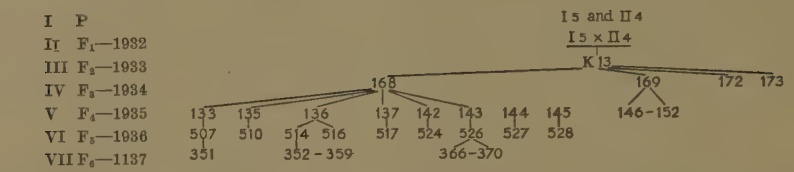

Fig. 2. Some beanyields, descendants of K_{13}, F_2-seedgeneration 1933.

Van *pl. 457* met uitgangsboon van pl. 121 (fig. 3, [2]) K 15) zijn alle bonen in cl 1 en cl 2. Van één peul met 3 bonen is de dikte der bonen klein (th = 6—5.7 mm). Deze 3 bonen van cl 2 komen geheel overeen met bonen van de I-lijn van 1936. We beschouwen ze als uitsplitsingen. Van *pl. 524* is de uitg. boon van pl. 142 (fig. 2, K 13). Volgens de classificatie van de bonenopbrengst van pl. 524 zijn er bonen in alle classen. Haar uitgangsboon van pl. 142 is in belangrijke mate heterozygoot voor de form. L B Th. Er zijn in 1937 van pl. 524 alle 6 bonen van een peul voortgekweekt. Ze leverden de pl. 360—365 F_6-1937. De uitgangsbonen hebben geringe verschillen. De classificaties der bonenopbrengsten van pl. 360—365 wijzen op kleine genotypische verschillen van de uitgangsbonen, die ook in hun phaenotype tot uitdrukking komen. De bevindingen van de 3 op elkaar volgende generaties kunnen met elkaar in overeenstemming gezien worden. *Pl. 517.* De uitg. boon is van pl. 137 (fig. 2, K 13). De form. van de gemiddelden van pl. 137 is L_1l_2 B Th. Volgens de classificatie van pl. 517 zijn er veel bonen in cl 1b (één in cl 1a) en in cl 2b. Van de bonenopbrengst van 19 bonen van pl. 517 zijn er in 1937 16 voortgekweekt; ze leverden de pl. 398—415, F_6-1937. Voor 6 planten is de form. van de uitg. boon L B Th, voor 5 planten L B th (4 behoren tot eenzelfde peul). Voor één plant is de form. van de uitg. boon L b th, cl 3 en voor de 4 overige planten is de form. L b th, cl 4 (3 behoren tot eenzelfde peul). Deze bonenopbrengsten hebben veel bonen in cl 1b, cl 2b en enkele in cl 4, ze verschillen onderling weinig, als uiting van kleine genotypische verschillen. De uitg. bonen voor de pl. 398—415 hebben alle een zekere mate van heterozygotie in overeenstemming met het phaenotype van pl. 517 en haar uitg. boon van pl. 137.

Pl. 776 (tab. 5d en 6d). De uitg. boon is van pl. 295 (fig. 1, K 33). Zie ook blz. 213. Volgens de classificatie zijn er in de bonenopbrengst van 40 bonen van pl. 776 17 bonen in cl 1a. Van de bonen in de overige classen zijn er enige minderwaardig „verschrompeld" en „misvormd". [3]) Pl. 776 is een goed voorbeeld van een bonenopbrengst met de form. L_1l_2 B Th, waarbij de uitgangsboon van pl. 295 met de form. L_1L_2 B Th, deze formule in een in hoge mate homozygote vorm heeft. Van pl. 295 (tab. 5c) is de formule der gemiddelden L B th, cl 2. Volgens de classificatie (tab. 6c) zijn er zeer veel bonen in cl 2 en ook vele in cl 1. Eén van deze bonen van (cl 1a) is de uitgangsboon voor pl. 776. De uitg. boon van pl. 272 voor pl 295 (tab. 5b en 6b) heeft de form. L_1L_2 B Th, cl 1a. Pl. 272 voert tot K 33 (tab. 5a en 6a). *Pl. 507*; de uitg. boon is

[2]) De Figg. 3, 6 en 7 worden hier niet gepubliceerd.

[3]) Tussen aanhalingstekens geplaatste woorden zijn ontleend aan de aantekeningen.

van pl. 133 (fig. 2, K 13). Volgens de classificatie heeft pl. 133 zeer veel bonen in cl 1a en 1b en een enkele in cl 2a en 2b; de form. van de gemiddelden is L_1l_2B th ($l_m = 15.2$ mm). De bonenopbrengst van pl. 507 is nog al samengesteld, evenals die van pl. 351, F_6-1937, die uit een uitgangsboon van pl. 507 gegroeid is. Ten slotte *pl. 441* (tab. 5d en 6d). De uitg. boon is van pl. 118 (fig. 3, K 15). Alle bonen van de peul van de uitg. boon hebben de form. L_1L_2 B th, cl 1a. Pl. 118 (tab. 5c en 6c) is een goede bonenopbrengst van het phaenotype L_1L_2 B Th. De uitgangsboon voor pl. 118 is van pl. 159, F_3-1934 (tab. 5b en 6b). Ze heeft de form. L_1L_2 B Th, cl 1a. Bij pl. 159 staat aangetekend „peulen iets te breed, bonen type I". De uitg. boon voor pl. 159 is van K 15 (tab. 5a en 6a) en behoort tot cl 8a. Ook de andere bonen van de peul van de uitg. boon hebben kleine afmetingen. De bonenopbrengst van K 15 bevat zeer veel kleine bonen.

Volgens de classificatie van de bonenopbrengst van pl. 441 (tab. 6d) zijn er zeer veel bonen in cl 1 en cl 2, ook meerdere in cl 1a en 2a. De boon met de grootste lengte van pl. 441 is in 1937 voortgekweekt; ze leverde pl. 279, F_6-1937 (tab. 5e en 6e). Van de 3 overige bonen van de peul van de uitg. boon behoort de 3de tot cl 1a ($l = 16.4$ mm) en de 2de en 4de tot cl 1b ($l = 15.3$ mm). De form van de gemiddelden van pl. 279 is $L_1 l_2$ B Th, cl 1b. Van de bonen met een zeer grote lengte is $l = 17.3$ mm (dan volgt $l = 16.7$, dan $l = 15.6$ mm). De bonenopbrengst van pl. 441, F_5-1936 toont ascenderend en descenderend erfelijkheid van bonen met de form. L B Th en van grote afmetingen.

Onder de gevallen van cl 1a, waarvan de formule van de gemiddelden van de bonenopbrengsten $L_1 L_2$ B Th (één geval) of $L_1 l_2$ B Th is, zijn er dus enkele, waarvan de uitgangsboon van F_4-1935 de form. L B Th, cl 1 in de homozygote of bijna homozygote vorm kan hebben. We vinden erfelijkheid van grote afmetingen en van de indices. Overigens blijkt uit de classificatie der bonenopbrengsten veel heterozygotie van de uitgangsbonen, wellicht in iets mindere mate dan in vorige generaties.

Er zijn *14 gevallen*, waar de uitgangsboon van F_4-1935 de form. $L_1 L_2$ B Th van cl 1a heeft en van de gemiddelden van de bonenopbrengsten van F_5-1936, de form. $L_1 l_2$ B th, cl 2b, is. Van 4 bonenopbrengsten *pl. 459, 451, 447* (fig. 3, K 15) en *785* (Fig. 1, K 33) is de gemiddelde lengte, $l_m = 15.4-15.1$ mm. De grootste gemiddelde lengte, $l_m = 15.4$ mm, is van pl. 447 en is iets kleiner dan de grootste gemidd. lengte ($l_m = 15.47$ mm, van pl. 81), die bij de bonenopbrengsten van de I-lijn van 1936 voorkomt. Van 6 bonenopbrengsten *pl. 449, 472, 474, 446, 438* (fig. 3, K 15) en *733* (fig. 7, K 4) is de gemidd. lengte $l_m = 15.0-14.6$ mm, en in de overige gevallen, pl. *452, 471* (fig. 3, K 15), *672* (fig. 7, K 4) en *939* (fig. 1, K 33) is $l_m = 14.2-13.8$ mm. Bijna alle 14 bonenopbrengsten hebben veel bonen in cl 2a en 2b, doch ook in cl 1a en 1b. Ze komen in meer en minder hoge mate met bonenopbrengsten van de I-lijn van 1936 overeen, ook de uitgangsbonen met bonen van

de I-lijn van 1935. Ook de bonenopbrengst van pl. 785 (tab. 5d) heeft zeer veel bonen in cl 2 (fig. 1, K 33 en blz. 213).

Zeer grote overeenkomst met bonenopbrengsten van de I-lijn heeft de bonenopbrengst van *pl. 438* met de uitg. boon van pl. 118 (fig. 3, K 15). De dikte van de uitgangsboon is echter groot (tab. 5d). Onder de vergelijkbonen van de I-lijn van 1935 zijn er niet met zulke grote dikte. Zoals de gemiddelden van pl. 438, F_5-1936 zijn, komen ze ook bij bonenopbrengsten van de I-lijn van 1936 voor. De classificatie (tab. 6d) verschilt niet van die van de I-lijn van 1936. Uit de krommen (fig. 4) van de dikten, de L Th- en de B Th-indices van de bonen van pl. 438 en de vergelijking met 3 bonenopbrengsten van de I-lijn van 1936, blijkt geen verschil: de krommen van pl. 438 liggen niet meer naar rechts dan die van de vergelijkplanten. We mogen zeggen, dat pl. 438 het phaenotype heeft van bonenopbrengsten van de I-lijn, d.w.z. van uitzonderlijke bonenopbrengsten van de I-lijn. Ook heeft de uitgangsboon van pl. 118, een te grote dikte.

Belangrijk is voor de vaststelling van de betekenis van het phaenotype van pl. 438, dat van pl. 438 een boon voortgekweekt is in 1937; ze leverde pl. 277, F_6-1937 (tab. 5e en 6e). De uitgangsboon is de boon met de grootste dikte en de hoogste indices van pl. 438. Ze behoort tot cl 1b met de form $L_1\, l_2$ B Th; de 4 overige bonen van de peul van de uitg. boon hebben alle de form. L B th, cl 2. De formule van de gemiddelden van pl. 278 is L B th. Bonen als de uitg. boon van pl. 438 voor pl. 277 komen onder de vgl. bonen van de I-lijn van 1936 ook voor. De gemiddelde dikte van de bonenopbrengst van pl. 277 is groter dan bij bonenopbrengsten van de I-lijn van 1937 voorkomt. Volgens de classificatie zijn er in de bonenopbrengst van pl. 277 te veel bonen in cl 1a en 1b. Het is duidelijk, dat de bonenopbrengst van pl. 277 niet beantwoordt aan een bonenopbrengst, waarvan de uitgangsboon de form. L B th in raszuivere vorm heeft. Niet alle bonen dus van de bonenopbrengst van pl. 438, — en die voor pl. 277 is de meest uitzonderlijke —, hebben blijkens het resultaat van het voortkweken in 1937, de form. L B th in de homozygote vorm. We hebben in pl. 438 niet een voorbeeld van een bonenopbrengst met het phaenotype L B th, waaraan een uitgangsboon met het genotype L B th in raszuivere vorm ten grondslag ligt.

Zeer grote overeenstemming, ten slotte met bonenopbrengsten van de I-lijn hebben, blijkens gemiddelden en classificatie, ook de bonenopbrengsten van pl. 672 en pl. 939 met uitg. bonen van pl. 210 en 304.

In *2 van de 25 gevallen*, pl. *521* en *531* (fig. 2, K 13) is de form. van de gemiddelden der bonenopbrengsten L b th, cl 4. Er is hier een groot verschil tussen de zeer grote afmetingen van de uitg. bonen (van pl. 139 is l = 17.2 mm, en van pl. 153 is l = 17.1 mm) en de kleine gem. afmetingen van de bonenopbrengsten (l_m = 13.5 en = 13.9 mm). De bonenopbrengst van pl. 531 heeft alle bonen in cl 4 en cl 2. Van de bonenopbrengst van *pl. 518* (fig. 2, K 13) ten slotte is de form. van de gemiddelden l B th, cl 6. Het is een kleine, niet kenmerkende bonenopbrengst.

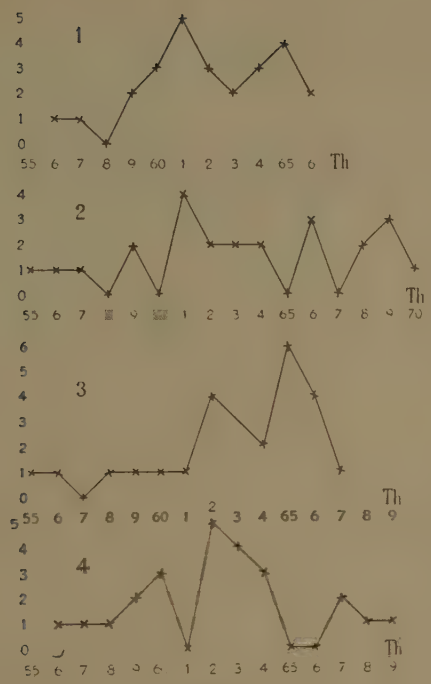

Fig. 4*a*

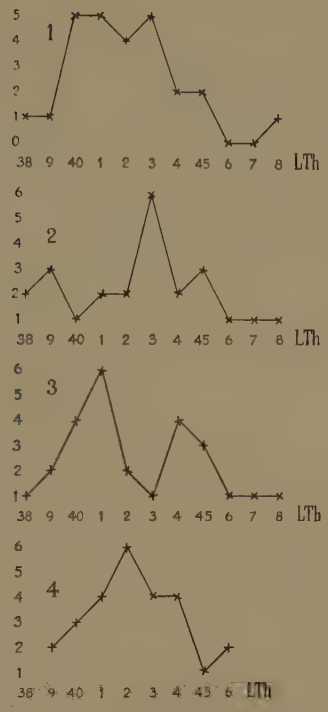

Fig. 4*b*

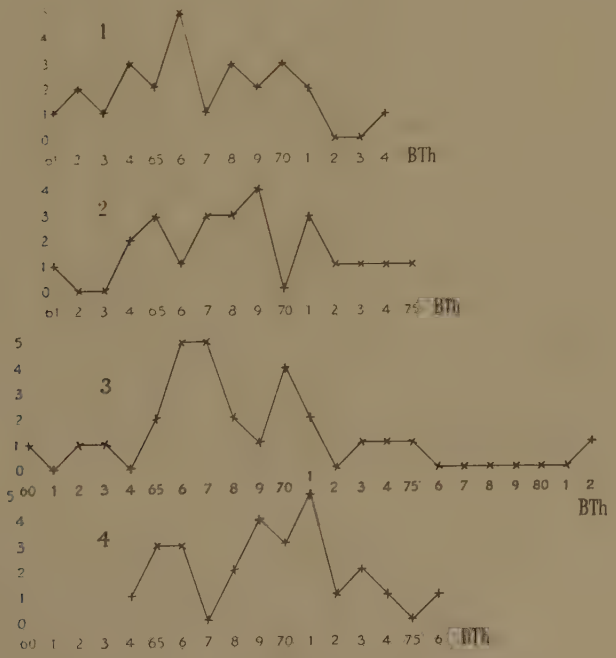

Fig. 4*c*

Fig. 4. The thickness, the L Th- and the B Th-index of pl. 438 F_5-1936 and of 3 comparable beanyields pl. 89, 92 and 94 of the I-line of 1936, Fig. 4*a*. The thickness. Fig. 4*b*. The L Th-index. Fig. 4*c*. The B Th-index. 1 = pl.438, n = 26, 2 = pl. 89, n = 24. 3 = pl. 92, n = 26. 4 = pl. 94, n = 26.

Cl 1b. Er zijn 34 *gevallen,* waar de uitgangsboon van F_4-1935 voor F_5-1936 de form. $L_1 l_2$ B Th, cl 1b, heeft (tab. 7). Van *5 bonenopbrengsten* is de formule der gemiddelden $L_1 l_2$ B Th, cl 1b.

Van *pl. 420* (fig. 5, K 16) is de bonenopbrengst samengesteld, met overwegend bonen in cl 1b; er is één boon in cl 7. In 1937 zijn 2 bonen van pl. 420 voortgekweekt. Ze leverden pl. 272 en 273, F_6-1937. Ook

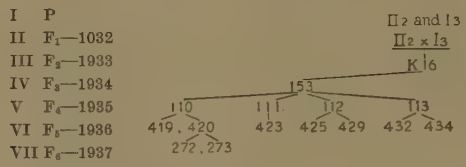

Fig. 5. Some beanyields, descendants of K 16, F_2-seedgeneration of 1933.

deze bonenopbrengsten zijn samengesteld. Voor pl. 272 is van de uitgangsboon van pl. 420, de form. L B Th, cl 1b en van pl. 272 is ze L b th, cl 4. De formule der gemiddelden is van beide bonenopbrengsten L B th, cl 2 met overwegend bonen in cl 1 en cl 2, enkele in cl 4 en ieder één boon in cl 7.

Van *pl. 489* (fig. 3, K 15 en tab. 5d en 6d) behoren alle bonen op één na van de bonenopbrengst tot cl 1b (één boon is in cl 2b). Eén boon met de form. L B Th is in 1937 voortgekweekt; ze leverde pl. 332, F_6-1937. De form. van de gemiddelden is L B th, cl 2, waarbij de dikte de grenswaarde heeft. De bonenopbrengst is samengesteld met veel bonen in cl 1, doch enige in cl 2, 3, 4 en 8. De uitg. boon van pl. 489 voor pl. 332 is dus heterozygoot voor de form. L B Th, cl 1b. Van de bonenopbrengst van pl. 489 met het phaenotype L B Th, cl 1b, is het genotype van de uitgangsboon van pl. 126, F_4-1935, niet homozygoot voor deze formule. Ook van *pl. 485* is de uitgangsboon van pl. 126. Er zijn zeer veel bonen in cl 1b; bovendien enkele in cl 7, 5, 3 en 8. Eén van de bonen van pl. 485 met de form. L B Th, cl 1b, leverde in 1937, pl. 328, F_6-1937 met zeer veel bonen in cl 1 en enkele in cl 7, 4, 3, 2 en 8. De bonenopbrengst van pl. 485 is meer samengesteld dan die van pl. 489; bevat enkele uitsplitsingen met de form. l b Th, cl 7, haar uitgangsboon van pl. 126 is meer heterozygoot.

Pl. 575, de uitg. boon is van pl. 257 (fig. 7, K 4). De bonenopbrengst is samengesteld en heeft zeer veel bonen in cl 1; ze komt overeen met de bonenopbrengst van pl. 257. De uitg. boon heeft de form. L B Th in heterozygote vorm. Van *pl. 556* is de uitg. boon van pl. 160 (fig. 6, K 10). De bonenopbrengst van pl. 160 heeft zeer veel bonen in cl 2. Pl. 556 heeft zeer veel bonen in cl 1. Dit is in overeenstemming met de form. L B Th, van de uitgangsboon. Deze uitg. boon, genomen uit een bonenopbrengst met zeer veel bonen in cl 2, form. L B th, mag dus als een uitsplitsing in de bonenopbrengst van pl. 160 beschouwd worden.

Van de 5 bonenopbrengsten met de form. L B Th van de uitgangsboon en van de gemiddelden van de bonenopbrengsten is pl. 489 (tab. 5d en 6d)

een goed voorbeeld van een bonenopbrengst van alleen bonen in cl 1b. Verder voortkweken moet in dergelijke gevallen leren in hoeverre we er met homozygotie van de form. L B Th te doen hebben.

Er zijn *8 bonenopbrengsten* van cl 1b, dus met de form. L B Th, cl 1b van de uitgangsboon, van welke de form. van de gemiddelden L B th, cl 2 is. Er zijn daarbij grensgevallen (pl. 819, fig. 8, K 64; pl. 572 en pl. 737, Fig. 7 (niet gepubl.), K 4; $th_m = 6.5$ en $= 6.4$ mm). Er zijn in deze gevallen veel bonen in cl 1 en cl 2. De gemiddelden van alle 8 bonenopbrengsten zijn niet groot: $l_m = 14.4-13.7$ mm, $b_m = 9-8.6$ mm. Van *pl. 1068* (fig. 6 (niet gepubl.), K 10) met de kleine gemidd. dikte $th_m = 5.9$ mm, zijn alle bonen op één na (deze is in cl 1b) in cl 2b en cl 4.

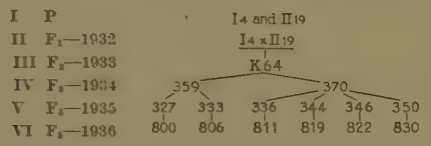

Fig. 8. Some beanyields, descendants of K 64, F_2-seedgeneration of 1933.

Het phaenotype van deze bonenopbrengst is L B th. De 2de boon van de peul van de uitgangsboon heeft een iets kleinere dikte; ze is de uitgangsboon voor *pl. 1069*. Onder de individuele bonen van pl. 1068 is er geen, zoals ze ook niet bij bonenopbrengsten van de I-lijn van 1936 voorkomen, ook pl. 1080 komt zeer met pl. 1068 en 1069 overeen. De bonenopbrengst van *pl. 580* komt zeer overeen met die van pl. 1068. Beide planten hebben uitgangsbonen van pl. 174, F_4-1935 (fig. 6, K 10). Er zijn bij pl. 580 zeer veel bonen in cl 2 en cl 4 en enkele in cl 1b en 8a. Van de bonenopbrengsten van 23 bonen van pl. 580 zijn er in 1937 21 uitgezaaid, daaronder ook 2 bonen van cl 1; ze leverden de pl. 489—509, F_6-1937. Deze bonenopbrengsten wijzen op een niet geheel raszuivere samenstelling van de bonenopbrengst van pl. 580. We nemen ook voor pl. 1068 aan, dat haar uitgangsboon van pl. 177 de formule L B th, in niet geheel raszuivere vorm heeft. *Pl. 941* (fig. 1, K 33) met de uitg. boon van pl. 304 heeft veel bonen in cl 1a en 1b, cl 2 en cl 4.

Van *4 bonenopbrengsten* van de 33 gevallen van cl 1b met de form. L B Th van de uitgangsboon is de formule van de gemiddelden L b th, cl 4.

De gemiddelde lengte van deze bonenopbrengsten (*pl. 726*, fig. 7, K 4; *pl. 533*, fig. 2, K 13; *pl. 540* en *585*, fig. 6, K 10) is niet groot ($l_m = 13.3-13.7$ mm). Opmerkelijk is de grote dikte van 3 dezer uitgangsbonen (7.1—7.4 mm). De gemiddelden van de bonenopbrengsten komen zeer overeen. Volgens de classificatie zijn er veel bonen in cl 2 en cl 4; van *pl. 533* (tab. 5d en 6d) bijna uitsluitend. De bonenopbrengst van pl. 153, waarvan voor pl. 533 de uitgangsboon genomen is, heeft zeer veel bonen in cl 1 en cl 2, ook meerdere in cl 4, behoort tot het gebied van cl 2, form. L B th (zie ook pl. 531, blz. 216). We nemen voor de uitgangsbonen niet-erfelijke variabiliteit en heterozygotie aan.

In *3 van de 33 gevallen* (*pl. 598*, fig. 6, K 10, *pl. 629*, fig. 3, K 15 en *pl. 503*, fig. 2, K 13) met de form. L B Th, cl 1b van de uitgangsbonen, is de form. der gemiddelden van de bonenopbrengsten l b Th, cl 7. *Pl. 598* behoort, blijkens de classificatie der bonenopbrengst tot het gebied van cl 7. De grote dikte (th = 7.3 mm) van de uitg. boon van pl. 181 past hierbij. *Pl. 629* en haar uitg. boon van pl. 193, voert tot een dergelijke conclusie.

Belangrijk is het 3de geval. Van *pl.* 503 (fig. 3, K 15 en tab. 5d en 6d) bevat de bonenopbrengst alleen bonen in cl 7 en cl 8b. De 2de boon van de peul van de uitg. boon is de uitgangsboon van pl. 504 (blz. 221). De 3de, laatste, boon van de peul heeft de form. l b Th, cl 7. Dergelijke gemiddelden als die van pl. 503, komen bij vergelijk-bonenopbrengsten van de II-lijn van 1936 niet voor: ten opzichte van de gemiddelde lengte zijn de gem. breedte en de gem. dikte te klein, de gem. L B- en B Th-indices te laag (vgl. deze Proceed. 1949, tab. 4b). Onder de individuele bonen zijn er, die geheel met bonen van de II-lijn van 1936 overeenkomen. Twee bonen van pl. 503, F_5-1936 zijn in 1937 voortgekweekt; ze leverden pl. 349 en 350, F_6-1937.

Pl. 349, F_6-1937, (tab. 5e en 6e). De uitg. boon van pl. 503 voor pl. 439 heeft de form. l b Th, cl 7 en behoort tot een peul van 4 bonen, alle met de form. l b Th. Zoals de uitg. boon is, komen er ook bonen van de II-lijn van 1936 voor. De B Th-index is niet hoog. De gemiddelden van de bonenopbrengst van pl. 349 zijn klein; ook de gemidd. dikte is niet groot en de gemidd. B Th-index niet hoog. Dergelijke kleine gemidd. breedten en dikten komen bij de gemiddelde van de bonenopbrengsten van de II-lijn van 1937 niet voor. Volgens de classificatie behoren alle bonen van de bonenopbrengst van pl. 349 tot cl 8b en 8c. We hebben hier te doen met een bonenopbrengst uit het gebied van cl 7, form. l b Th, doch met opmerkelijk kleine afmetingen der bonen. Er is transgressieve variabiliteit van minusvariaties. De transgressieve variabiliteit, nemen we ons voor, nog in samenhang te bezien.

Pl. 350, F_6-1937. De uitgangsboon van pl. 503 voor pl. 350 heeft de grootste breedte van alle bonen op één na van de bonenopbrengst. Ze komt zo ook onder de bonen van de II-lijn van 1936 voor (deze Proceed. 1949, tab. 8). De form. van de gemiddelden van pl. 350 is l b th, cl 8 met vooral een kleine gemidd. dikte. De kleine bonenopbrengst van 19 bonen heeft zeer veel bonen in cl 8. De grootste lengte der bonen is l 13.7 mm. Ze bevat „zeer vlekkige", „vlekkige en gerimpelde", „zeer gerimpelde" en „misvormde" bonen. De kleine en slechte bonenopbrengst is samengesteld. Ze behoort tot het gebied van cl 7 met bonen in cl 6 en 5 (volgens de indices; het zijn bonen van cl 8c).

In *13 van de 34 gevallen*, waar de formule van de uitgangsboon van F_4-1935 voor F_5-1936, $L_1\, l_2$ B Th, cl 1b is, is de formule van de gemiddelden van de bonenopbrengsten van F_5-1936, l b th, cl 8.

Pl. 725 met uitg. boon van pl. 227, (fig. 7, K 4) heeft een samengestelde bonenopbrengst met ook enkele bonen in cl 7. Van de *pl. 1066*

en *1061* (fig. 6, K 10) zijn de uitg. bonen van pl. 174, waarvan zoveel uitg. bonen genomen zijn (blz. 219); ze behoren tot het gebied van cl 2, form. L B th). *Pl. 748* met uitgangsboon van pl. 234 (fig. 7, K 4) behoort tot het gebied van cl 4, form. L b th. Van de *pl. 516* met uitg. boon van pl. 136 (fig. 2, K 13), *pl. 535* met uitg. boon van pl. 154 (fig. 6, K 10) en *pl. 944* met uitg. boon van pl. 304 (fig. 1, K 33) volstaan we met de vermelding. (Pl. 944 heeft 3 peulen met bonen met zeer kleine afmetingen en zeer klein gewicht (gew. = 18—38 cG.). Van de laatste 5 of 6 van de 13 gevallen is de lengte van de uitg. boon met de form. L B Th niet groot; l = 13.7—13.1 mm. De uitgangsbonen naderen daardoor in grootte tot cl 8, d.i. tot die van hun bonenopbrengsten. De dikte van de uitgangsbonen is vaak groot; in de classificatie treffen we enige bonen van cl 7 aan. Voor *pl. 504* is de uitg. boon van pl. 131 (fig. 3, K 15 en tab. 5d en 6d) evenals voor pl. 503, waarmee ze zeer overeenkomt (blz. 220). Er zijn enige bonen in cl 7 en de overige zijn alle in cl 8, vooral cl 8b. Een dergelijke kleine gemiddelde breedte en dikte tezamen met de gemidd. lengte, treffen we bij bonenopbrengsten van de II-lijn van 1936 niet aan. De grote dikte van de uitgangsboon van pl. 131 voor pl. 504 wijst mede er op, dat pl. 504 behoort tot het gebied van cl 7.

Pl. 419. De uitg. boon is van pl. 110 (fig. 5, K 16). Zie ook pl. 420 (blz. 218). Er zijn zeer veel bonen in cl 8; ze zijn „slecht, onvolgroeid". Van pl. 419 zijn 2 bonen in 1937 uitgezaaid; ze leverden de pl. 267 en 269, F_6-1937. De 2 uitg. bonen zijn van een peul met 5, alle overeenkomstige kleine bonen (gew. 27.5—37 cG). De gemiddelden van de bonenopbrengsten van de pl. 267 en 269 zijn klein. Er zijn bonen in bijna alle classen. Er is hier enige erfelijkheid van bonen met kleine afmetingen.

Van de *pl. 658*, met uitg. boon van pl. 202 (fig. 6, K 10), *pl. 423* met uitg. boon van pl. 111, *pl. 434* met uitg. boon v. pl. 113 (beide fig. 5, K 16) en *pl. 644* met uitg. boon van pl. 200 (fig. 6, K 101) volstaan we met de vermelding. Van een uitg. boon van pl. 202, F_2-1935, groeide in 1937 *pl. 696*, F_5-1937, met vrij samengestelde bonenopbrengst.

De vele gevallen, die we bespraken, waar de uitgangsboon de form. L B Th, (cl 1a en 1b) heeft en de formule van de gemiddelden de form. L B Th of ook andere formules is, wijzen op de erfelijkheid van bonen met deze formule, doch ook op de grote mate van heterozygotie, die bonen met deze formule eigen is.

Cl 2a. 12 gevallen. De formule van de uitgangsboon is L_1L_2 B th, cl 2a (tab. 7). In 5 gevallen is de formule van de gemiddelden van de bonenopbrengsten $L_1 l_2$ B th, cl 2b.

Pl. 751. De uitg. boon is van pl. 274 (fig. 7, K 4). Dergelijke bonen komen ook bij de I-lijn van 1935 voor. Dergelijke gemiddelden als van de bonenopbrengst van pl. 751 treffen we ook aan bij de gemiddelden van de bonenopbrengsten van de I-lijn van 1936. Dit geldt ook voor de classificatie. Onder de individuele bonen van pl. 751 is er geen, zoals ze ook niet bij de I-lijn van 1936 voorkomen. Volgens de gegevens van de bonenopbrengst van pl. 751 is de uitgangsboon voor pl. 751 van pl. 274,

F_4-1935 homozygoot voor de form. L B th als van bonen van de I-lijn. Pl. 274, F_4-1935 is in een vorige publicatie beschreven (Genetica, 1948). In *pl. 730* met de uitgangsboon van pl. 228 (fig. 7, K 4) hebben we een tweede voorbeeld van de gevallen van cl 2a, dat geheel overeenkomt met bonenopbrengsten van de I-lijn van 1936. Van de bonenopbrengst van pl. 228, F_4-1935 (Genetica, 1948) is de uitgangsboon voor pl. 730 een uitsplitsing met de form. L B th in homozygote of bijna homozygote vorm.

Pl. 666. De uitg. boon is van pl. 206 (fig. 7, K 4 en tab. 5d en 6d). Dgl. bonen komen ook bij de I-lijn van 1935 voor. Dit geldt niet geheel en al voor de gemiddelden van pl. 666 ten opzichte van de gemiddelden van bonenopbrengsten van de I-lijn van 1936; de gemidd. dikte is iets te groot. Ook volgens de classificatie zijn er te veel bonen in cl 1, form. L B Th. Er zijn in de bonenopbrengst van pl. 666 er geen, zoals ze ook niet bij de bonenopbrengsten van de I-lijn van 1936 voorkomen, maar er zijn te veel van deze uitzonderingsbonen. Van de bonenopbrengst van pl. 666 zijn in 1937 8 bonen uitgezaaid, die de pl. 1039—1046, F_6-1937 leverden. Twee uitgangsbonen hebben de form. L B th, cl 2a en 2b en de form. der gemiddelden van de bonenopbrengsten cl 4 en cl 2, (pl. 1046, tab. 5e en 6e). Deze bonenopbrengsten komen geheel overeen met vgl.-bonenopbrengsten van de I-lijn van 1937. Dit geldt ook voor pl. 1044, waarvan de form. van de uitg. boon L b th, cl 4 is (pl. 1044, tab. 5e en 6e) en die der gemiddelden L B th, cl 2. In 4 gevallen is de form. van de uitg. boon $L_1 L_2$ B Th cl 1a en $L_1 l_2$ B Th cl 1b en die van de gemiddelden der bonenopbrengsten $L_1 l_2$ B th, cl 2b en $L_1 L_2$ B Th, cl 1a (pl. 1039, tab. 5e en 6e). De eerstgenoemde komen overeen met bonenopbrengsten van de I-lijn van 1937. Dergelijke grote gemiddelden als van pl. 1039 komen als hoge uitzondering ook onder de gemiddelden van de ruim 90 bonenopbrengsten van de I-lijn van 1937 voor. De classificatie van pl. 1039 stemt ook zeer overeen met de classificatie van een uitzonderlijke bonenopbrengst van de I-lijn van 1937. In één geval is de form. van de uitgangsboon en die van de gemiddelden der bonenopbrengst l b th, cl 8, (tab. 5e en 6e, pl. 1045). Ook dit geval behoort tot het gebied van cl 2. Er is hier erfelijkheid van kleine afmetingen. Waarschijnlijk is de uitgangsboon van pl. 206, F_4-1935 voor pl. 666, F_5-1936 niet geheel homozygoot voor de form. L B th van de bonen van de I-lijn. Ook van de *pl. 719* en *675* (fig. 7, K 4) komen de bonenopbrengsten zeer overeen met die van de I-lijn.

In *3 gevallen* is de form. van de uitg. boon $L_1 L_2$ B th, cl 2a en die van de gemiddelden van de bonenopbrengsten L B Th cl 1, waarvan in één geval $L_1 L_2 B_1 b_2 Th_1$, th_2, cl 1a.

Pl. 450. De uitg. boon is van pl. 119 (fig. 3, K 15). Ze is een uiterste plus-variant (gew. = 108.4 cG). Volgens de aantekeningen is ze „wat gevlekt"; de 2de boon van dezelfde peul is „donker en wat verschrompeld"; de 3de en laatste — zonder aantekening — is ook zeer groot (gew. = 102.6 cG) en is de uitg. boon voor pl. 452 (blz. 215). Zoals de uitg. boon van pl. 119 voor pl. 450 is, zijn er ook enkele bonen van de I-lijn van 1935. Zo grote gemiddelde afmetingen als van pl. 450 komen bij de bonenopbrengsten van de I-lijn van 1936 niet voor. Er is hier transgressieve variabiliteit. Er zijn wél bonenopbrengsten met overeenkomstige gemidd. indices bij de I-lijn van 1936. De „spurious correlation" heeft hier ook betekenis. Daardoor is ook het beeld van de classificatie beheerst. Het gemidd. gewicht van de bonenopbrengst van

pl. 450 is belangrijk groter dan het grootste gemidd. gewicht van bonenopbrengsten van de I-lijn van 1936. De grootste lengte van de individuele bonen van de bonenopbrengst van pl. 450 is l = 18.1 mm, die van de individuele bonen van de I-lijn is l = 21.6 (dan = 18.1 mm). De op één na langste boon van F_5-1936 is ook van pl. 450; de lengte is l = 18.0 mm. Deze boon is in 1937 voortgekweekt en leverde pl. 292, F_6-1937 (tab. 5e en 6e). Een 2de uitgangsboon van pl. 450, leverde pl. 289, F_6-1937 (tab. 5e en 6e). Zoals de uitgangsboon voor pl. 289, zijn er ook onder de I-bonen van 1936. Van de bonenopbrengst van pl. 289 is de form. der gemiddelden L_1 L_2 B Th cl 1a. De gemiddelde lengte (l_m = 16.3 mm) is zo groot als de grootste gemiddelde lengte van de bonenopbrengsten van de I-lijn van 1937, (pl. 58, n = 18, l_m = 16.5 mm). Dit geldt ook m.m. voor de gemidd. breedte en de gemidd. dikte van pl. 289. Zoals de gemidd. indices van pl. 289 zijn, komen ze ook onder de gemidd. indices van de bonenopbrengsten van de I-lijn van 1937 voor. Als we de gemiddelden van de bonenopbrengsten van pl. 289 en 292 vergelijken met bonenopbregnsten van de I-lijn van 1937, dan blijken de gemidd. dikten toch iets te groot te zijn (van de pl. 58 en 66 van de I-lijn van 1937 met l_m = 16.5 en 16.4 mm, zijn de gemiddelde dikten, th_m = 6.5 en 6.6 mm) De bonenopbrengsten van de pl. 289 en 292, F_6-1937 zijn niet geheel gelijk te stellen met bonenopbrengsten van de I-lijn van 1937. Er is in de as- en descendentie van pl. 450 erfelijkheid van de grote afmetingen en de indices. Ze behoort tot het gebied van de I-lijn.

Van *pl. 708* (fig. 7, K 4) en *pl. 510* (fig. 2, K 13) met de form. L_1 l_2 B Th, cl 1b der gemiddelden is de bonenopbrengst samengesteld. Er zijn bonen in alle classen. Er zijn 4 gevallen, waarvan de formule der gemiddelden van de bonenopbrengsten L b th, cl 4 is (*pl. 759* en *673*, fig. 7, K 4; *pl. 559* en *548*, fig. 6, K 10). Deze 4 bonenopbrengsten hebben veel bonen in cl 2, 4 en 8 en behoren ook blijkens hun ascendentie tot het grensgebied van cl 2 en 4. We vermelden nog de *pl. 669—672*, F_5-1937, waarvoor de uitgangsbonen in 1937 van pl. 210, F_4-1935 genomen zijn.

TABLE 1. Heredity as correlation of the F_4-seedgeneration of 1935 and the F_5-seedgeneration of 1936.

	m ± m	σ ± m	V	gr. var.	sm. var.	var r.	m ± m	σ ± m	V	gr. var.	sm. var.	var. r.	
	168 initial beans of the F_4-seed-generation of 1935. The means						The averages of the 168 beanyields of the F_5-seedgeneration of 1936. The means						r [3]) ± m
L [1])	13.99 ± 0.14	1.89 ± 0.1	13.5	19.1	9.7	9.4	13.28 ± 0.1	1.24 ± 0.7	9.3	16.2	10.5	5.7	+ 0.65 ± 0.04
B	9.06 ± 0.1	0.92 ± 0.07	10	11.7	7.1	4.6	8.57 ± 0.05	0.59 ± 0.03	7	10.0	6.9	3.1	+ 0.53 ± 0.06
Th	6.63 ± 0.05	0.58 ± 0.03	8.8	8.2	4.7	3.5	6.23 ± 0.03	0.41 ± 0.02	6.6	7.3	4.7	2.6	+ 0.35 ± 0.07
W [2])	60.3 ± 1.5	18.8 ± 1	30	112	26	86	51.4 ± 0.8	9.95 ± 0.6	19.3	83	30	53	+ 0.57 ± 0.05
L B	65 ± 0.3	4.3 ± 0.2	6.6	76.5	53.5	23	64.8 ± 0.3	3.4 ± 0.2	5.2	74	58.5	15.5	+ 0.73 ± 0.04
L Th	48 ± 0.5	6.5 ± 0.4	13.5	64.5	33	31.5	47.5 ± 0.4	5 ± 0.3	10	61.5	36	25.5	+ 0.71 ± 0.04
B Th	74 ± 0.5	7 ± 0.4	9.4	89	54	35	73 ± 0.4	4.8 ± 0.3	6.6	87	59.5	27.5	+ 0.62 ± 0.05

[1]) In 0.1 mm. [2]) In cg. [3]) The coefficient of correlation after the formula of Bravais.

TABLE 2. Heredity as correlation of the F_3-seedgeneration of 1934 and the F_4-seedgeneration of 1935.

	m ± m	σ ± m	V	gr. var.	sm. var.	var. r.	m ± m	σ ± m	V	gr. var.	sm. var.	var. r.	
	108 initial beans of the F_3-seed-generation of 1934. The means.						The averages of the 108 beanyields of the F_4-seedgeneration of 1935. Means.						r [3]) ± m
L [1])	13.54 ± 0.15	1.6 ± 0.1	11.9	17.3	9.6	7.7	13.5 ± 0.11	1.24 ± 0.08	9.2	16.3	11.0	5.3	+ 0.6 ± 0.06
B	8.79 ± 0.06	0.68 ± 0.04	7.7	10.9	6.7	4.2	8.79 ± 0.05	0.51 ± 0.04	6	10.3	7.7	2.6	+ 0.4 ± 0.07
Th	6.68 ± 0.06	0.59 ± 0.04	9	8.2	4.5	3.7	6.47 ± 0.04	0.4 ± 0.03	6.2	7.2	5.4	1.8	+ 0.34 ± 0.08
W [2])	56.3 ± 1.3	13.4 ± 0.9	23.7	94	21	73	55.5 ± 1	10 ± 0.7	18	85	39	46	+ 0.31 ± 0.08
L B	65.3 ± 0.4	4.8 ± 0.3	7.3	75	53	22	65.5 ± 0.3	3.4 ± 0.2	5.2	77	60	17	+ 0.51 ± 0.07
L Th	49 ± 0.6	6.3 ± 0.4	12.8	62	33	29	48.4 ± 0.5	5.6 ± 0.4	11.7	65	38	27	+ 0.72 ± 0.05
B Th	76.5 ± 0.7	7 ± 0.5	9.2	98	54	44	74.2 ± 0.5	5.5 ± 0.4	7.4	85	62	23	+ 0.6 ± 0.06

[1]) In 0.1 mm. [2]) In 0.1 cg. [3]) The coefficient of correlation after the formula of Bravais.

TABLE 3. Heredity as correlation of the F_2-seedgeneration of 1933 and the F_3-seedgeneration of 1934.

	m ± m	σ ± m	V	gr. var.	sm. var.	var. r.	m ± m	σ ± m	V	gr. var.	sm. var.	var. r.	
	87 initial beans of the F_2-seed-generation of 1933. The means.						The averages of the 87 beanyields of the F_3-seedgeneration of 1934. Means.						r [3]) ± m
L [1])	14.24 ± 0.1	1.01 ± 0.08	7.1	17.1	11.5	5.6	13.62 ± 0.1	1.00 ± 0.08	7.3	16.1	10.9	5.3	+ 0.44 ± 0.09
B	8.5 ± 0.06	0.52 ± 0.04	6	9.7	6.8	2.9	8.77 ± 0.04	0.4 ± 0.03	4.5	9.8	7.8	2	+ 0.11 ± 0.09
Th	6.66 ± 0.04	0.34 ± 0.03	5.2	7.5	5.5	2	6.69 ± 0.04	0.37 ± 0.03	5.5	7.6	5.8	1.8	+ 0.1 ± 0.1
W [2])	54.9 ± 0.9	8.6 ± 0.7	15.4	75	30	45	55.1 ± 0.8	7.9 ± 0.6	14	75	36	39	+ 0.2 ± 0.1
L B	59.6 ± 0.3	2.7 ± 0.2	4.5	66.5	53	13.5	64.6 ± 0.3	2.95 ± 0.2	4.6	72	59	13	+ 0.1 ± 0.1
L Th	46.6 ± 0.3	3.1 ± 0.2	6.6	55	37	18	49.5 ± 0.5	3.8 ± 0.4	7.5	58.5	38.5	20	+ 0.14 ± 0.1
B Th	78.6 ± 0.5	4.5 ± 0.3	5.7	91	64	27	76.9 ± 0.4	3.9 ± 0.3	5	84	65.5	18.5	+ 0.2 ± 0.1

[1]) In 0.1 mm. [2]) In 0.1 cg. [3]) The coefficient of correlation after the formula of Bravais.

TABLE 4. The mean length, the standard deviation and the variation coefficient of some beanyields of F_5-1936 and of the I- and the II-line of 1936.

Pl	n	M ± m	σ ± m	V	gr. var.	sm. var.	var. r.
			F_5-1936				
450	28	16.16 ± 0.2	0.91 ± 0.12	5.7	18.1	14.6	3.5
449	27	14.63 ± 0.3	1.29 ± 0.18	8.8	16.8	12.8	4.0
452	27	14.22 ± 0.2	1.05 ± 0.15	7.4	16.3	12.4	3.9
1053	98	13.97 ± 0.1	1.17 ± 0.08	8.4	16.1	9.9	6.2
1061	36	12.85 ± 0.2	1.27 ± 0.15	10	16.1	10.0	6.1
800	32	11.96 ± 0.1	0.8 ± 0.1	6.7	14.1	10.4	3.7
565	25	11.56 ± 0.1	0.56 ± 0.08	4.8	12.6	10.5	2.1
565	52	11.48 ± 0.1	0.67 ± 0.08	5.9	12.8	10.1	2.7
492	24	10.45 ± 0.1	0.36 ± 0.05				
			I-line				
22 S	23	14.64 ± 0.14	0.66 ± 0.1	4.6	16.9	13.9	3
68	63	13.33 ± 0.13	1.03 ± 0.09	7.7	16.2	10.2	6
total	1617	14.22 ± 0.02	0.99 ± 0.02	7	18.1	10.2	7.9
total	2350	14.28 ± 0.02	1.15 ± 0.02	8.1	19.2	8.5	11.7
			II-line				
113 S	36	12.01 ± 0.12	0.7 ± 0.08	5.8	13.0	10.6	2.4
101 S	46	11.08 ± 0.12	0.82 ± 0.09	7.4	12.3	8.0	4.3
117 S	65	11.66 ± 0.08	0.66 ± 0.06	5.6	13.1	9.9	3.2
total	605	11.16 ± 0.03	0.81 ± 0.02	7.3	13.3	7.3	6

ON REAL PERIODIC FUNCTIONS AND FUNCTIONS WITH PERIODIC DERIVATIVES

BY

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§ 1. *Introduction.*

B. MEULENBELD and the author of the present paper recently introduced the notions of C^I-, C^{II}- and C^{III}-uniform distribution (mod 1) of measurable functions $f(t)$ defined for $t \geqslant 0$ [1]. We repeat the definition of C^I-uniform distribution (mod 1).

Definition.

For any given pair of real numbers α, β, satisfying $0 \leqslant \alpha \leqslant \beta \leqslant 1$, we denote by $\theta(\alpha, \beta; f(t))$ ($f(t)$ being a measurable function of t ($t \geqslant 0$)) the following characteristic function

$$\theta(\alpha, \beta;\ f(t)) = 1 \quad \text{if } \alpha \leqslant f(t) < \beta \pmod 1,$$
$$\theta(\alpha, \beta;\ f(t)) = 0 \quad \text{elsewhere.}$$

Then, if for any pair α, β the function $f(t)$ satisfies

$$\lim_{T\to\infty} (1/T) \int_0^T \theta(\alpha, \beta;\ f(t))\, dt = \beta - \alpha,$$

$f(t)$ is said to be C^I-uniformly distributed (mod 1).

In the note mentioned above some tests are deduced in order to study the C-distribution (mod 1) of any measurable function. The most useful test is the following one:

C^I-test.

Necessary and sufficient for the C^I-uniform distribution (mod 1) *of the measurable function $f(t)$ is that, for any integer $h \neq 0$, $f(t)$ satisfies*

$$\lim_{T\to\infty} (1/T) \int_0^T e^{2\pi i h f(t)}\, dt = 0.$$

The present note is concerned with the study of functions which are periodic or have a first or second derivative which is periodic.

§ 2. *Periodic functions.*

In my dissertation [2] I proved the following

Theorem 1.

A periodic function $f(t)$, defined for $t \geqslant 0$, with period p, is C^I-uniformly distributed (mod 1) *if and only if for any integer $h \neq 0$ $f(t)$ satisfies*

$$\int_0^p e^{2\pi i h f(t)}\, dt = 0. \tag{1}$$

The simple proof runs as follows: Applying the test it is no restriction to limit ourselves to the case that T increases through a sequence of positive numbers, increasing with a fixed difference.

For every integer $m \geqslant 1$ we have now

$$(1/mp) \int_0^{mp} e^{2\pi i h f(t)}\, dt = (1/mp) \sum_{l=0}^{m-1} \int_0^{p} e^{2\pi i h f(t+lp)}\, dt = (1/p) \int_0^{p} e^{2\pi i h f(t)}\, dt.$$

Remark.

In [3] it has been shown that the function $\sin t$ is not C^1-uniformly distributed (mod 1). In this case the integral in (1) with $h = 1$ led to the value $J_0(2\pi) \neq 0$. ($J_0(2\pi)$ is the Bessel coefficient of argument 2π and order zero).

§ 3. *Functions with a periodic first derivative.*

Theorem 2.

If $f(t)$ $(t \geqslant 0)$ is a differentiable function, and if $f'(t)$ is periodic with period p, then the following statements holds:

1°. *If the fixed number $\tau = f(t+p) - f(t)$ is irrational, then $f(t)$ is C^1-uniformly distributed* (mod 1).

2°. *If $\tau = f(t+p) - f(t)$ is rational, then $f(t)$ is C^1-uniformly distributed* (mod 1) *if and only if*

$$\int_0^{p} e^{2\pi i h f(t)}\, dt = 0$$

for any integer $h \neq 0$ with $h\tau$ integer.

Remark. This theorem is a generalisation of a theorem proved in [2].

Proof:

We put $T = kp$ (k is a positive integer). Then we have

$$\int_0^{kp} e^{2\pi i h f(t)}\, dt = \sum_{l=0}^{k-1} \int_{lp}^{(l+1)p} e^{2\pi i h f(t)}\, dt = \sum_{l=0}^{k-1} \int_0^{p} e^{2\pi i h\{f(t)+l\tau\}}\, dt =$$

$$\begin{cases} = \dfrac{e^{2\pi i h k\tau} - 1}{e^{2\pi i h\tau} - 1} \cdot \displaystyle\int_0^{p} e^{2\pi i h f(t)}\, dt & \text{for } h\tau \text{ not integer} \\ = k \displaystyle\int_0^{p} e^{2\pi i h f(t)}\, dt & \text{for } h\tau \text{ integer.} \end{cases}$$

1°. If τ is irrational, then for any fixed integer $h \neq 0$

$$\left| \frac{e^{2\pi i h k\tau} - 1}{e^{2\pi i h\tau} - 1} \right| < \frac{1}{|\sin \pi h\tau|},$$

hence

$$(1/kp) \int_0^{kp} e^{2\pi i h f(t)}\, dt \to 0 \quad \text{for} \quad k \to \infty,$$

so $f(t)$ is C^1-uniformly distributed (mod 1).

2°. If τ is rational, then the argument of 1°. holds for $h\tau$ not integer. For $h\tau$ integer, then

$$(1/kp)\int_0^{kp} e^{2\pi i h f(t)}\,dt = (1/p)\int_0^{p} e^{2\pi i h f(t)}\,dt,$$

so that the condition

$$\int_0^{p} e^{2\pi i h f(t)}\,dt = 0 \quad (h\tau \text{ integer})$$

is necessary and sufficient for $f(t)$ being C^1-uniformly distributed (mod 1).

§ 4. *Application of Theorem* 2.

We consider the function $f(t) = \alpha t + \beta \sin t$ ($t \geqslant 0$, α and β real). $f'(t) = \alpha + \beta \cos t$ has the period $p = 2\pi$.

$\tau = f(t+p) - f(t) = 2\pi\alpha$. Hence:

$f(t)$ is C^1-uniformly distributed (mod 1) *if $\pi\alpha$ is irrational.*

Now we assume that $\pi\alpha$ is rational. We shall prove that $f(t)$ is not C^1-uniformly distributed (mod 1).

In this case we have to consider the integral

$$\int_0^{2\pi} e^{2\pi i h(\alpha t + \beta \sin t)}\,dt$$

for any integer $h \neq 0$ with $2\pi h\alpha$ integer.

This integral however, in the terminology of the theory of Bessel-functions, may be represented by

$$2\pi J_{2\pi h\alpha}(-2\pi h\beta).$$

From the properties of Besselfunctions, that is, from

(1°.) $J_{-\nu}(x) = (-1)^{\nu} J_{\nu}(x)$ (ν integer)

(2°.) $x^{-\nu} J_{\nu}(x)$ is an even function of x (ν integer), it follows that it is no restriction if we limit ourselves to consider the zeros of

$$J_{2\pi h|\alpha|}(2\pi h|\beta|) \qquad (h \text{ positive}).$$

From the theory of the zeros of Besselfunctions [4] we know that the smallest positive zero j_{ν} of $J_{\nu}(x)$ (ν integer and > 0) satisfies the inequality

$$j_{\nu} > \sqrt{\nu(\nu+2)}.$$

Hence it follows that

$$J_{2\pi h\alpha}(2\pi h\beta) \neq 0$$

for

$$2\pi h|\beta| < \sqrt{2\pi h|\alpha|\,(2\pi h|\alpha| + 2)},$$

where $2\pi h|\alpha|$ is a positive integer.

Now we assume

$$2\pi h|\beta| \geqslant \sqrt{2\pi h|\alpha|\,(2\pi h|\alpha| + 2)},$$

where again $2\pi h|\alpha|$ is a positive integer. The last inequality implies $|\beta| > |\alpha|$. We put $|\beta/\alpha| = \sec\varphi$ (φ is an acute angle) and $2\pi h|\alpha| = \nu$.

From the theory of the zeros of Besselfunctions [5] of unrestrictedly large order we know that at the positive zeros of

$$J_\nu(\nu \sec \varphi) \qquad (\nu \text{ sufficiently large})$$

with a possible exception of the first zero the function

$$\nu(\operatorname{tg}\varphi - \varphi)$$

has a value between

$$(m - \tfrac{1}{4})\pi \quad \text{and} \quad (m - \tfrac{1}{6})\pi + \varepsilon,$$

ε being arbitrarily small and > 0, and $m \geqslant 2$. If now for every positive integer h with $\nu = 2\pi h \mid \alpha \mid$ a positive integer the relation $J_\nu(\nu \sec \varphi) = 0$ is satisfied, then the relation $J_{2\nu}(2\nu \sec \varphi) = 0$ would hold, and ν and φ would satisfy the inequalities

$$(2) \qquad \begin{cases} l\pi - \frac{1}{4}\pi < 2\nu(\operatorname{tg}\varphi - \varphi) < l\pi - \frac{1}{6}\pi + \varepsilon \\ 2m\pi - \frac{1}{2}\pi < 2\nu(\operatorname{tg}\varphi - \varphi) < 2m\pi - \frac{1}{3}\pi + 2\varepsilon, \end{cases}$$

with l integer and $\geqslant 2$, m integer and $\geqslant 2$, and ε arbitrarily small and > 0. It is obvious that for l and m integer the inequalities (2) cannot be satisfied simultaneously.

It is impossible furthermore, that, if $\nu \sec \varphi$ is the smallest positive root of $J_\nu(x) = 0$, $2\nu \sec \varphi$ is a root of $J_{2\nu}(x) = 0$ (ν sufficiently large), and that, if $2\nu \sec \varphi$ is the smallest positive root of $J_{2\nu}(x) = 0$, $\nu \sec \varphi$ satisfies the equation $J_\nu(x) = 0$. This follows from the inequality (2) and from the fact that the smallest positive zero of $J_\nu(x)$ occurs for a value of φ for which

$$\tfrac{3}{4}\pi < \nu(\operatorname{tg}\varphi - \varphi) < \tfrac{5}{4}\pi$$

(ν sufficiently large) [6].

So there exists always an integer $h \neq 0$ for which

$$J_{2\pi h\alpha}(2\pi h\beta) \neq 0.$$

Hence we have the following result:

The function $f(t) \equiv \alpha t + \beta \sin t$ $(\beta \neq 0)$ *is* C^I*-uniformly distributed* (mod 1) *if and only if* $\pi\alpha$ *is an irrational number.*

Remark. If $\alpha = 0$ and $\beta = 1$ we find again that $\sin t$ is not C^I-uniformly distributed (mod 1) in accord with § 2.

§ 5. *C^{II}-uniform distribution* (mod 1).

In the present paragraph we prove that the function $f(t) \equiv \alpha t + \beta \sin t$ with $\pi\alpha$ irrational is C^{II}-uniformly distributed (mod 1) if moreover $|\beta| < \alpha$. We repeat the

Definition.

Let $\mathfrak{S}_T(\mathfrak{B}_k)$ *be the set of points* t *in* $0 \leqslant t \leqslant T$ *with* $f(t) - [f(t)]$ *in* $\mathfrak{B}_k$, *where* $\mathfrak{B}_k$ *is an interval of* $\mathfrak{I}$ $(0 \leqslant u \leqslant 1)$ *and where again* $f(t)$ *is a measurable*

function for $t \geqslant 0$. Then $f(t)$ is called C^{II}-uniformly distributed (mod 1) *if, for any sum $\sum_{k=1}^{\infty} \mathfrak{V}_k$ of non-overlapping intervals, $f(t)$ satisfies*

$$\lim_{T\to\infty} (1/T) \, \| \sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{V}_k) \| = \| \sum_{k=1}^{\infty} \mathfrak{V}_k \| .$$

($\| \mathfrak{E} \|$ *is the measure of the pointset* $\mathfrak{E}$).

We apply the

C^{II}-test [7]:

Necessary and sufficient for the C^{II}-uniform distribution (mod 1) *of the measurable function $f(t)$ is that, for any integer $h \neq 0$, $f(t)$ satisfies the relation*

$$\lim_{T\to\infty} (1/T) \int_0^T e^{2\pi i h f(t)} \, dt = 0, \tag{3}$$

and that for any system of intervals $\sum_{k=1}^{\infty} \mathfrak{V}_k$ on $\mathfrak{F}$ the relation

$$\lim_{T\to\infty} (1/T) \sum_{k=1}^{\infty} \| \mathfrak{S}_T(\mathfrak{V}_k) \| = \sum_{k=1}^{\infty} \lim_{T\to\infty} (1/T) \, | \mathfrak{S}_T(\mathfrak{V}_k) | \tag{4}$$

is satisfied.

First we prove the following general

Theorem 3.

If $f(t)$ $(t \geqslant 0)$ is a differentiable function with $f(t)/t$ bounded, if $f'(t) \geqslant \lambda > 0$ with fixed λ, and if $f(t)$ is C^{I}-uniformly distributed (mod 1) then $f(t)$ is also C^{II}-uniformly distributed (mod 1).

Proof.

Let $\sum_{k=1}^{\infty} \mathfrak{V}_k$ be a set of intervals on $\mathfrak{F}$. We shall prove that the series $\sum_{k=1}^{\infty} (1/T) \, \| \mathfrak{S}_T(\mathfrak{V}_k) \|$ is uniformly convergent in T.

For any $T \geqslant 0$ the integer N exists such that

$$N \leqslant f(T) < N+1,$$

so

$$F(N) \leqslant T < F(N+1),$$

where $t = F(u)$ is the inverse function of $u = f(t)$.

Let $\mathfrak{V}_k$ be the interval (α_k, β_k). Then we have

$$(1/T) \, \| \mathfrak{S}_T(\mathfrak{V}_k) \| \leqslant (1/T) \sum_{n=0}^{N} \{ F(n+\beta_k) - F(n+\alpha_k) \} =$$

$$= (1/T) \sum_{\substack{n=0 \\ \alpha_k < \xi_{n,k} < \beta_k}}^{N} (\beta_k - \alpha_k) \, F'(n+\xi_{n,k}) \leqslant \frac{(N+1)(\beta_k - \alpha_k)}{T\lambda}$$

$$\leqslant \frac{(\beta_k - \alpha_k)\{f(T)+1\}}{T\lambda} < c(\beta_k - \alpha_k) \text{ for } T \geqslant T_0 \text{ and with } c \text{ independent of } k.$$

The series $\sum_{k=1}^{\infty} (1/T) \, \| \mathfrak{S}_T(\mathfrak{V}_k) \|$ is therefore uniformly convergent in T for $T \geqslant T_0$, so that the relation (4) holds. From our assumption that

$f(t)$ is C^I-uniformly distributed (mod 1) follows (3). So our theorem is proved.

The function $f(t) \equiv \alpha t + \beta \sin t$ with $|\beta| < \alpha$ and irrational $\pi\alpha$ satisfies the conditions of Theorem 3, and this function is therefore C^{II}-uniformly distributed (mod 1).

§ 6. *Functions with a periodic second derivative.*

Theorem 4.

If $f(t)$ $(t \geqslant 0)$ has a periodic second derivative and a non-periodic first derivative, then $f(t)$ is C^I-uniformly distributed (mod 1).

Proof.

Let p be the period of $f''(t)$. Putting $f'(t+p) - f'(t) = \tau \neq 0$ and $f(t+p) - f(t) = \tau t + \omega$ (τ and ω fixed), we have for l integer and > 0

$$f(t+lp) = f(t) + l\tau t + \tfrac{1}{2} l(l-1)\tau p + l\omega.$$

Put $T = kp$ (k is a positive integer). For $h = 1, 2, \ldots$ we have

$$\left|(1/kp)\int_0^{kp} e^{2\pi i h f(t)}\,dt\right| = (1/kp)\left|\sum_{l=0}^{k-1}\int_0^{p} e^{2\pi i h f(t+lp)}\,dt\right| =$$

$$= (1/kp)\left|\sum_{l=0}^{k-1}\int_0^{p} e^{2\pi i h\{f(t)+l\tau t+{}^1\!/_2 l(l-1)\tau p+l\omega}\,dt\right| =$$

$$= (1/kp)\left|\sum_{l=0}^{k-1} e^{2\pi i h\{{}^1\!/_2 l(l-1)\tau p+l\omega\}}\int_0^{p} e^{2\pi i h\{f(t)+l\tau t\}}\,dt\right|$$

$$\leqslant (1/kp)\sum_{l=0}^{k-1}\left|\int_0^{p} e^{2\pi i h\{f(t)+l\tau t\}}\,dt\right|.$$

The sequence Σ on the right side is a sequence of k positive terms which we denote by S_k. If we can prove that $S_k - S_{k-1}$ tends to zero for $k \to \infty$, then by a well-known theorem of CAUCHY the right side tends to zero for $k \to \infty$. Now

$$S_k - S_{k-1} = \left|\int_0^{p} e^{2\pi i h\{f(t)+(k-1)\tau t\}}\,dt\right|, \quad (\tau \neq 0),$$

and applying a theorem of RIEMANN which asserts that

$$\lim_{m\to\infty}\int_a^b F(x)\sin mx\,dx = \lim_{m\to\infty}\int_a^b F(x)\cos mx\,dx = 0,$$

if $F(x)$ is integrable, we find that $S_k - S_{k-1} \to 0$ if $k \to \infty$. This completes the proof.

Application.

The function $f(t) \equiv \alpha t^2 + \beta t + \gamma \sin t$ $(\alpha \neq 0)$ satisfies the conditions of Theorem 4, so that $f(t)$ is C^I-uniformly distributed (mod 1).

Bandung, 31 December 1949. *University of Indonesia.*

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ELASTIC VISCOUS OLEATE SYSTEMS CONTAINING KCl. IX [1])

a. *General remarks on $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams.*

b. *The use of this kind of diagrams in discussing the results obtained with the homologous series of the fatty acid anions.*

BY

H. G. BUNGENBERG DE JONG AND H. J. VAN DEN BERG

(Communicated at the meeting of February 25, 1950)

1. *The simultaneous change of G and λ (or $1/\Lambda$ or n) by addition of organic substances of the types A, B and C at various KCl concentrations.*

In Part VIII of this series we showed the usefulness of $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams for the discussion of the results, obtained with the homologous series of the n-primary alcohols. The curves representing the experimental results all went upwards and to the left. These alcohols belong to the substances of the type A (see for the three possible types A, B and C Part VI of this series). The influence of the alcohols was investigated at (or very near) the KCl concentration of minimum damping of the blank.

It will be usefull to know the course of $G - \lambda$ (or $G - 1/\Lambda$ or $G - n$) curves for all nine imaginable cases, that is for substances of the types A, B and C added at KCl concentrations smaller than, equal to or higher than the KCl concentration corresponding with the minimum damping of the blank.

In Part VII of this series (see there, fig. 4) we gave a scheme for the sign of the change of G, likewise for the sign of the chauge of λ (or $1/\Lambda$ or n) for each of the above nine cases. From this scheme, comprising 18 diagrams, one obtains the scheme given here as fig. 1 by combining again and again the two diagrams which are characteristic for each case. In the diagrams of fig. 1 the blank oleate system is represented by a black dot, and the curve starting from this dot gives the simultaneous changes of G and λ (or $1/\Lambda$ or n). The curves have been represented in the diagrams 2 — 8 by straight lines ending in an arrow point. Of course we do not mean that these curves are actually straight lines. They only indicate into which direction the curves proceed from their starting points. For this reason we have drawn through the black dots two coordinate axis,

[1]) Part I has appeared in these Proceedings **51**, 1197 (1948); Parts II-VI in these Proceedings **52**, 15, 99, 363, 465 (1949); Parts VII and VIII in these Proceedings **53**, 7, 109 (1950).

so that it may be seen into which quadrant the curve takes its initial course. In diagram 5 the curve runs along the x-axis between the quadrants II and III. In the diagrams 2, 3 and 6 the curve takes its course into quadrant II, in the diagrams 4, 7 and 8, however, into quadrant III. More complicated is the course of the curve in the diagrams 1 and 9. In 1 the curve starts its course in quadrant I but afterwards proceeds to quadrant II. In diagram 9 the curve starts its course in quadrant IV but afterwards proceeds to quadrant III.

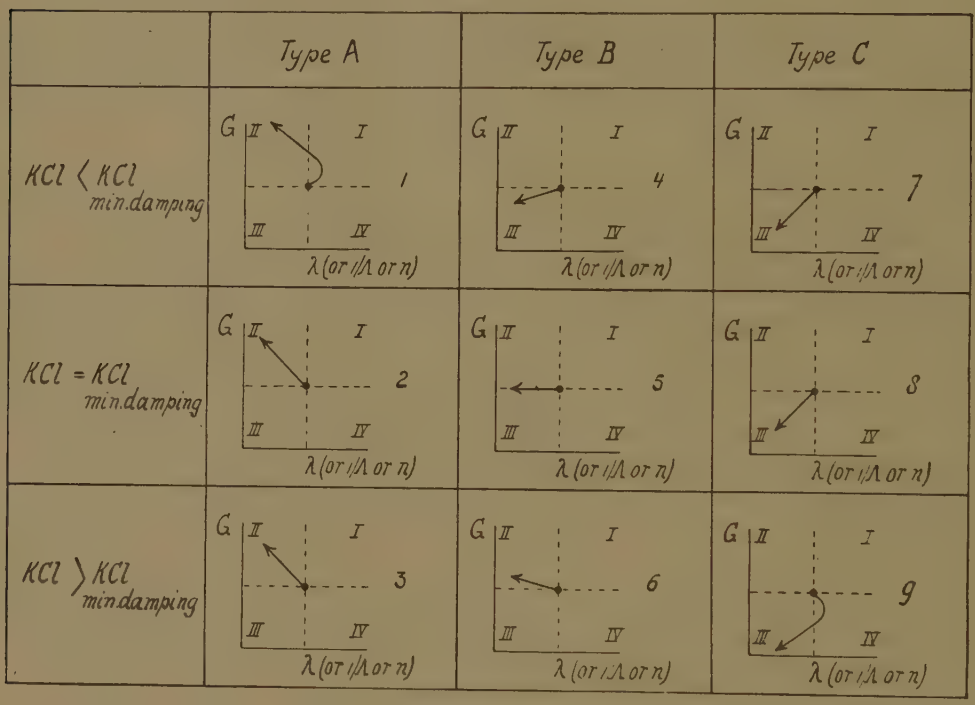

Fig. 1

Fig. 1 gives an answer to the question which will be the most favourable KCl concentration for performing experiments about the action of added organic substances on the elastic behaviour of the oleate system. This is the KCl concentration of minimum damping of the blank, for the diagrams 2, 5 and 8 are different, so that we at once can discern to which of the types A, B and C the organic substance belongs.

Less information is gained at other KCl concentrations because two diagrams are alike each time. At lower KCl concentrations the Types B and C cannot be discriminated (the diagrams 4 and 7 being in principle the same) and the same holds for the Types A and B at higher KCl concentrations (the diagrams 3 and 6 being in principle the same).

In the next section we will give examples of the $G - \lambda$ (or $G - 1/\Lambda$, or $G - n$) curves as it was discussed in the above. We will see, that there exist gradual transitions between diagrams which in fig. 1 are very dis-

similar, e.g. if the KCl concentration, though still too low, comes already near the one of minimum damping or if a substance still belonging to type *A* stands already near to type *B*.

It does not seem superfluous to remark that the scheme of fig. 1 quite as the scheme of fig. 4 in Part VII, on which it is based, gives a simplification of the experimental facts. It is assumed that a given organic substance when acting on the elastic oleate system can under all circumstances be classed under one of the types *A*, *B* and *C*. We indeed have in many cases no indications against this assumption. So, *n*-hexylalcohol belongs typically to class *A* and the undecylic anion typically to class *C*. Ethanol, however, is a substance which behaves as a weak representant of class *A* at the KCl concentration of minimum damping of the blank. This is also the case at smaller KCl concentrations. At much higher KCl concentrations, however, ethanol behaves as a weak representant of class *C* (compare fig. 4 in Part VI and the present communication section 2). We gather from this that the action of an organic substance may still be a function of the KCl concentration.

From fig. 1, we concluded that, working at the KCl concentration of minimum damping of the blank, the conditions for gaining information are most favourable. Therefore we will compare in the future the action of organic substances at or very near to this KCl concentration of minimum damping. If therefore from such experiments we come to the conclusion, that a certain substance belongs to type *A* or *B* or *C*, this will mean: investigated at the KCl concentration of minimum damping of the blank.

2. *Examples: n-hexylalcohol and ethanol acting at various KCl concentrations.*

We will now give examples of the form of the $G - \lambda$ (or $1/\Lambda$ or n) curves, as it was discussed in the above. We choose therefore first the experiments from Part VI of this series dealing with the influence of *n*-hexylalcohol on the elastic behaviour of the 1.2 % oleate system as a function of the KCl concentration. From the data stated in Table I and fig. 2 of Part VI *n*-hexylalcohol appears to be a good example of a substance of Type *A*. The experimental data have been plotted in a $G - \lambda$ diagram in fig. 2, and curves have been drawn through these experimentally determined points in two different ways. In fig. 2*A* curves *A*, *B*, *C* and *D* have been drawn every time through points with equal hexylalcohol concentration (*A* = blank: *B* = 0,0016 moles/l: *C* = 0,0042 moles/l; *D* = 0,0094 moles/l), so that along each curve going from the left end at the bottom to the left end at the top the KCl concentration increases (the arrows indicate this direction of increasing KCl concentration).

In fig. 2*B*, however, curves have been drawn through experimentally determined points with the same KCl concentration (the values have been indicated). Along each curve the hexylalcohol concentration increases in the direction of the arrow.

Each curve in fig. 2*B* has as starting point the blank oleate system. These starting points (black dots) are therefore those, which in fig. 2*A* are situated on curve *A*.

In what follows now the curves of fig. 1*A* do not play a particular part. They form a separate class of $G - \lambda$ ($G - 1/\Lambda$ or $G - n$) curves and will be discussed in the small print at the end of this section.

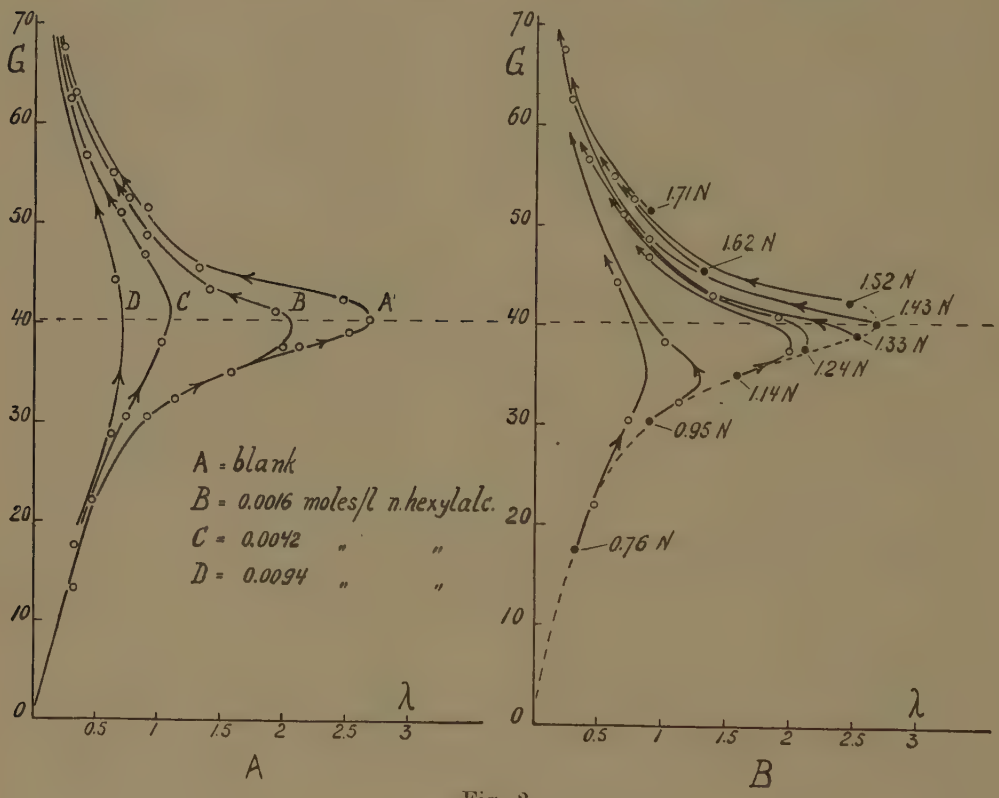

Fig. 2

In fig. 2*B* we have drawn a horizontal line (dotted) through the blank point lying at 1.43 N KCl. This is the KCl concentration of minimum damping of the blank (λ, $1/\Lambda$ and n have a maximum here).

We perceive that the $G - \lambda$ curve which starts here proceeds indeed as is given in diagram 2 of fig. 1.

We see further that all the $G - \lambda$ curves proceeding at higher KCl concentrations [1]) have the same form, which is in accordance with diagram 3 of fig. 1.

Now turning to the form of the $G - \lambda$ curves at KCl concentrations lower than 1.43 N KCl, we should expect curves of the type as is depicted in diagram 1 of fig. 1. The $G - \lambda$ curve for 0.57 N KCl

[1]) Because of the crowding together of these curves, we have drawn some of them only over a short distance, e.g. the one of 1.52 N KCl.

takes its initial course in the expected direction. It is not reproduced in the graph as the bend upwards and to the left cannot be controlled here for lack of further experimentally determined points. The curves for 0.76 N and 0.95 N KCl certainly belong to the expected form, though here too few experimentally determined points on the branches pointing upwards and to the left are present to draw their exact course. The expected form is realised in its whole in the curve for 1.14 N KCl, though now the initial course which goes upwards and to the right extends itself over a short distance only.

Two more curves, those for 1.24 N and 1.33 N KCl, no longer show that the initial course goes upwards and to the right, though we are still below the KCl concentration of minimum damping of the blank (1.43 N).

We learn from this example — and after all that was said about it, it might have been foreseen — that here is no abrupt change in the course of the curves at the exact moment that the KCl concentration has risen to the one of minimum damping. Instead of this we have a gradual transition, and at KCl concentrations already near but still below 1.43 N KCl the initial course of the curve upwards and to the right may extend itself over so small a distance that it stays within the experimental errors.

What has been said about the transition between the diagrams 1 and 2, of course will also be true for the analogous transition between the diagrams 8 and 9.

We will now discuss the influence of ethanol on the elastic behaviour of the 1.2 % oleate system. From this we will get acquainted with another kind of transition between diagrams which in fig. 1 seem to be fundamentally different.

The results of these experiments have been given in Part VI of this series by graphical representations only (see fig. 4 in Part VI). For our present aims we give in Table I the data [2]), which may be used for constructing $G - \lambda$: $G - 1/\Lambda$ and $G - n$ diagrams. As in this case too each of these three kinds of diagrams show the same general aspects, we give here only one of them. Compare fig. 3.

Fig. 3 *A* which is analogous to fig. 2 *A* plays no part in what follows and will be shortly discussed in the small print at the end of this section. Fig. 3 *B* is analogous to fig. 2 *B* and therefore represents the simultaneous change of G and λ, by addition of ethanol to the blank system, at various

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[2]) In consequence of the method for adding ethanol followed in Part VI the KCl concentration did not remain constant (compare fig. 4 in Part VI, in which figure the experimentally determined points, referring to systems containing 0.195 and 0.39 moles/l of ethanol, lie at smaller KCl concentrations than the corresponding blanks). The data given in Table I have been corrected graphically for this shift of the KCl concentration, so that the G, λ, $1/\Lambda$ and n values of the ethanol containing systems refer to the same KCl concentration as the corresponding blank oleate system.

KCl concentrations. The starting points (blank systems) of these $G - \lambda$ curves are situated on curve A of fig. 3 A.

Measurements exactly at the KCl concentration of minimum damping of the blank (at 1.43 N KCl) have not been achieved in this series. The course of the corresponding $G - \lambda$ curve follows from the $G - \lambda$ curve for a slightly higher (= 1.52 N) and a slightly lower (= 1.33 N) KCl concentration. The latter curves ascend both slightly going to the left, so that the $G - \lambda$ curve at the KCl concentration of minimum damping must be of the same type. This $G - \lambda$ curve therefore will proceed from its starting point into quadrant II, that is to say it takes the course as given in principle by diagram 2 of fig. 1.

Ethanol is therefore a substance of type A.

Compared with n-hexylalcohol (fig. 2 B for 1.43 N KCl) the ascent of the curve is very slight, so that ethanol is only a weak representant of Type A, standing already near to a substance of Type B.

If we direct our attention to the course of the $G - \lambda$ curves at other KCl concentrations, we encounter nothing which is in striking contradiction with the expected course for KCl concentrations higher than the one of minimum damping of the blank. The $G - \lambda$ curves for 1.52 N and 1.71 N KCl [3]) take their initial course into quadrant II, as is to be expected from diagram 3 of fig. 1.

If however, we consider in fig. 3 B the course of the $G - \lambda$ curves at KCl concentrations which are much lower than that of the minimum damping of the blank, we perceive a striking difference in course with the one of the curve in diagram 1 of fig. 1. Instead of first proceeding into quadrant I and thereafter continuing their course to quadrant II, the $G - \lambda$ curves for 1.14 N and 0.95 N KCl in reality take their course into quadrant III, which a course is to be expected for a substance of type B (diagram 4 of fig. 1).

This must be caused by the fact that ethanol though still belonging to Type A, is only a weak representant of this Type and stands already near to a substance of Type B. We learn from this once more, that abrupt changes between the diagrams in fig. 1 which seem fundamentally different, do not occur, but that instead of this a gradual change takes place.

In the above (n-hexylalcohol) we saw that such a transition between diagram 1 and diagram 2 did not occur abruptly the moment the KCl concentration had risen exactly to that of the minimum damping of the blank, but that this transition was already taking place gradually at lower KCl concentrations. Similarly an abrupt change of diagram 1 into diagram 4 does not exist. Comparing at these lower KCl concentrations the

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[3]) The bend downwards at the left end of these curves is probably caused by the complication already discussed in the above in section 2, that at higher KCl concentrations ethanol may act as a weak representant of Type C. The abnormal course of the $G - \lambda$ curve in fig. 3 B at the highest KCl concentration (1.90 N) may be a direct evidence of this phenomenon.

TABLE I

The elastic behaviour as a function of the KCl concentration and the influence of ethanol thereon

KCl concentration mol/l	G (dynes/cm²)			λ (sec.)		
	Ethanol concentr. mol/l			Ethanol concentr. mol/l		
	blank	0.195	0.39	blank	0.195	0.39
0.95	30.4	26.3	22.6	0.91	0.57	0.31
1.14	34.9	35.0	32.8	1.63	0.93	0.45
1.33	39.0	40.3	40.5	2.59	1.00	0.44
1.52	42.2	46.5	45.3	2.58	0.76	0.35
1.71	51.7	54.0	50.8	0.91	0.47	0.22
1.90	63.3	61.5	63.0?	0.35	0.24	0.11

KCl concentration mol/l	η			η		
	Ethanol concentr. mol/l			Ethanol concentr. mol/l		
	blank	0.195	0.39	blank	0.195	0.39
0.95	1.42	0.82	0.41	18.2	12.2	5.0
1.14	2.73	1.56	0.74	39.6	25.0	11.5
1.33	4.44	1.74	0.79	48.0	27.8	13.0
1.52	4.61	1.44	0.63	48.3	23.5	8.8
1.71	1.78	0.92	0.43	29.7	14.2	5.3
1.90	0.78	0.57	0.28	14.0	7.3	3.0

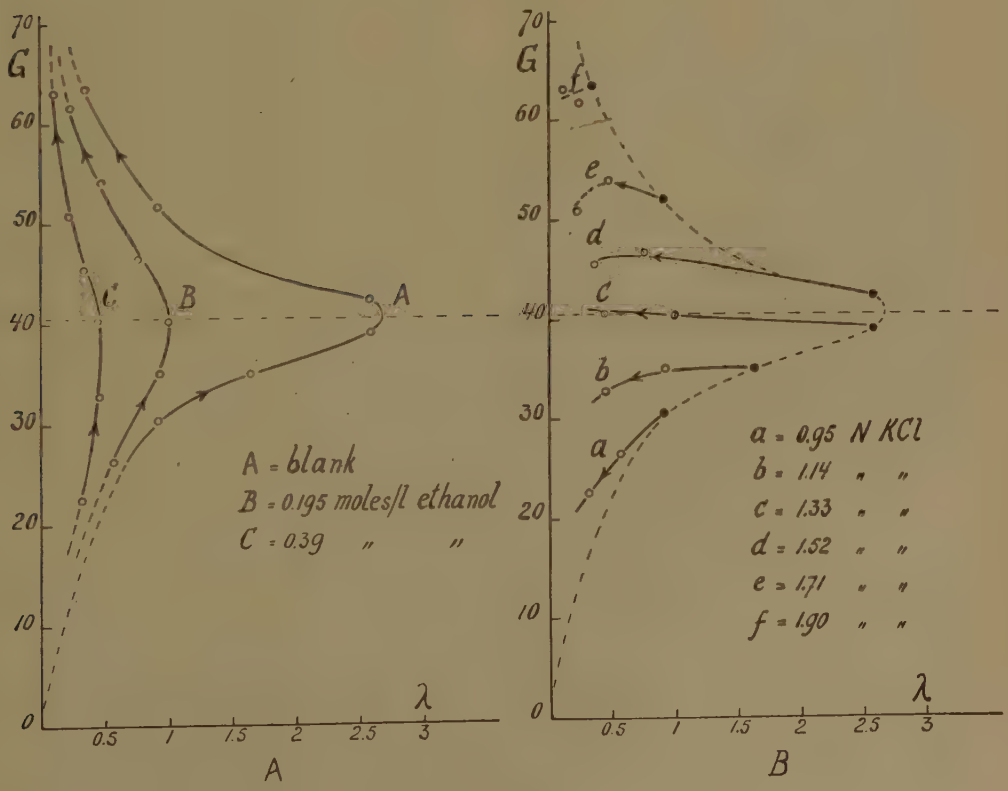

Fig. 3

$G - \lambda$ curves for a number of n-primary alcohols, it must be expected that somewhere there will occur a gradual change between diagram 1 (typical of hexylalcohol) and diagram 4 if we go in the sequence of a decreasing number of carbon atoms. This change has already taken place with ethanol, though this is still a weak representant of Type A, and not a substance belonging to type B.

At this place we will add a few remarks about the curves of the figures 2 A and 3 A. They represent the simultaneous change of G and λ (or $1/\Lambda$, or n) when the KCl concentration is increased. The arrows give the direction in which this increase takes place. The curves A refer to the blank 1.2 % oleate system, the other curves to the 1.2 % oleate systems which contain each a constant concentration of n-hexylalcohol ($B = 0.0016$; $C = 0.0042$ and $D = 0.0094$ moles/l) or of ethanol ($B = 0.195$ and $C = 0.39$ moles/l).

In both figures 2 A and 3 A the dotted horizontal line gives the value of G for the blank oleate system at the KCl concentration of minimum damping. We observe that these horizontal dotted lines practically intersect the other curves B, C and D at the maximum value of λ (or $1/\Lambda$ or n). Therefore we find expressed in the $G - \lambda$ (G—$1/\Lambda$ or $G - n$) curves of fig. 2 A and fig. 3 A a fact which we have already discovered previously, namely that in displacing the G, λ, $1/\Lambda$ and n curves by addition of an organic substance, the value of G at the inflexion point of the $G - C_{KCl}$ curve remains practically constant (compare fig. 3 in Part VI of this series).

3. *A comparison of the successive terms of the homologous series of the fatty acid anions with the aid of $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams.*

It was shown in part VIII of this series that for substances of type A, acting upon the elastic oleate system at the KCl concentration of minimum damping, the *increase* of G accompanying an arbitrary chosen but constant decrease in λ (or in $1/\Lambda$ or in n) is a measure for the ratio of the horizontal component of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curve (in the direction of *smaller* KCl concentrations) and the vertical component (in downward direction) of this displacement.

It is easy to deduce in a quite analogous way, that for substances of the Type C, the *decrease* in G accompanying an arbitrary chosen but constant decrease in λ (or in $1/\Lambda$ or in n) is a measure for the ratio of the horizontal component (in the direction of *higher* KCl concentrations) of displacement of the maximum of the λ (or $1/\Lambda$ or n) curve and the vertical component (in downward direction) of this displacement.

We will not repeat here the analogous reasoning, but refer at once to figure 4, where this correlation is given schematically.

The decrease in λ (or $1/\Lambda$ or n) represented by the distance ab, is the same here for the three cases B, C_1 and C_2. The decrease in G, represented by the distance fg, is nil in case B (representing a substance of type B), has a certain value in C_1 and an even larger value in C_2.

We perceive further, in fig. 4, that the ratios of the horizontal and the vertical components of the displacement of the maximum of the λ (or $1/\Lambda$ or n) curves, viz. the ratios ad/dc, have also different values. This

ratio is zero for case B, has a finite value for C_1 and a larger value for C_2. So fg, the decrease in G (at the KCl concentration of minimum damping) accompanying an arbitrary chosen decrease in λ (or $1/\Lambda$ or n), is a measure for the numerical value of the above ratio ad/dc.

We will now applie this to the investigation about the influence of the fatty acid anions $C_8 - C_{14}$ on the elastic behaviour of the 1.2 % oleate

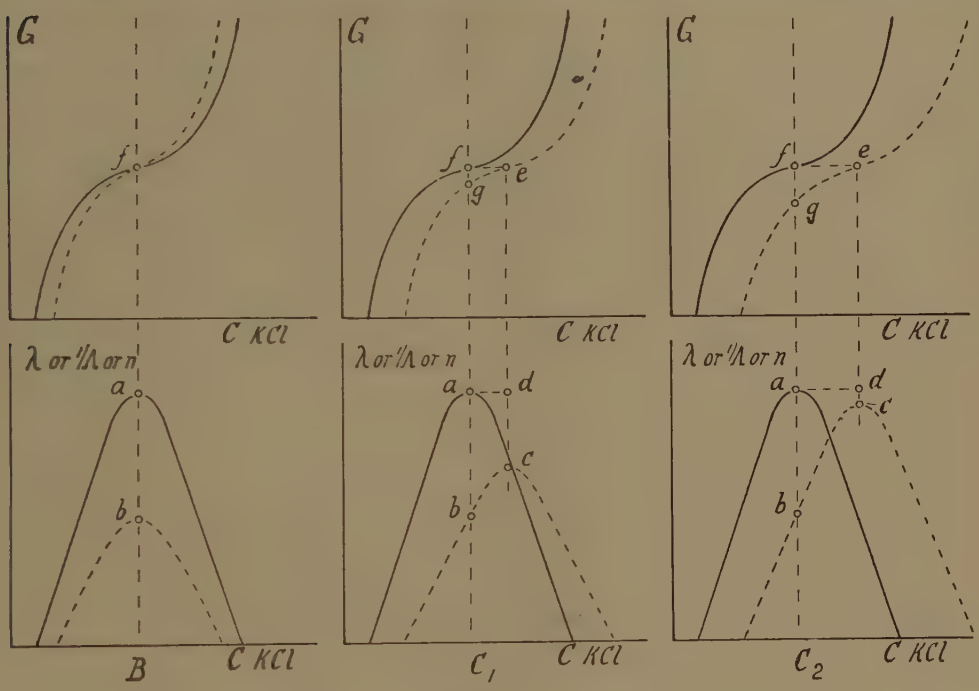

Fig. 4

system, the results of which were given in Part VI of this series by graphical representations only. For our present aims we give the data which are referring to these measurements in Table II.

For the application of the above, we could determine graphically which G values correspond to an arbitrary chosen, but constant decrease (e.g. of 50 %) in λ (or $1/\Lambda$ or in n). It is simpler to construct at once $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams (using the data of Table II) which give the decrease of G for any decrease of λ (or $1/\Lambda$ or n).

This has been done in fig. 5, from which we see that the curves for $C_8 - C_{12}$ start their course into quadrant IV and afterwards continue their course to quadrant III. This course need not surprise us, as it may occur for substances of Type C if the KCl concentration is higher than that of minimum damping of the blank (compare diagram 9 in fig. 1). Indeed the actual KCl concentration in these experiments was 1.52 N, instead of 1.43 N.

The curve for the myristate ion, C_{14}, takes its course into quadrant I,

a case which does not occur in one of the diagrams 3, 6 and 9 of fig. 1.

Let us imagine for a moment that we add more oleate to the original oleate system. The $G - \lambda$, $G - 1/\Lambda$ and $G - n$ curves would then take their course into this quadrant I (because increase of the oleate concentration results in an increase of G, λ, $1/\Lambda$ and n; compare Part III, section 2).

TABLE II

Influence of fatty acid anions on the elastic behaviour of the 1.2 % oleate system at approximately the KCl concentration of minimum damping (at 15°).

fatty acid anion	values of the blanks	conc. m. moles/l	100 G/G_0	100 λ/λ_0	100 Λ_0/Λ	100 n/n_0
C_8	$G_0 = 42.5$	0.45	99.6	104.0	104.2	106.4
	$\Lambda_0 = 0.224$	1.34	97.2	108.4	107.2	109.0
	$\lambda_0 = 2.50$	4.90	91.3	106.4	101.8	106.6
	$n_0 = 46.8$	9.80	83.2	40.8	37.2	52.4
C_9	$G_0 = 42.8$	0.40	99.3	104.4	103.7	107.1
	$\Lambda_0 = 0.222$	1.20	95.2	105.6	102.8	107.9
	$\lambda_0 = 2.51$	4.40	86.5	49.8	46.4	63.6
	$n_0 = 46.7$	8.80	35.3	9.6	5.8	8.6
C_{10}	$G_0 = 41.9$	0.25	100.0	106.4	106.2	108.0
	$\Lambda_0 = 0.224$	0.84	93.2	109.2	105.2	109.9
	$\lambda_0 = 2.50$	3.05	87.7	52.4	48.9	65.2
	$n_0 = 46.3$	6.16	52.5	10.4	7.5	10.8
C_{11}	$G_0 = 42.0$	0.35	99.6	105.9	105.8	111.6
	$\Lambda_0 = 0.219$	0.84	94.9	109.4	106.8	112.3
	$\lambda_0 = 2.56$	2.88	93.9	49.2	47.9	67.7
	$n_0 = 46.5$	5.70	63.5	12.9	10.2	15.1
C_{12}	$G_0 = 42.2$	0.19	98.9	105.6	104.7	107.3
	$\Lambda_0 = 0.222$	0.96	96.2	106.8	104.7	108.4
	$\lambda_0 = 2.51$	2.89	100.0?	76.5	76.6	93.1
	$n_0 = 46.7$	5.74	94.5	35.9	34.7	45.0
C_{14}	$G_0 = 42.5$	0.18	100.0	104.7	104.7	108.0
	$\Lambda_0 = 0.221$	0.77	97.2	105.1	103.8	110.3
	$\lambda_0 = 2.54$	2.58	102.5	117.3	118.8	115.7
	$n_0 = 46.4$	5.14	113.3	130.3	139.0	128.0

Mean values of the blanks $G_0 = 42.3$; $\Lambda_0 = 0.222$; $\lambda_0 = 2.52$; $n = 46.6$.

The effect of added C_{14} in fig. 5 is therefore of the same kind as that of extra added oleate. This clearly shows that myristate can no longer be considered as a "foreign substance". Its addition gives rise to a mixed elastic system, in which both oleate and myristate contribute to the elastic properties.

We now consider, in fig. 5, the relative position of the curves for C_8,

C_9, C_{10}, C_{11} and C_{12} in quadrant III. We perceive that these curves form a bundle of non intersecting lines.

At any arbitrary chosen decrease in λ (or in $1/\Lambda$ or in n) the decrease in G is different for the successive terms of the homologous series of the fatty acid anions. As in the $G - \lambda$, $G - 1/\Lambda$ or $G - n$ diagrams, the concentrations of the added substances are not plotted these different

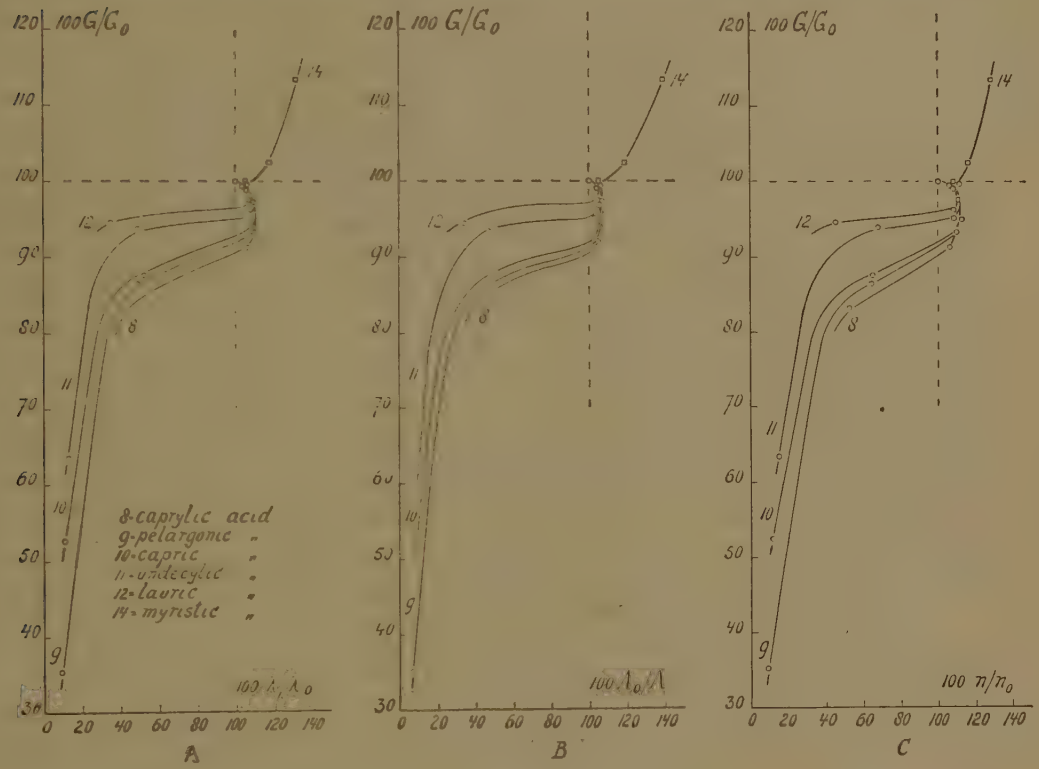

Fig. 5

positions point to a gradual change in the nature of the action of the fatty acid anions which are taken up by the oleate micelles, every time the carbon chain is lengthened by one carbon atom.

The discussions at the beginning of this section enable us to predict what would happen with the ratio of the horizontal and vertical components of the shifts occurring in diagrams (fig. 4) in which G, λ, $1/\Lambda$ and n are plotted as functions of the KCl concentration, this ratio being an expression for the specific nature of the action of the fatty acid anion taken up by the oleate micelles. We therefore (see fig. 5) must arrange the fatty acid anions $C_8 - C_{12}$ in the subjoined sequence

$$C_{12} < C_{11} < C_{10} < C_9^- < C_8$$

in which from right to left this ratio decreases.

In the above sequence C_8 has the greatest ratio, C_{11} a much smaller one, that is, C_8 is a much more pronounced representant of substances of Type C than C_{11}. Exactly the reverse would follow from fig. 6 in Part VI, in which the action of the fatty acid anions is laid down in graphs representing G and λ (or $1/\Lambda$ or n) as functions of the *gross* concentrations of the fatty acid anions. From this figure one might deduce as sequence of "increasing activity"

$$C_{11} > C_{10} > C_9 > C_8$$

In Part VIII (section 3) of this series it has been discussed that for substances sufficiently soluble in water such a series of "increasing activity" may only indicate that the ease with which the successive terms of a homologous series are taken up by the oleate micelles increases with lengthening of the carbon chain. This is also applicable here, as BOOIJ and BUNGENBERG DE JONG [4]) have shown that the uptake of the undecylate ion is indeed much easier than the uptake of the nonylate ion.

4. *A comparison between the homologous series of the n-primary alcohols and of the fatty acid anions.*

It may be worthwhile, to compare the results obtained with the alcohols $C_1 - C_6$ (Parts VI, VII and VIII) and with the fatty acid anions $C_8 - C_{12}$ (Part VI and the present one). The following points of resemblance and of contrast can be formulated:

A. In both homologous series the distribution of the added molecules

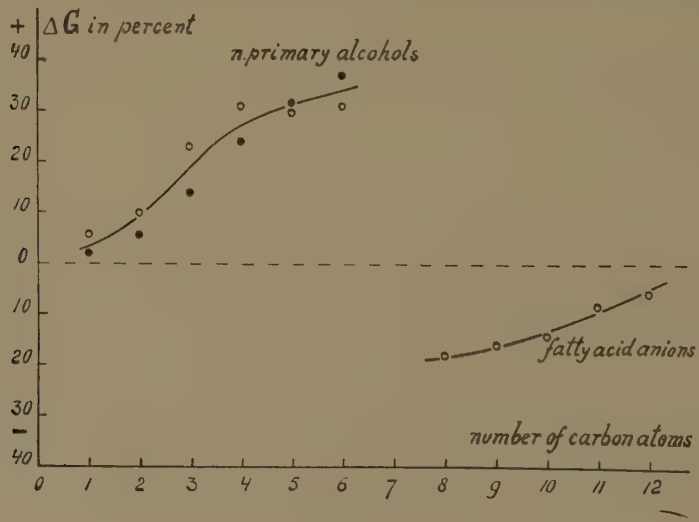

Fig. 6

[4]) H. L. BOOIJ and H. G. BUNGENBERG DE JONG, Bioch. Bioph. Acta **3**, 242 (1949).

[5]) H. L. BOOIJ, these Proceedings **53**, 59 (1950).

between oleate micelles and medium show a shifting in favour of the oleate micelles by lenghtening of the carbon chain.

B. The alcohols belong to Type *A*, (less KCl is needed to obtain the elastic system), the fatty acid anions to Type *C* (more KCl is needed).

C. If the carbon chain is lenghtened the ratio (not taking into account its sign) of the horizontal and vertical components of the displacement (in the graphs representing G, $1/\lambda$, $1/\Lambda$ and n as functions of the KCl concentration) increases in the case of the alcohols and decreases in the case of the fatty acid anions.

Thus of the alcohols $C_1 - C_6$ it is methanol, but of the fatty acid anions $C_8 - C_{12}$ it is just C_{12} which stands nearest to a substance of Type *B* (for which type the above ratio is zero).

Compare also fig. 6, in which the increase or decrease of G in per cent. at a decrease of 50 per cent. of n is plotted as a function of the number of carbon atoms of the alcohols or the fatty acid anions. The data for the alcohols are taken from fig. 3 and 4 in Part VIII of this series (black dots refer to the series with 0.6 % oleate, white dots to that with 1.2 % oleate), the data for the fatty acid anions are taken from fig. 5 of the present communication.

5. *Summary.*

1. For the oleate system a scheme has been given for the simultaneous changes of G and λ (or $1/\Lambda$ or n) by addition of substances of the Types *A*, *B* and *C* at KCl concentrations lower than, equal to or higher than the KCl concentration of minimum damping of the blank. The scheme has been illustrated by some examples (*n*-hexylalcohol and ethanol acting at various KCl concentrations).

2. The results obtained in Part VI with the homologous series of the fatty acid anions $C_8 - C_{14}$ have been discussed with aid of $G - \lambda$, $G - 1/\Lambda$ and $G - n$ diagrams. These diagrams show that the nature of the action exerted by the fatty acid anions which are bound to the oleate micelles is not the same for the successive terms of the homologous series, but that a gradual shift occurs when the carbon chain is lenghtened.

3. For $C_8 - C_{12}$ the gradual shift of sub 2) consists of a change of the ratio of the horizontal and the vertical component of the displacement of the maximum of the λ, $1/\Lambda$ or n curve (in diagrams which represent λ, $1/\Lambda$ or n as functions of the KCl concentration). The fatty acid anions may by arranged in the sequence $12 < 11 < 10 < 9 < 8$, in which sequence the above mentioned ratio decreases from right to left.

4. The effects of added C_{14} (myristate) on the elastic properties of the oleate system are of the same kind as occurs when the oleate concen-

tration of the oleate system is increased. Addition of C_{14} gives rise to an mixed elastic system, in which both oleate and myristate contribute to the elastic properties.

5. The results hitherto obtained with the homologous series of the alcohols ($C_1 - C_6$) and of the fatty acid anions ($C_8 - C_{12}$) have been compared.

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CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. I.

BY

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(*Mededeling no. 65a uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated at the meeting of February 25, 1950)

1. *Introduction.* [1])

In the following pages formulae will be derived concerning correlations in a particular type of solutions of the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} \tag{1}$$

We suppose the coefficient ν to be small. Before describing the solutions to be considered, a few general properties of eq. (1) will be mentioned.

If v is interpreted as a velocity, an "equation of momentum" can be formed by integrating (1) with respect to y. For a domain at the limits of which v vanishes and $\partial v/\partial y$ is of such magnitude that $\nu(\partial v/\partial y)$ can be neglected, we obtain:

$$\frac{d}{dt}\int v\,dy = 0 \tag{2}$$

An "equation of energy" can be formed by multiplying eq. (1) with v and integrating it with respect to y. For any domain at the ends of which v vanishes, we obtain

$$\frac{d}{dt}\int \frac{v^2}{2}\,dy = -\nu\int\left(\frac{\partial v}{\partial y}\right)^2 dy \tag{3}$$

When $v^2/2$ is considered as kinetic energy per unit length, eq. (3) shows a loss of energy through "internal friction", the amount of which per unit of length is determined by $\nu(\partial v/\partial y)^2$.

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[1]) This paper is a continuation of that on "The formation of vortex sheets in a simplified type of turbulent motion", these Proceedings 53, 122 (1950) (Mededeling no. 64). Equation (1) has already been considered to some extent in the papers: "Application of a model system to illustrate some points of the statistical theory of free turbulence", these Proceedings 43, 8 (1940); and "A mathematical model illustrating the theory of turbulence", Advances in Applied Mechanics 1, 182 (1948). In those papers a coefficient 2 had been introduced before the term $\nu(dv/dy)$ of the equation; this factor has now been dropped in connection with the considerations of the paper on the formation of vortex sheets, which causes a slight difference in some of the formulae.

Equation (1) is invariant with respect to a shift over an arbitrary distance along the y-axis. We consider a domain of unlimited extent and assume that the initial course of v as function of y is such that mean values can be defined, *e.g.*:

$$\overline{v} = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} v \, dy \quad ; \quad \overline{v^2} = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} v^2 \, dy, \quad \text{etc.},$$

which for sufficiently large values of the interval $y_2 - y_1$ do not change when this interval is shifted along the y-axis. This property will remain valid throughout the development in the course of time and thus applies to $v(y)$ at any instant. We shall say that the state of our system [meaning the course of $v(y)$ at a given value of t] is *statistically homogeneous*.

The equation of energy (3) will then also apply to the mean values of both members. We write:

$$E = \tfrac{1}{2} \overline{v^2} \quad \text{(mean kinetic energy per unit length)}$$

$$\varepsilon = \nu \overline{(\partial v / \partial y)^2} \quad \text{(mean dissipation per unit length)},$$

and obtain:

$$\varepsilon = - d\dot{E}/dt \tag{4}$$

Equation (1) is moreover invariant with respect to a simultaneous change of sign of both v and y. When mean values calculated for the initial state of the system possess this property, it again will be preserved during the development of the system. The property implies that $\overline{v} = 0$.

2. *Correlation functions.* — Following the example of all modern authors on hydrodynamic turbulence, we introduce the quantities:

$$\overline{v_1 v_2} = \overline{v(y)\, v(y+\eta)} = \overline{v^2} f(\eta) \tag{5}$$

$$\overline{v_1^2 v_2} = \overline{v(y)^2 \, v(y+\eta)} = \overline{(v^2)^{3/2}} \, k(\eta) \tag{6}$$

(mean values to be taken with respect to y, with a fixed value of η, at a given instant of time). As is known, $f(\eta)$ is an even function, while $k(\eta)$ is an odd function, the development of which in the neighbourhood of $\eta = 0$ begins with a term in η^3.

If the development of $f(\eta)$ in the neighbourhood of $\eta = 0$ is written:

$$f(\eta) = 1 - \eta^2 / 2\lambda^2 + \ldots \tag{7}$$

where λ is a quantity having the dimensions of a length, we obtain:

$$\overline{(\partial v / \partial y)^2} = - \overline{v^2} \cdot f''_{\eta=0} = \overline{v^2} / \lambda^2 \tag{8}$$

Hence (4) can be written:

$$\varepsilon = - dE/dt = 2\nu E / \lambda^2 \tag{9}$$

When the course of $v(y)$ presents a sufficiently random character, $f(\eta)$ and $k(\eta)$ will become zero for η increasing without limit. We assume that both are integrable and put:

$$L = \int_0^\infty f(\eta)\, d\eta \tag{10}$$

In this way L and λ are two linear dimensions connected with the course of $v(y)$. Both are functions of t. We introduce two dimensionless parameters, of the nature of REYNOLDS' numbers:

$$Re_I = \overline{(v^2)}^{1/2} L/\nu \quad ; \quad Re_{II} = \overline{(v^2)}^{1/2} \lambda/\nu \tag{11}$$

We suppose that ν is so small that Re_I is a large quantity.

If again we write v_1, v_2 for $v(y)$, $v(y+\eta)$ respectively, we can form the equation:

$$\frac{\partial}{\partial t}(v_1 v_2) = -v_1 v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y}\right) + \nu \left(v_2 \frac{\partial^2 v_1}{\partial y^2} + v_1 \frac{\partial^2 v_2}{\partial y^2}\right).$$

Taking mean values from both sides with respect to y, we obtain, by a well known procedure, the fundamental equation:

$$\frac{\partial}{\partial t}\overline{(v_1 v_2)} = \frac{\partial}{\partial \eta}\overline{(v_1^2 v_2)} + 2\nu \frac{\partial^2}{\partial \eta^2}\overline{(v_1 v_2)} \tag{12}$$

If this equation is applied to $\eta = 0$, we come back to (4). On the other hand, since we have assumed that Re_I is large, it is possible that for values of η comparable with L or exceeding L, the second term on the right hand side of (12) can be discarded and:

$$\frac{\partial}{\partial t}\overline{(v_1 v_2)} \cong \frac{\partial}{\partial \eta}\overline{(v_1^2 v_2)} \tag{12a}$$

By integrating eq. (12) with respect to η from 0 to ∞, we find:

$$dJ_0/dt = 0 \tag{13}$$

where

$$J_0 = \int_0^\infty \overline{v_1 v_2}\, d\eta = \overline{v^2}\, L \tag{14}$$

This proves that J_0 is the analogue of LOITSIANSKY's invariant in hydrodynamic turbulence.

3. *Fourier analysis of* $\overline{v_1 v_2}$ *and* $\overline{v_1^2 v_2}$. — Following BATCHELOR we put:

$$\overline{v_1 v_2} = \int_0^\infty dn\ \Gamma(n) \cos n\eta \tag{15}$$

$$\overline{v_1^2 v_2} = \int_0^\infty dn\ \Psi(n) \sin n\eta \tag{16}$$

so that:

$$\pi \Gamma(n) = \int_{-\infty}^{+\infty} \overline{v_1 v_2} \cos n\eta \, d\eta \tag{17a}$$

$$\pi \Psi(n) = \int_{-\infty}^{+\infty} \overline{v_1^2 v_2} \sin n\eta \, d\eta \tag{17b}$$

The function Γ will be even in n and for small n we can write:

$$\Gamma = \Gamma_0 - \tfrac{1}{2} n^2 \Gamma_2 + \ldots . \tag{18a}$$

From (14) we obtain $\Gamma_0 = 2 J_0/\pi$.

The function Ψ will be odd in n, so that for small n:

$$\Psi = n \Psi_1 - \tfrac{1}{6} n^3 \Psi_3 + \ldots . \tag{18b}$$

The FOURIER transform of the fundamental equation (12) becomes:

$$\frac{\partial \Gamma}{\partial t} = n \Psi - 2 \nu n^2 \Gamma \tag{19}$$

This gives:

$$\begin{cases} d\Gamma_0/dt = 0 \\ d\Gamma_2/dt = -2 \Psi_1 + 4 \nu \Gamma_0 \\ d\Gamma_{2m}/dt = -2m \Psi_{2m-1} + 4 \nu m (2m-1) \Gamma_{2m-2} \end{cases} \tag{20}$$

4. *Particular solution of eq.* (1). — We write: $y^* = (y - y_0)/(t - t_0)^{\frac{1}{2}}$ and assume: $v = V(y^*)/(t - t_0)^{\frac{1}{2}}$. Equation (1) transforms into:

$$\nu V'' - V V' + \tfrac{1}{2} y^* V' + \tfrac{1}{2} V = 0,$$

accents denoting derivatives with respect to y^*. Integration gives:

$$\nu V' - \tfrac{1}{2} V^2 + \tfrac{1}{2} y^* V = 0,$$

the integration constant having been adjusted so that V can vanish at infinity. We now put:

$$V = -2 \nu \, d(\ln u)/dy^*$$

and obtain the following equation for u:

$$\frac{d^2 u}{dy^{*2}} = -\frac{y^*}{2\nu} \frac{du}{dy^*}.$$

This equation can be integrated. The resulting expression for V can be brought into the form:

$$V = \frac{2 \nu A \exp\{(A^2 - y^{*2})/4\nu\}}{4\nu - A \int_A^{y^*} dy_1 \exp\{(A^2 - y_1^2)/4\nu\}} \tag{21}$$

where A is an integration constant.

The following approximations are valid when A^2/ν is large:

(*a*) when $y^* = A + \delta$, where $\delta \ll A$:

$$V \cong \tfrac{1}{2} A - \tfrac{1}{2} A \tanh (A\delta/4\nu);$$

(*b*) when $0 < y^* < A$ and y^* not too near to one of the endpoints:

$$V \cong y^*;$$

(*c*) when y^* is near zero or is negative:

$$V \cong 2 \sqrt{\frac{\nu}{\pi}} \frac{\exp(-y^{*2}/4\nu)}{1 - \mathrm{Erf}(y^*/2\sqrt{\nu})}.$$

With the aid of these approximations we can obtain a general picture of the course of V as a function of y^*, and thus of the course of v as a function of y and t (see fig. 1). The curve for v approaches to a rectangular triangle

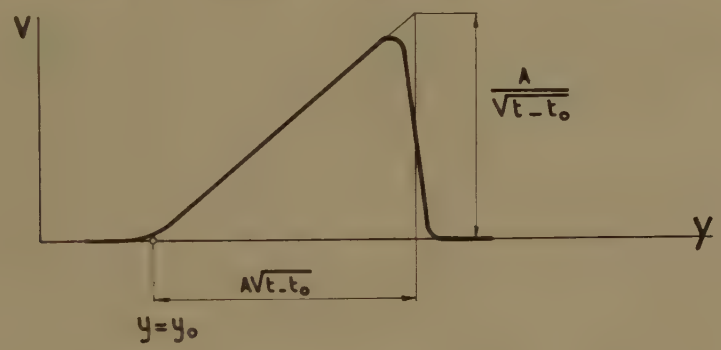

Fig. 1.

with base length $A(t-t_0)^{1/2}$ and height $A/(t-t_0)^{1/2}$. The area is equal to $\frac{1}{2}A^2$ and remains constant (conservation of momentum). The kinetic energy amounts to:

$$\tfrac{1}{6}(A\sqrt{t-t_0})\,(A/\sqrt{t-t_0})^2 = \tfrac{1}{6} A^3 (t-t_0)^{-1/2},$$

while the value of the dissipation integral is found to be:

$$\nu \int (\partial v/\partial y)^2 \, dy = \tfrac{1}{12} A^3 (t-t_0)^{-3/2}.$$

In the approximation used here, only the steep part at the end of the curve contributes to the integral.

It should be noted that there is also a solution:

$$v = -V(y^*)/(t-t_0)^{1/2} \quad \text{with} \quad y^* = -(y-y_0)/(t-t_0)^{1/2}.$$

I have considered these results (and similar ones found in previous work) as an indication that an investigation of the behaviour of approximate solutions, formed out of almost rectilinear parts interspersed with nearly vertical jumps, will be sufficiently interesting to make it worth while. These approximate solutions belong to the category of

those constructed in the preceding paper when we considered the properties of certain two-dimensional fields. An important property of these solutions is that they present features which can be easily *counted*; this is a distinct advantage when we have to do with statistical problems [2]).

5. The approximate method of solution is based upon the circumstance that in a domain where $\partial v/\partial y$ and $\partial^2 v/\partial y^2$ are of normal order of magnitude, we can simplify eq. (1) to:

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial y} = 0 \tag{22}$$

We will suppose that the initial course of v can be represented by a series of straight segments, forming a broken line. Consider a particular segment, given by:

$$v = \beta\,(y - \sigma),$$

where β and σ are functions of t. When this expression is substituted into (22), the equation is satisfied identically in y if:

$$\beta = 1/(t + t_0), \quad \text{and} \quad \sigma = \text{constant} \tag{23}$$

t_0 being another constant. Hence every straight segment will turn to the right, while its point of intersection with the horizontal axis (its "hinge point") remains unchanged, the angle α (compare fig. 2) increasing according to: $\tan \alpha = t + \textit{constant}$.

This result can immediately be applied to a series of segments. — The point of intersection of two consecutive segments moves along a horizontal line with a velocity equal to the height of the point above the

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[2]) It has been pointed out to the author by J. D. Cole and V. Bargmann at Pasadena, Cal., that the substitution

$$v = -\,2\nu\,\partial\,(ln\,u_1)/\partial y$$

can be used to transform the original equation (1) into a linear equation of the third order, which can immediately be integrated into:

$$\frac{\partial u_1}{\partial t} = \nu\,\frac{\partial^2 u_1}{\partial y^2} + u_1\,C\,(t),$$

$C(t)$ being an arbitrary function of the time. If we eliminate this function with the aid of the substitution: $u_1 = u_2 \cdot \exp\,\{\int C(t)\,dt\}$, we are left with:

$$\frac{\partial u_2}{\partial t} = \nu\,\frac{\partial^2 u_2}{\partial y^2}.$$

In this way it becomes possible to write down in explicit form solutions of eq. (1) starting from given initial conditions, *e.g.* from an arbitrarily distributed system of concentrated loads. However, the circumstance that ν is supposed to be very small, makes these solutions not always convenient in further deductions.

The case of a single concentrated load leads automatically to the particular solution considered in the text.

axis. If the point is below the axis, the velocity is negative and the point moves to the left. A graphical construction can be easily executed.

When the initial slope of the segments was positive, it will remain so for ever, the slope gradually decreasing to zero. If the initial slope is

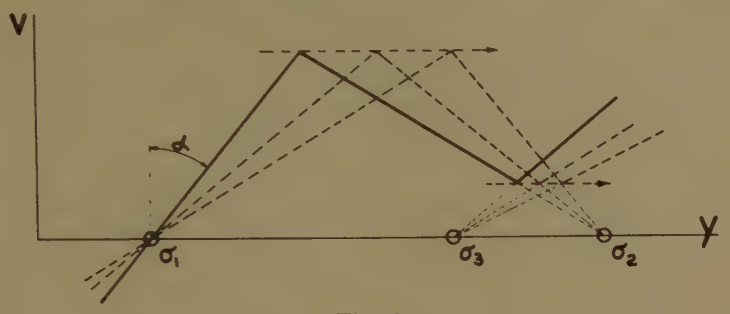

Fig. 2.

negative, the slope will increase in absolute measure and after a finite lapse of time the segment will approach to a vertical position. Equation (22) can then no longer be used and we must have recourse to an appropriate solution of the original equation. In the same way as in the preceding paper it is found that a better approximation to the course of v is given by the expression:

$$(24) \qquad v = \tfrac{1}{2}(v_- + v_+) - \tfrac{1}{2}(v_- - v_+)\tanh\{(v_- - v_+)(y-\xi)/4\nu\}$$

together with:

$$(25) \qquad d\xi/dt = \tfrac{1}{2}(v_- + v_+)$$

v_- being the limiting value of v on the left hand side and v_+ that on the right hand side of the vertical segment.

Two conclusions will be drawn from this result. One is, that the approximate picture in which the course of v is taken as a chain of rectilinear segments, can still be used when segments originally sloping downwards have acquired a vertical direction. From that instant onward they do not rotate any more, but move with the velocity $d\xi/dt$ given by (25), keeping their vertical direction. This feature can be introduced into the graphical construction alluded to before (see fig. 3). It must be observed that vertical segments following each other, can overtake one another. When this occurs, they combine to form a single segment, moving from then onward with a velocity again given by (25), v_- and v_+ now referring to the limiting velocities at the end points of the new segment.

The second conclusion is, that when greater accuracy is needed with regard to the "rounding off" at the ends of a vertical segment, we can use the hyperbolic tangent function indicated in (24) as a convenient approximation. In this way it is found that the dissipation integral referring to a single vertical segment has the value $\frac{1}{12}(v_- - v_+)^3$. — It

should be kept in mind that $(v_- - v_+)$ is positive for every vertical segment, as will be evident from the process by which these segments are generated.

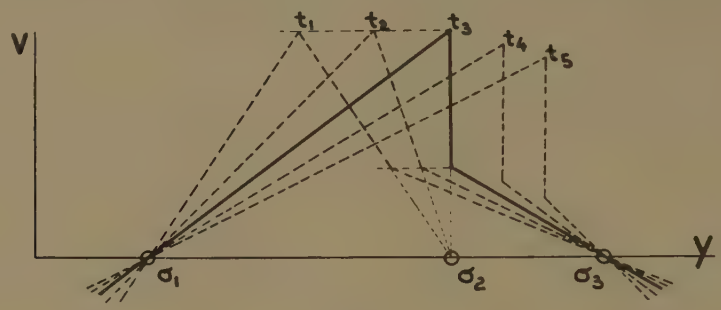

Fig. 3.

6. With a solution starting from a course of $v(y)$ given by a chain of straight segments, several periods can be distinguished in its history. There is first the period in which segments with a negative slope change into vertical segments, so that an increasing number of apparent discontinuities in the course of v is developed. When the number of vertical segments has sufficiently increased, there will arise an appreciable chance for segments to overtake each other and the number of vertical segments will again decrease. At the same time the slope of the positively inclined segments steadily becomes smaller, so that the average amplitude of the curve will become less and less. This may bring us to a third period, in which the REYNOLDS' number Re_I, defined in (11), will no longer be large enough for our approximation to remain valid. We should then have recourse to a more accurate solution of the original equation (see footnote 2).

In the following lines I will give attention to the *second* period, in which there is a gradual coalescence of vertical segments. In order to study this phenomenon in a pure form, not influenced by the formation of new vertical segments, it will be assumed that the initial state of the system is already given by a *series of parallel straight segments*, all having the same slope β upward to the right, separated from each other by vertical segments. During the development of the system in the course of time the slope of the inclined segments will decrease according to (23); their parallelism will be retained. To simplify the equations, we assume $\beta = 1/t$; the "initial state" of the system can be defined as the state for $t = 1$.

A description of the state of the system at a given instant of time is obtained (compare fig. 4) by stating

(1) the values of the ξ_i which define the positions of the vertical segments at that instant, and

(2) the values of the σ_i which define the locations of the "hinge points".

These data are subject to the evident conditions: $\xi_{i-1} < \xi_i$; $\sigma_{i-1} < \sigma_i$. We introduce the following derived quantities:

$$\text{(26)} \qquad \begin{cases} \lambda_i = \xi_i - \xi_{i-1} \quad ; \quad \tau_i = \sigma_i - \sigma_{i-1} \\ \zeta_i = \xi_i - \frac{1}{2}(\sigma_i + \sigma_{i-1}) \end{cases}$$

Thus λ_i is the length of a domain in which v changes linearly with y; $\beta\tau_i$ is the height of the vertical segment on the right hand side of λ_i; and

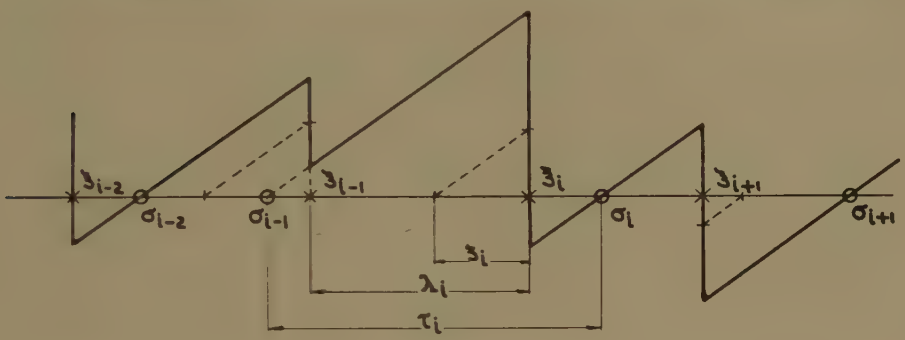

Fig. 4.

$\beta\zeta_i$ is the height above the y-axis of the point midway on this vertical segment. The quantities λ_i and τ_i are always positive; the ζ_i can be positive as well as negative.

7. In the present description the laws of motion of the system are:

(I) $\beta = 1/t$.

(II) $d\sigma_i/dt = 0$.

(III) $d\xi_i/dt = d\zeta_i/dt = \beta\zeta_i$.

(IV) When two consecutive ξ's (say ξ_{i-1} and ξ_i) come to coincidence, the corresponding vertical segments combine and the hinge point σ_{i-1} disappears (see fig. 5).

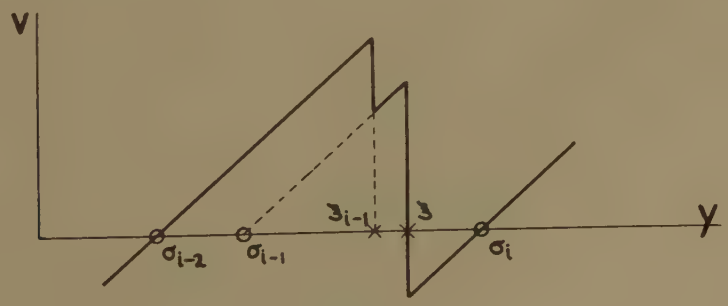

Fig. 5.

(V) The τ_i are constants, until two consecutive ξ_i coincide and a σ disappears. When σ_{i-1} disappears, τ_{i-1} and τ_i combine to form a single

entity and the number of segments τ in a domain of the y-axis of length S decreases by one. The average length of these segments correspondingly increases.

(VI) The λ_i are functions of the time:

$$d\lambda_i/dt = \beta\{\lambda_i - \tfrac{1}{2}(\tau_i + \tau_{i-1})\} = \beta(\zeta_i - \zeta_{i-1}).$$

This quantity can be negative as well as positive. When λ_i is small, the chance for a negative value will be large; thus it is possible that λ_i decreases to zero and disappears from the system. Hence there is a progressive decrease of the number of segments λ, keeping exactly pace with the reduction of the number of segments τ. The average length of the λ_i correspondingly increases.

(VII) The ζ_i increase proportionally with the time, so that $\beta\zeta_i = \zeta_i/t$ remains constant, until two consecutive ξ_i come to coincidence. At the instant of coincidence of ξ_{i-1} and ξ_i we have $\zeta_{i-1} - \frac{1}{2}\tau_{i-1} = \zeta_i + \frac{1}{2}\tau_i$; ζ_{i-1} and ζ_i disappear as such and are replaced by a single new value of ζ, equal to $\zeta^0 = \zeta_{i-1} - \frac{1}{2}\tau_i = \zeta_i + \frac{1}{2}\tau_{i-1}$ [3]).

8. We can consider various initial states of the system, that is, various initial arrangements of the successive values of λ_i and τ_i. Also one ζ must be given initially; the other ζ_i then follow from the λ_i and τ_i. We assume that the system is statistically homogeneous, so that mean values can be defined. Along with mean values taken over a certain length of the y-axis, it is convenient to introduce another type, obtained by summation with respect to i over a large number (N) of consecutive values, and division by N. Such mean values will be indicated by the sign $\overline{}$.

We introduce the assumption that the mean values of the λ_i and of the τ_i are equal:

$$\overline{\lambda_i} = \overline{\tau_i} = l \tag{27}$$

The number N_λ of segments λ in a certain large domain S of the y-axis

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[3]) The laws of motion and the tendency towards reduction of the number of segments can be illustrated with the aid of a molecular analogue, in which molecules colliding with each other immediately combine. We give to the molecules

coordinates	:	ξ_i
velocities	:	$d\xi_i/dt = \beta\zeta_i$
masses	:	τ_i
momenta	:	$\beta\tau_i\zeta_i$.

At every collision masses and momenta are added; kinetic energy is lost. The description of the process obtains slightly greater clarity, if instead of τ_i, ζ_i we introduce the notation $\tau_{i-1,i}$, $\zeta_{i-1,i}$. Then in a collision $\tau_{i-1,i}$ and $\tau_{i,i+1}$ combine to $\tau_{i-1,i+1}$, etc.

A simple graphical construction can be worked out, starting from the initial values of the $\xi_{i-1,i}$ and those of $\frac{1}{2}(\sigma_{i-1} + \sigma_i)$.

and the number N_τ of segments τ in this domain will then be practically the same:

$$N_\lambda = N_\tau = N = S/l \tag{27a}$$

When this relation is fulfilled in the initial state, it will be maintained in consequence of the laws of motion.

It follows that mean values obtained by summation with respect to i can be reduced to mean values defined with respect to unit length of the y-axis, by division through l.

The property of statistical homogeneity entails that in calculating mean values by summation, we may start from an arbitrary value of i. There is thus no difference between expressions like $\overline{\tau_i^2}$ and $\overline{\tau_{i+k}^2}$.

The property of statistical invariance with respect to a simultaneous change of sign of v and y, now takes the form that any statistical quantity formed out of the λ_i, τ_i and ζ_i, will remain unchanged when the signs of all ζ_i are changed simultaneously with a change in the direction in which i is counted, no change of sign being made with respect to λ_i and τ_i. For instance:

$$\overline{\lambda_i \tau_i} = \overline{\lambda_i \tau_{i-1}} \; ; \; \overline{\lambda_i \zeta_i} = -\overline{\lambda_i \zeta_{i-1}} \; ; \; \overline{\tau_i \zeta_{i+k}} = -\overline{\tau_i \zeta_{i-k}} \quad (k = 0, 1, 2, \ldots).$$

This rule implies that:

$$\overline{\zeta_i} = 0 \quad ; \quad \overline{\tau_i^p \zeta_i} = 0 \quad (p = 1, 2, \ldots) \tag{28}$$

We cannot exclude the possibility that there may exist correlations between the values of consecutive λ_i or of consecutive τ_i, or between λ_i and τ_i of the same or only slightly differing indices. Such correlations may arise from the laws of motion. It is also to be observed that we require the property of statistical homogeneity to be valid with respect to expressions involving the ζ_i. The values of the ζ_i would become large, if over some large section of the y-axis the λ_i and τ_i should get more and more out of step; we assume that on the long run this is always corrected in such a way that it has a sense to speak of mean values like $\overline{\zeta_i}$ and $\overline{\zeta_i^2}$, etc.

We can neither assume that expressions of the type $\overline{\zeta_i \zeta_j}$ with $i \neq j$ will necessarily be zero in consequence of (28). However, it is possible to assume that such quantities will vanish when the difference between i and j increases without limit, in such a way that

$$\sum_{k=-\infty}^{k=+\infty} \overline{\zeta_i \zeta_{i+k}} \quad \text{converges} \tag{29}$$

This equation expresses the ultimate randomness of phase differences at large distances.

9. *Calculation of mean values.*

I. *Mean value of* v. — In order to show the method to be used in calculating mean values, we start with that of v, although it is evident that the result must be zero. We observe that over a segment λ_i the variable v changes linearly with y; hence for a single segment:

$$\int v\,dy = \frac{1}{\beta}\int v\,dv = \tfrac{1}{2}\beta\left\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2\right\}.$$

This expression must be summed with respect to i over all segments contained in a domain of length S; the sum must be divided by $S = Nl$. Having regard to the relations: $\overline{\zeta_i^2} = \overline{\zeta_{i-1}^2}$; $\overline{\tau_i \zeta_i} = 0$, etc., which follow from the rules of invariance mentioned in section 8, it will immediately be found that $\overline{v} = 0$.

II. *Mean value of* v^2. — The same procedure is applied. Integration over the length of the segment λ_i gives:

$$\int v^2\,dy = \frac{1}{\beta}\int v^2\,dv = \tfrac{1}{3}\beta^2\left\{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^3\right\}.$$

Hence taking the mean value:

$$\overline{v^2} = (\beta^2/l)\,(\overline{\tau_i \zeta_i^2} + \tfrac{1}{12}\overline{\tau_i^3}) \tag{30}$$

We shall write:

$$\overline{\tau_i^2} = l^2(1+\omega)\ ;\ \overline{\tau_i^3} = l^3(1+\omega^*)\ ;\ \overline{\tau_i \zeta_i^2} = l^3\,\tilde{\omega} \tag{31}$$

Then:

$$E = \tfrac{1}{2}\overline{v^2} = \tfrac{1}{2}\beta^2 l^2\left\{\tilde{\omega} + \tfrac{1}{12}(1+\omega^*)\right\} \tag{32}$$

III. *Value of* ε. — It has been mentioned at the end of section 5 that contributions to the dissipation integral exclusively derive from the vertical segments, each segment giving an amount which in the present notation is equal to $\frac{1}{12}\beta^3\tau_i^3$.

Hence:

$$\varepsilon = \tfrac{1}{12}(\beta^3/l)\,\overline{\tau_i^3} = \tfrac{1}{12}\beta^3 l^2(1+\omega^*) \tag{33}$$

It follows that

$$\lambda^2 = \nu t\left\{1 + \frac{12\,\tilde{\omega}}{1+\omega^*}\right\} \tag{34}$$

10. To find the mean value of the correlation product $v_1 v_2$ an elaborate calculation is necessary, which will be given in sections 13 and 14. However, it is a simple matter to find the value of J_0. It is convenient to write:

$$2J_0 = \lim \int_a^b \overline{v_1 v_2}\,d\eta \quad \text{with } a \to -\infty,\ b \to +\infty.$$

Since the quantity to be obtained depends on two integrations, one

with respect to y to find the mean value of $v_1 v_2$, the other one with respect to η, we interchange the order of integration and first calculate:

$$v(y) \int v(y+\eta)\, d\eta$$

with a fixed value of y, which we suppose to be situated in the segment λ_i. The integral with respect to η, taken over the length of a segment λ_{i+k}, has the value:

$$\tfrac{1}{2}\beta\, v(y)\,\{(\zeta_{i+k}+\tfrac{1}{2}\tau_{i+k})^2-(\zeta_{i+k-1}-\tfrac{1}{2}\tau_{i+k-1})^2\}.$$

We must take a sum of such expressions, from $k=-p$ until $k=+q$, p and q being large numbers. The domain of integration will not end precisely at vertical segments, and there will remain at each end an integral referring to a part of a segment only. We shall indicate the contributions from these parts by

$$\tfrac{1}{2}\beta\, v(y)\,\{(\zeta_{i-p}+\tfrac{1}{2}\tau_{i-p})^2-v_{\mathrm{I}}^2\}$$

and

$$\tfrac{1}{2}\beta\, v(y)\,\{v_{\mathrm{II}}^2-(\zeta_{i+q}-\tfrac{1}{2}\tau_{i-q})^2\}.$$

Summing up we obtain:

$$\beta\, v(y)\left\{\sum_{-p}^{+q}\tau_{i+k}\,\zeta_{i+k}-\tfrac{1}{2}v_{\mathrm{I}}^2+\tfrac{1}{2}v_{\mathrm{II}}^2\right\}.$$

It should be observed that v_{I} and v_{II} are dependent on y.

We now integrate with respect to y over the length of the segment λ_i. This gives

$$\tfrac{1}{2}\beta^2\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_{i-1}-\tfrac{1}{2}\tau_{i-1})^2\}\sum_{-p}^{+q}\tau_{i+k}\,\zeta_{i+k}-\delta_{\mathrm{I}}+\delta_{\mathrm{II}},$$

where δ_{I}, δ_{II} are the integrals of $\tfrac{1}{2}\beta\, v(y)\, v_{\mathrm{I}}^2$, $\tfrac{1}{2}\beta\, v(y)\, v_{\mathrm{II}}^2$ respectively.

When now the mean value is calculated by summation with respect to i, we can safely assume that the mean values of δ_{I} and δ_{II} cancel, since for large p and q there is no relation between $v(y)$ and either v_{I} or v_{II}. As regards the other terms, the mean value of quantities containing uneven powers of the ζ_i will either vanish or cancel on account of the second invariant property, so that there will remain:

$$\tfrac{1}{2}\beta^2\sum_{k=-p}^{k=-q}\overline{(\tau_i\,\zeta_i+\tau_{i-1}\,\zeta_{i-1})\,\tau_{i+k}\,\zeta_{i+k}}.$$

In extension of (29) we assume that this sum converges. Hence we obtain:

$$J_0=\frac{\beta^2}{2l}\sum_{k=-\infty}^{k=+\infty}\overline{\tau_i\,\tau_{i+k}\,\zeta_i\,\zeta_{i+k}} \tag{35}$$

Since this must be independent of the time, we must have:

$$(1/l)\sum\overline{\tau_i\,\tau_{i+k}\,\zeta_i\,\zeta_{i+k}}\sim t^2.$$

It can be proved in a direct way that this sum is not affected by the coalescence of segments and increases only in consequence of the relation $d\zeta_i/dt = \beta\zeta_i = \zeta_i/t$. We shall come to this in section 18.

It is even true that

$$\beta \Sigma \tau_i \zeta_i$$

is not affected by the coalescence of segments (compare footnote 3, where this quantity occurs as the momentum of a system of combining molecules). This sum, however, is not convergent, while its mean value with respect to i is zero.

(To be continued).

Note. — The expression for J_0 can also be written:

$$J_0 = \frac{\beta^2}{l} \{\tfrac{1}{2} \overline{\tau_i^2 \zeta_i^2} + \sum_{k=1}^{k=\infty} \overline{\tau_i \tau_{i+k} \zeta_i \zeta_{i+k}}\} \tag{35a}$$

There may be cases where the terms with $k \neq 0$ all vanish.

ON THE GEOMETRY OF SPIN SPACES. IV

BY

J. A. SCHOUTEN

(Communicated at the meeting of January 28, 1950)

§ 8. *The group G_s for $v = 1$ and $v = 2$.*

In this and the following sections we distinguish between the following groups in R_n and their corresponding groups of normalized (Det. $= \pm 1$, cf. § 4) transformations in $\mathfrak{S}_N$:

R_n ; $n = 2v$			$\mathfrak{S}_N$; $N = 2^v$	
G_{or}	orthogonal,	$\Delta = \pm 1$	G_s	
G'_{or}	real orthogonal,	$\Delta = \pm 1$		G'_s
G_{ro}	rotational,	$\Delta = +1$	G_s^+	
G'_{ro}	real rotational,	$\Delta = +1$		G'^+_s
G^*_{ro}	proper real rotations, neither + - nor — - reflexional	$\Delta = +1$,		G^*_s

The transformations of G_{ro}, G^*_{ro}, G_s^- and G^*_s can be reached continuously starting from the identical transformation and accordingly these groups can be dealt with conveniently by means of infinitesimal transformations.

Let $A^A_B + T^{.A}_{.B}\,dt$ or shortly $A + T dt$ be an infinitesimal transformation of G_s^-. For these transformations R and C are invariant (cf § 4). From the invariance of R

$$(A + T dt)\, R\, (A - T dt) = R \tag{8.1}$$

we get

$$\boxed{TR - RT = 0}\,. \tag{8.2}$$

From the invariance of C

$$({}'A + {}'T dt)\, C\, (A + T dt) = C \tag{8.3}$$

we get

$$\boxed{{}'TC + CT = 0}\,. \tag{8.4}$$

Ω is invariant for all transformations of G^*_s (cf § 4). Hence, if $A + T dt$ belongs to G^*_s, from the invariance of Ω

$$({}'\bar{A} + {}'\bar{T} dt)\, \Omega\, (A + T dt) = \Omega \tag{8.5}$$

it follows that

$$\boxed{{}'\bar{T}\Omega + \Omega T = 0}\,. \tag{8.6}$$

For $\nu = 1$, $(n = 2,\ N = 2)$, we get from (8.2), (8.4) and (8.6), writing $1,1'$ instead of $1,2$:

$$T^{1}_{\cdot 1'} = 0 \ ; \ T^{1'}_{\cdot 1} = 0 \tag{8.7}$$

$$T^{1}_{\cdot 1} + T^{1'}_{\cdot 1'} = 0 \tag{8.8}$$

$$\begin{cases} a.\ \text{for } \underset{1}{\varepsilon} = \underset{2}{\varepsilon} : \ \bar{T}^{\bar{1}}_{\cdot\bar{1}} = -T^{1}_{\cdot 1} \ ; \ \bar{T}^{\bar{1}'}_{\cdot\bar{1}'} = -T^{1'}_{\cdot 1'} \\ b.\ \text{for } \underset{1}{\varepsilon} = \pm i \underset{2}{\varepsilon} : \ \bar{T}^{\bar{1}}_{\cdot\bar{1}} = -T^{1'}_{\cdot 1'} \ ; \ \bar{T}^{\bar{1}'}_{\cdot\bar{1}'} = -T^{1}_{\cdot 1}. \end{cases} \tag{8.9}$$

From (8.7) we see that the group G_s^+ splits up into two groups each in one of the two invariant E_1's in $\mathfrak{S}_2$. But from (8.8) we see that there is a one to one correspondence between the transformations of these two groups. (But not between the vectors in the two E_1's!) If this occurs for some value of ν we always say that the groups are *coupled*. In this case we need only one of the E_M's $(M = \frac{1}{2}N)$ and we call this E_M the *reduced spinspace* $\mathfrak{S}_M$.

If we write $T^{1}_{\cdot 1} = i\gamma$ we get from (3.10) for $\lambda = 1$ the transformation in R_2

$$\begin{cases} i \to \underset{1}{i} + 2\gamma \underset{1}{\varepsilon}\underset{2}{\varepsilon}\underset{2}{i}\,dt \\ \underset{2}{i} \to -2\gamma \underset{2}{\varepsilon}\underset{1}{\varepsilon}\underset{1}{i}\,dt + \underset{2}{i} \end{cases} \tag{8.10}$$

and from (8.9) it follows that this transformation is real because γ is real for $\underset{1}{\varepsilon} = \underset{2}{\varepsilon}$ and purely imaginary for $\underset{1}{\varepsilon} = i\underset{2}{\varepsilon}$. Hence, for $\underset{1}{\varepsilon} = \underset{2}{\varepsilon}$, G_s^* consists of the homogeneous affine transformations in one E_1 leaving invariant the part of Ω in this E_1 (Ω is (\)) and the transformations coupled with them in the other E_1. But, for $\underset{1}{\varepsilon} = \pm i\underset{2}{\varepsilon}$, G_s^* consists of the coupled homogeneous affine transformations in both E_1's, satisfying the condition that together each pair leaves invariant Ω (Ω is (/)). In both cases the coupled transformations depend on one *real* parameter.

From the finite proper rotations we get the reflexotations by adding one real reflexotation for instance a reflexion at a real axis. The transformation $\underset{1}{i}^{1}_{\cdot R}$ in $\mathfrak{S}_N$ represents for any value of ν a reflexion at the 1-axis because $\underset{1}{i}^{-1} = \varepsilon^2 \underset{1}{i}$ and

$$\underset{1\,1}{i}\,\underset{1}{i}\,\underset{1}{i}^{-1} = +\underset{1}{i} \ ; \ \underset{1\,x}{i}\,\underset{1}{i}\,\underset{1}{i}^{-1} = -\underset{x}{i} \ ; \ x = 2, \ldots, n \tag{8.11}$$

More generally $\underset{i_1 \ldots i_p}{i}$ represents for any choice of the unit vectors in R_n a real reflexotation for p odd. For $\nu = 1$ the reflexions at the 1-axis and the 2-axis correspond to the following transformations in $\mathfrak{S}_2$

$$\begin{cases} \underset{1}{i} \to \underset{1}{i} \ ; \ \underset{2}{i} \to -\underset{2}{i} : \ 1 \to \underset{1}{\varepsilon}\,1' \ ; \ 1' \to \underset{1}{\varepsilon}\,1 \\ \underset{1}{i} \to -\underset{1}{i} \ ; \ \underset{2}{i} \to \underset{2}{i} : \ 1 \to -\underset{1}{i}\varepsilon 1' \ ; \ 1' \to \underset{2}{i}\varepsilon 1 \end{cases} \tag{8.12}$$

Each of them interchanges the two E_1's in some way.

The improper rotations can be obtained by adding two suitably chosen reflexotations. Hence G_s' consists of all transformations that can be obtained by combining a transformation of G_s^* with *0,1* or *2* times interchanging the two E_I's in a way corresponding to a real reflexotation. The two E_I's are not individually invariant for G_s' and accordingly there does not exist a reduced spinspace. For each of the cases (*a*), (*b*), (*c*) and (*d*) the transformations of G_s' depend on one *real* parameter and the invariance or invariance but for the sign of R and Ω is given in the table on page (695) 225. From $G_s'^-$ we get G_s^+ by dropping the invariance of Ω. Hence G_s^- consists of the coupled homogeneous affine transformations in both E_I's. For G_s^- a reduced spinspace exists and the transformations depend on one *complex* parameter. G_s consists of all transformations of G_s^- and all those that can be obtained by combining a transformation of G_s^- with an interchanging of the two E_I's corresponding to a reflexotation. No reduced spinspace exists and the transformations with $\Delta = -1$ depend also on one complex parameter.

For $\nu = 2$ we get from (8.2) writing *1, 2, 1', 2'* for *11, 22, 21, 12*

$$T^a_{\cdot b'} = 0 \;;\; T^{a'}_{\cdot b} = 0 \;;\; a, b = 1,2 \;;\; a', b' = 1', 2'. \tag{8.13}$$

For C we have (cf. (6.2))

$$C :: (12 + 21)(12 - 21) = 12 - 21 + 1'2' - 2'1', \tag{8.14}$$

hence from (8.4) we get

$$T^1_{\cdot 1} + T^2_{\cdot 2} = 0 \;;\; T^{1'}_{\cdot 1'} + T^{2'}_{\cdot 2'} = 0. \tag{8.15}$$

For Ω we have from the rule of § 6 for $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$

$$\Omega :: \omega(1, 1)\,\omega(1, 1) :: \bar{1}1 + \bar{2}2 + \bar{1}'1' + \bar{2}'2' \tag{8.16}$$

and for $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = i;\ \varepsilon_4 = 1$ (the case of relativity)

$$\Omega :: \omega(i, i)\,\omega(i, 1) :: \bar{1}2' + \bar{2}1' - \bar{1}'2 - \bar{2}'1. \tag{8.17}$$

Hence from (8.6) and (8.16) we get

$$\bar{T}^{\bar{a}}_{\cdot \bar{b}} = -T^b_{\cdot a} \;;\; \bar{T}^{\bar{a}'}_{\cdot \bar{b}'} = -T^{b'}_{\cdot a'} \;;\; \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1 \tag{8.18}$$

and from (8.6) and (8.17):

$$\begin{cases} a) & T^{1'}_{\cdot 1'} = -\bar{T}^{\bar{2}}_{\cdot \bar{2}} = \bar{T}^{\bar{1}}_{\cdot \bar{1}} \\ b) & T^{2'}_{\cdot 2'} = -\bar{T}^{\bar{1}}_{\cdot \bar{1}} = \bar{T}^{\bar{2}}_{\cdot \bar{2}} \\ c) & T^{1'}_{\cdot 2'} = -\bar{T}^{\bar{1}}_{\cdot \bar{2}} \\ d) & T^{2'}_{\cdot 1'} = -\bar{T}^{\bar{2}}_{\cdot \bar{1}} \end{cases} \tag{8.19}$$

from which only (8.19 *a, c, d*) are independent because of (8.15).

Because of (8.13) the group G_s^+ splits up into two groups each in one of the invariant E_2's in $\mathfrak{S}_4$. But here C is [$\diagdown$] and accordingly the two groups are not coupled. (8.15) expresses the fact that both groups are unimodular and that for each of them three complex parameters are left.

For the definite case, (8.18) reduces the parameters in each E_2 to three real ones but there is no coupling and no reduced spin space. The same holds for $\nu = 2$ for every even value of s because it is only due to the fact that Ω is ($\diagdown$). As we will see later the lack of a reduced spinspace for G_s^* is an exception only occurring for $\nu = 2$ and s even. G_s^* consists in this case of two independent groups in the two E_2's each consisting of the unimodular homogeneous affine transformations leaving invariant the part of Ω and depending on three real parameters.

For the case with $s = 3$, (8.19) couples the groups, so *the coupling is here due to the invariance of* Ω. Every transformation in one E_2 is (in *this* representation) coupled with the conjugate transformation in the other E_2 (for $-1, 2$) and the coupled pairs depend on three complex, that is six real, parameters. There is a reduced spin space. The same holds for every odd value of s for $\nu = 2$ because it is only due to the fact that Ω is ($\diagup$). G_s^* consists now of the complex unimodular homogeneous affine transformations in one E_2 and the transformations coupled with them in the other E_2.

The group G_s' consists of all transformations that can be obtained by combining a transformation of G_s^* with one or two times interchanging the two E_2's in a way corresponding to a real reflexotation. There is no longer a reduced spinspace. In each of the cases (a), (b), (c) and (d) the transformations of G_s' depend on six real parameters and R and C are invariant or invariant to within the sign according to the table on page (695) 225.

The group G_s^+ can be obtained by dropping the invariance of Ω. *For* G_s^+ *there is no coupling and no reduced spinspace.* G_s^- consists of the unimodular homogeneous affine transformations in both E_2's and the total number of complex parameters is six. G_s consists of all transformations of G_s^- and all those that can be obtained by combining a transformation of G_s^+ with an interchanging of the two E_2's corresponding to a reflexotation. No reduced spinspace exists and the transformations of G_s with $\triangle = -1$ depend also on six complex parameters.

§ 9. *The group* G_s *for* $\nu > 2$.

According to the first invariant condition (cf § 3) we have

$$R = \pm ({}^1{}_1 - {}^2{}_2)\ ({}^1{}_1 - {}^2{}_2) \ldots \qquad (\nu \text{ factors}). \tag{9.1}$$

In order to get R in the form (3.8) we number all M combinations of ν factors *1* or *2*, among which an *even* number of *2*'s occurs, in their natural order from *1* to M and the other M combinations with

an *odd* number of 2's in the *inverted natural order* from $1'$ to M', for instance for $\nu = 2, 3$ and 4:

(9. 2)

$a)$ $11 \to 1 \quad 21 \to 1'$; $22 \to 2 \quad 12 \to 2'$

$b)$ $111 \to 1 \quad 222 \to 1'$; $122 \to 2 \quad 211 \to 2'$; $212 \to 3 \quad 121 \to 3'$; $221 \to 4 \quad 112 \to 4'$

$c)$ $1111 \to 1 \quad 2221 \to 1'$; $1122 \to 2 \quad 2212 \to 2'$; $1212 \to 3 \quad 2122 \to 3'$; $1221 \to 4 \quad 2111 \to 4'$; $2112 \to 5 \quad 1222 \to 5'$; $2121 \to 6 \quad 1211 \to 6'$; $2211 \to 7 \quad 1121 \to 7'$; $2222 \to 8 \quad 1112 \to 8'$

The indices $A, B, \ldots$ in $\mathfrak{S}_N$ now take the values $1, \ldots, M, 1', \ldots, M'$. Note that by interchanging of 1 and 2, $a = 1, \ldots, M$ and $a' = 1', \ldots, M'$ interchange for ν *odd* and a and $M + 1 - a$ as well as a' and $(M + 1 - a)'$ for ν *even*. If A stands for ν numbers 1 or 2 we call t_A the number of 2's occurring at an *even* place, for instance for $\nu = 4$: $t_3 = 2$; $t_{6'} = 1$; $t_{7'} = 0$.

For R we get now (cf (6.14))

$$(9.3) \quad R = \pm (E - E) = \pm ({}^{1}{}_{1} + {}^{2}{}_{2} + \ldots + {}^{M}{}_{M} - {}^{1'}{}_{1'} - {}^{2'}{}_{2'} - \ldots - {}^{M'}{}_{M'})$$

and from (8.2) we get for transformations of G_s^+

$$(9.4) \quad T^{a}_{\cdot b'} = 0 \; ; \; T^{a'}_{\cdot b} = 0 \; ; \; a, b = 1, \ldots, M \; ; \; a', b' = 1', \ldots, M'$$

from which we see that T is *reducible* (cf. § 6) and that the group G_s^+ in $\mathfrak{S}_N$ splits up into two groups in the two invariant E_M's. (9.4) represents $2\,M^2$ conditions, reducing the number of complex parameters from $4\,M^2$ to $2\,M^2$, that is M^2 for each subgroup.

For C we get (cf. (6.1))

$$(9.5) \quad C :: (12 + 21)\,(12 - 21)\,(12 + 21) \ldots \quad (\nu \text{ factors}).$$

If C_{AB} is written out in terms of the form $\overset{C}{e}_A \overset{D}{e}_B$; $A, B, C, D = 1, \ldots, M, 1', \ldots, M'$, the coefficient of $\overset{C}{e}_A \overset{D}{e}_B$ is $(-1)^{t_C}$. Accordingly we have for C the general formulae

$$(9.6) \quad \nu \text{ even}: \; C_{AB} :: \sum_a (-1)^{t_a} \overset{a}{e}_A \overset{M+1-a}{e}_B + \sum_{a'} (-1)^{t_{a'}} \overset{a'}{e}_A \overset{(M+1-a)'}{e}_B$$

(C is reducible)

$$(9.7) \quad \nu \textit{ odd}: \; C_{AB} :: \sum_a (-1)^{t_a} \overset{a}{e}_A \overset{a'}{e}_B + \sum_{a'} (-1)^{t_{a'}} \overset{a'}{e}_A \overset{a}{e}_B$$

(C is connecting)

For ν even we have

$$(9.8) \quad t_{M+1-a} = {}^{\nu}/_{2} - t_a \quad ; \quad t_{(M+1-a)'} = {}^{\nu}/_{2} - t_{a'},$$

hence, according to (8.4), (9.4) and (9.6)

$$(9.9)\quad \nu \text{ even: } \begin{cases} a) & (-1)^{\nu/2+t_d}\, T^{M+1-d}_{\cdot\, c} + (-1)^{t_c}\, T^{M+1-c}_{\cdot\, d} = 0;\ c, d = 1,\ldots, M \\ b) & (-1)^{\nu/2+t_{d'}}\, T^{(M+1-d)'}_{\cdot\, c'} + (-1)^{t_{c'}}\, T^{(M+1-c)'}_{\cdot\, d'} = 0;\ c', d' = 1',\ldots, M'. \end{cases}$$

For $c = d$ (9.9a) takes the form

$$(9.10)\qquad ((-1)^{\nu/2} + 1)\ T^{M+1-c}_{\cdot\, c} = 0$$

and these are M independent equations for $\nu/2$ *even* and identities for $\nu/2$ *odd*. For $c = M + 1 - d$ (9.9a) takes the form

$$(9.11)\qquad T^{c}_{\cdot\, c} + T^{M+1-c}_{\cdot\, M+1-c} = 0$$

and these are $^1/_2\, M$ independent equations. Among the other equations of (9.9a) there are $\binom{M}{2} - {}^1/_2\, M$ independent ones. Hence (9.9a) represents $\binom{M+1}{2}$ independent equations for $\nu/2$ *even* and $\binom{M}{2}$ for $\nu/2$ *odd*. The same is true for (9.9b). That means that for ν *even*, after the introduction of the invariance of R and C, that is for the group G_s^-, the number of complex parameters is reduced from $4\,M^2$ to $2\binom{M}{2}$ for $\nu/2$ *even* and to $2\binom{M+1}{2}$ for $\nu/2$ *odd*. So far the two groups in the two E_M's are entirely independent and each depends on $\binom{M}{2}$ or $\binom{M+1}{2}$ parameters. There is not yet any coupling or reduced spinspace. This is due to the fact that C is reducible for ν even (cf. (6.22)).

For ν *odd* we have

$$(9.12)\qquad t_{a'} = {}^1/_2\,(\nu - 1) - t_a,$$

hence, according to (8.4), (9.4) and (9.7)

$$(9.13)\quad (-1)^{t_d}\, T^{d}_{\cdot\, c} + (-1)^{t_{c'}}\, T^{c'}_{\cdot\, d'} = 0\ ;\ c, d = 1,\ldots, M;\ c', d' = 1',\ldots, M'.$$

These are M^2 independent equations. Hence for ν *odd* the number of complex parameters for G_s^- is reduced to M^2. The two groups are now coupled and each group is the full homogeneous affine group of an E_M.

As we have seen in § 7 there must exist additional conditions for $\nu > 2$. We use the same method as in § 7 but now for infinitesimal transformations. T can always be written in the form

$$(9.14)\qquad T = \overset{0}{a}\, A + \overset{j}{a}\, \underset{j}{i} + \overset{j_1 j_2}{a}\, \underset{j_1 j_2}{i} + \ldots$$

The first term can be dropped because it has no influence at all. Now we know that

$$(9.15)\qquad \underset{j_1 \ldots j_p}{i}\, R = (-1)^p\, R\, \underset{j_1 \ldots j_p}{i},$$

hence, if (9.14) is substituted in (8.2) it follows from the invariance of R that the coefficients a in (9.14) must vanish for $p = 1, 3, 5, \ldots$. In the same way, because of (4.20a)[1]) and (8.4), the invariance of C implies that the a's must vanish for $p = 0, 1, 4, 5, \ldots$. This gives for the invariance of R exactly $\binom{n}{1} + \binom{n}{3} + \ldots = 2M^2$ complex conditions and for the invariance of C just $1 + \binom{n}{4} + \binom{n}{8} + \ldots$ complex conditions, that is in total $2\binom{M}{2}$ for $\frac{1}{2}\nu$ *odd*, $2\binom{M+1}{2}$ for $\frac{1}{2}\nu$ *even* and M^2 for ν *odd*. This agrees with the results obtained above. $A + Tdt$ corresponds to a rotation in R_n. But for every rotation the transforms of $\underset{1}{i}, \ldots, \underset{n}{i}$ must be linear homogeneous in $\underset{1}{i}, \ldots, \underset{n}{i}$ and not depend on the other $n^2 - n$ hypercomplex numbers of M_N. Now let $\overset{1\ldots p}{a}$ in (9.14) be $\neq 0$ for some even value of p. Then the transform of $\underset{1}{i}$ contains a term

$$\overset{1\ldots p}{a}\,(\underset{1}{i}\ldots\underset{p}{i}\,\underset{1}{i} - \underset{1}{i}\,\underset{1}{i}\ldots\underset{p}{i})\,dt = -2\,\overset{1\ldots p}{a}\,\varepsilon^2\,\underset{1}{i}\,\underset{2}{i}\ldots\underset{p}{i} \tag{9.16}$$

and this term can never be cancelled by another term. That proves that all terms in (9.14) must vanish except those with $p = 2$. But from the invariance of R and C it only follows that all terms must vanish except those with $p = 2, 6, 10, \ldots$ Once more we see that only for $n = 2$ and $n = 4$ there are no additional conditions, but now we have found the form of these conditions for $\nu > 2$. Their number is $\binom{n}{6} + \binom{n}{10} + \ldots - \binom{n}{n-2}$ or $\binom{n}{n}$. This is in accordance with the fact that a rotation in R_n depends on $\binom{n}{2}$ complex parameters. We give here a table for $\nu = 1, \ldots, 8$ (cf (6.22)):

Conditions of invariance

(9.17)

ν	n	M	N^2	R: $2M^2$	C: $2\binom{M}{2}$ / $2\binom{M+1}{2}$ / M^2	rest	$\binom{n}{2}$	$\binom{n}{6}$	$\binom{n}{10}$	$\binom{n}{14}$
1	2	1	4	2	1 (/)	1	1			
2	4	2	16	8	2 [\]	6	6			
3	6	4	64	32	16 [/]	16	15	1		
4	8	8	256	128	72 (\)	56	28	28		
5	10	16	1024	512	256 (/)	256	45	210	1	
6	12	32	4096	2048	992 [\]	1056	66	924	66	
7	14	64	16384	8192	4096 [/]	4096	91	3003	1001	1
8	16	128	65536	32768	16512 (\)	16256	120	8008	8008	120

[1]) Note the errata at the end of this paper.

The general formula for the additional conditions is

$$(9.18)\quad \underset{j_1\ldots j_p}{i}\,{}^{A}_{\cdot B}\,T^{B}_{\cdot A}=0;\ p=6,\ 10,\ \ldots,\ n-2 \text{ or } n;\ j_1,\ldots,j_p=1,\ldots,n$$

For ν *odd* the last of these conditions has the form

$$(9.19)\quad R^{A}_{\cdot B}\,T^{B}_{\cdot A}=T^{a}_{\cdot a}-T^{a'}_{\cdot a'}=0\ ;\ a=1,\ldots,M\ ;\ a'=1',\ldots,M'$$

but according to (9.13) this is equivalent to

$$(9.20)\qquad T^{a}_{\cdot a}=0\ \ ;\ \ T^{a'}_{\cdot a'}=0,$$

expressing the fact that in each E_M the transformations are unimodular or in other words that the two M-vectors in these E_M's

$$(9.21)\quad \begin{cases} \underset{1}{V}{}^{A_1\ldots A_M}=\underset{1}{e}{}^{[A_1}\ldots\underset{M}{e}{}^{A_M]}\ ;\quad \underset{2}{V}{}^{A_1\ldots A_M}=\underset{1'}{e}{}^{[A_1}\ldots\underset{M'}{e}{}^{A_M]}\ ; \\ A_1,\ldots,A_M=1,\ldots,M,1',\ldots,M' \end{cases}$$

are invariant. The other $\binom{n}{6}+\binom{n}{10}+\ldots\binom{n}{n-4}$ conditions single out a subgroup of the unimodular homogeneous affine group, depending on $\binom{n}{2}$ complex parameters. Hence for ν *odd* G_s^+ consists of the transformations of a certain subgroup of the unimodular homogeneous affine group in one E_M (for $\nu=3$ this group itself [1])) and the transformations coupled with them in the other E_M.

In order to deal with higher *even* values of ν we remind that $\underset{j_1\ldots j_p}{i}$ is a reducible quantity for all even values of p (cf (6.22)). Now $\underset{j_{p+1}\ldots j_n}{i}::R\,\underset{j_1\ldots j_p}{i}$ if $j_1,\ldots,j_n$ is a permutation of $1,\ldots,n$, hence if $\underset{j_1\ldots j_n}{i}=P+Q'$, P lying in one E_M and Q' in the other E_M, we have $\underset{j_{p+1}\ldots j_n}{i}::P-Q'$. That implies that, if among the additional conditions (9.18) both $p=q$ and $p=n-q$ occur for some value of q, the conditions belonging to this value can be written as $\binom{n}{q}$ conditions in one E_M and $\binom{n}{q}$ conditions in the other E_M. Now for all even values of ν we get $\left\{\binom{n}{6}+\binom{n}{10}+\ldots+\right.$ $\left.+\binom{n}{n-6}\right\}+\binom{n}{n-2}$ conditions and all these sets of conditions occur in pairs, except the last set. The conditions occurring in pairs can be written as $\frac{1}{2}\left\{\binom{n}{6}+\ldots+\binom{n}{n-6}\right\}$ conditions in one E_M and just as many in the other E_M. They have to be added to the $\binom{M}{2}$ or $\binom{M+1}{2}$ conditions in each

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[1]) This is especially important for conformal relativity (6 supernumerary coordinates in spacetime) because it shows that in this case we need not an eight-dimensional spin space but only a four dimensional reduced spin space.

E_M arising from the invariance of R and C and all these conditions together reduce the number of complex parameters in each E_M exactly to $\binom{n}{2}$. Because $R\,i$ is reducible, the last $\binom{n}{n-2}$ conditions just fix a coupling between the remaining parameters, that is a coupling of the remaining groups in the two E_M's. *That proves that for $\nu > 2$ the two subgroups of G_s^- are always coupled and that for G_s^+ these is always a reduced spinspace except for $\nu = 2$, s even.* The E_M-part of C is a tensor of rank M for $\nu/2$ even and a bivector of rank M for $\nu/2$ *odd*. Hence, for ν even and > 2, G_s^- consists of the transformations of a certain subgroup with $\binom{n}{2}$ parameters of the orthogonal group (for $\nu/2$ even) or symplectic group (for $\nu/2$ odd) with respect to this part of C, in one E_M and the transformations coupled with them in the other E_M.

In order to get the transformations of G_s^* we substitute (9.14) in (8.6.) Then according to (4.26γ) [1]) it follows that the coefficients α must be real for $p = 2, 3, 6, 7, \ldots$, and purely imaginary for $p = 0, 1, 4, 5, \ldots$. This gives exactly $\binom{n}{2}$ conditions, reducing the number of remaining parameters to $\binom{n}{2}$ real ones in all cases. If the signatures of the i are given, these conditions can be written out explicitly. For instance for a positive definite fundamental tensor in R_n we have according to (5.2) and (6.5) for $\lambda = 1$

$$\Omega = \overset{0}{\Omega} = \bar{I}1 + \ldots + \bar{M}M + \bar{I}'1' + \ldots + \bar{M}'M'. \tag{9.22}$$

And from this and (8.6) we get

$$a)\quad \bar{T}^{\bar{a}}_{.\bar{b}} = -T^{b}_{.a}\,; \qquad b)\quad \bar{T}^{\bar{a}'}_{.\bar{b}'} = -T^{b'}_{.a'}\,. \tag{9.23}$$

Here Ω is reducible and accordingly does not establish a coupling of the two groups. The same is true for all even values of s (cf. 6.22). But this is only important for $\nu = 2$ because in all other cases a coupling already exists. If s is odd, Ω is connecting and the equations expressing the invariance of Ω establish a coupling by themselves. This is important for $\nu = 2$ because here we get now a coupling at least for s odd. In all other cases there is already a coupling and the invariance of Ω in connection with the equations resulting from the invariance of R and C and with the additional equations gives in each E_M the number of conditions necessary for reducing the number of real parameters to $\binom{n}{2}$.

From (6.22) we see that for ν odd the invertible hybrid quantity $\Pi^{\bar{I}}{}_{B}$ is always connecting (reducible) if $\Omega_{\bar{I}B}$ is reducible (connecting). Hence for ν odd and > 2 the transformations of G_s^* are those transformations of G_s^- that leave in each E_M the part of Ω invariant for s even

[1]) Note the errata at the end of this paper.

and the part of II for s odd. [2]) For s even this is the *unitary group* with respect to this part of Ω. [3])

Because of the importance of the case $\nu = 3$ for conformal relativity we give here the equations for this case more detailed. For the $\underset{i}{i}$, R and C we have

$$(9.\,24)\qquad \begin{cases} \underset{1}{i} = \underset{1}{\varepsilon}\,(^{1}_{2'} + {}^{2}_{1'} + {}^{3}_{4'} + {}^{4}_{3'} + {}^{1'}_{2} + {}^{2'}_{1} + {}^{3'}_{4} + {}^{4'}_{3}) \\ \underset{2}{i} = \underset{2}{\varepsilon}\,(^{1}_{3'} + {}^{2}_{4'} - {}^{3}_{1'} - {}^{4}_{2'} - {}^{1'}_{3} - {}^{2'}_{4} + {}^{3'}_{1} + {}^{4'}_{2}) \\ \underset{3}{i} = \underset{3}{\varepsilon}\,(^{1}_{4'} - {}^{2}_{3'} - {}^{3}_{2'} + {}^{4}_{1'} + {}^{1'}_{4} - {}^{2'}_{3} - {}^{3'}_{2} + {}^{4'}_{1}) \\ \underset{4}{i} = i\,\underset{4}{\varepsilon}\,(^{1}_{2'} + {}^{2}_{1'} - {}^{3}_{4'} - {}^{4}_{3'} - {}^{1'}_{2} - {}^{2'}_{1} + {}^{3'}_{4} + {}^{4'}_{3}) \\ \underset{5}{i} = i\,\underset{5}{\varepsilon}\,(^{1}_{3'} - {}^{2}_{4'} - {}^{3}_{1'} + {}^{4}_{2'} + {}^{1'}_{3} - {}^{2'}_{4} - {}^{3'}_{1} + {}^{4'}_{2}) \\ \underset{6}{i} = i\,\underset{6}{\varepsilon}\,(^{1}_{4'} + {}^{2}_{3'} + {}^{3}_{2'} + {}^{4}_{1'} - {}^{1'}_{4} - {}^{2'}_{3} - {}^{3'}_{2} - {}^{4'}_{1}) \end{cases}$$

$$(9.\,25)\qquad R = \pm\,(^{1}_{1} + {}^{2}_{2} + {}^{3}_{3} + {}^{4}_{4} - {}^{1'}_{1'} - {}^{2'}_{2'} - {}^{3'}_{3'} - {}^{4'}_{4'})$$

$$(9.\,26)\qquad C :: (11' - 22' + 33' - 44' - 1'1 + 2'2 - 3'3 + 4'4)$$

and for some different signatures we have for Ω according to (6.9)

$$(9.\,27)\qquad \begin{cases} \text{signature} & & \Omega \\ +\,+\,+\,+\,+\,+\,+ & \pm & (\bar{1}1 + \bar{2}2 + \bar{3}3 + \bar{4}4 + \bar{1}'1' + \bar{2}'2' + \bar{3}'3' + \bar{4}'4') \\ -\,+\,+\,+\,+\,+\,+ & \pm & (\bar{1}2' + \bar{2}1' + \bar{3}4' + \bar{4}3' + \bar{1}'2 + \bar{2}'1 + \bar{3}'4 + \bar{4}'3) \\ -\,-\,+\,+\,+\,+\,+ & \pm\,i & (-\bar{1}4 - \bar{2}3 + \bar{3}2 + \bar{4}1 + \bar{1}'4' + \bar{2}'3' - \bar{3}'2' - \bar{4}'1') \\ -\,-\,-\,+\,+\,+\,+ & \pm\,i & (-\bar{1}1' + \bar{2}2' - \bar{3}3' + \bar{4}4' + \bar{1}'1 - \bar{2}'2 + \bar{3}'3 - \bar{4}'4) \\ -\,-\,-\,-\,+\,+\,+ & \pm\,i & (\bar{1}2 - \bar{2}1 - \bar{3}4 + \bar{4}3 + \bar{1}'2' - \bar{2}'1' - \bar{3}'4' + \bar{4}'3') \\ -\,-\,-\,-\,-\,+\,+ & \pm & (-\bar{1}4' - \bar{2}3' - \bar{3}2' - \bar{4}1' - \bar{1}'4 - \bar{2}'3 - \bar{3}'2 - \bar{4}'1) \\ -\,-\,-\,-\,-\,-\,+ & \pm & (\bar{1}1 + \bar{2}2 + \bar{3}3 + \bar{4}4 - \bar{1}'1' - \bar{2}'2' - \bar{3}'3' - \bar{4}'4') \end{cases}$$

The invariance of C gives the 16 conditions

$$(9.\,28)\qquad \begin{cases} T^{1}_{.1} = -T^{1'}_{.1'};\ T^{1}_{.2} = T^{2'}_{.1'};\ T^{1}_{.3} = -T^{3'}_{.1'};\ T^{1}_{.4} = T^{4'}_{.1'} \\ T^{2}_{.1} = T^{1'}_{.2'};\ T^{2}_{.2} = -T^{2'}_{.2'};\ T^{2}_{.3} = T^{3'}_{.2'};\ T^{2}_{.4} = -T^{4'}_{.2'} \\ T^{3}_{.1} = -T^{1'}_{.3'};\ T^{3}_{.2} = T^{2'}_{.3'};\ T^{3}_{.3} = -T^{3'}_{.3'};\ T^{3}_{.4} = T^{4'}_{.3'} \\ T^{4}_{.1} = T^{1'}_{.4'};\ T^{4}_{.2} = -T^{2'}_{.4'};\ T^{4}_{.3} = T^{3'}_{.4'};\ T^{4}_{.4} = -T^{4'}_{.4'} \end{cases}$$

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[2]) For $\nu = 3$ this was proved in J. A. SCHOUTEN and J. HAANTJES, Konforme Feldtheorie II, R_6 und Spinraun, Ann. di Pisa II, 4, 175–189, (1935).

[3]) The group leaving invariant an invertible hybrid quantity of valence 2 seems not to have got a name. For the projective case Veblen and Givens called these quantities anti-involutions. L. c. p. 180 (599) footnote 9).

and the additional condition is here

$$(9.29)\quad T^1_{.1}+T^2_{.2}+T^3_{.3}+T^4_{.4}=0;\ \text{hence}\ T^{1'}_{.1'}+T^{2'}_{.2'}+T^{3'}_{.3'}+T^{4'}_{.4'}=0$$

G_s^+ consists of the two unimodular affine groups in the two E_4's, coupled by (9.28). The number of complex parameters is now *15*.

For $---+$ the invariance of the two parts of Ω gives

$$(9.30)\quad \begin{cases} T^1_{.1}=-\overline{T}^{\bar 2}_{.\bar 2}\,;\ T^1_{.2}=+\overline{T}^{\bar 1}_{.\bar 2}\,;\ T^1_{.3}=+\overline{T}^{\bar 4}_{.\bar 2}\,;\ T^1_{.4}=-\overline{T}^{\bar 3}_{.\bar 2} \\ T^2_{.1}=+\overline{T}^{\bar 2}_{.\bar 1}\,;\ T^2_{.2}=-\overline{T}^{\bar 1}_{.\bar 1}\,;\ T^2_{.3}=-\overline{T}^{\bar 4}_{.\bar 1}\,;\ T^2_{.4}=+\overline{T}^{\bar 3}_{.\bar 1} \\ T^3_{.1}=+\overline{T}^{\bar 2}_{.\bar 4}\,;\ T^3_{.2}=-\overline{T}^{\bar 1}_{.\bar 4}\,;\ T^3_{.3}=-\overline{T}^{\bar 4}_{.\bar 4}\,;\ T^3_{.4}=+\overline{T}^{\bar 3}_{.\bar 4} \\ T^4_{.1}=-\overline{T}^{\bar 2}_{.\bar 3}\,;\ T^4_{.2}=+\overline{T}^{\bar 1}_{.\bar 3}\,;\ T^4_{.3}=+\overline{T}^{\bar 4}_{.\bar 3}\,;\ T^4_{.4}=-\overline{T}^{\bar 3}_{.\bar 4} \end{cases}$$

in one E_4 and just the same equations with accents in the other E_4.

For $-----+$ we get for Π (cf (4.27))

$$(9.31)\qquad \Pi :: (\bar 1_4-\bar 2_3+\bar 3_2-\bar 4_1-\bar 1_{4'}+\bar 2_{3'}-\bar 3_{2'}+\bar 4_{1'})$$

and the invariance of the two parts of Π gives

$$(9.32)\quad \begin{cases} T^1_{.1}=-\overline{T}^{\bar 4}_{.\bar 4}\,;\ T^1_{.2}=\ \ \overline{T}^{\bar 4}_{.\bar 3}\,;\ T^1_{.3}=-\overline{T}^{\bar 4}_{.\bar 2}\,;\ T^1_{.4}=\ \ \overline{T}^{\bar 4}_{.\bar 1} \\ T^2_{.1}=\ \ \overline{T}^{\bar 3}_{.\bar 4}\,;\ T^2_{.2}=-\overline{T}^{\bar 3}_{.\bar 3}\,;\ T^2_{.3}=\ \ \overline{T}^{\bar 3}_{.\bar 2}\,;\ T^2_{.4}=-\overline{T}^{\bar 3}_{.\bar 1} \\ T^3_{.1}=-\overline{T}^{\bar 2}_{.\bar 4}\,;\ T^3_{.2}=\ \ \overline{T}^{\bar 2}_{.\bar 3}\,;\ T^3_{.3}=-\overline{T}^{\bar 2}_{.\bar 2}\,;\ T^3_{.4}=\ \ \overline{T}^{\bar 2}_{.\bar 1} \\ T^4_{.1}=\ \ \overline{T}^{\bar 1}_{.\bar 4}\,;\ T^4_{.2}=-\overline{T}^{\bar 1}_{.\bar 3}\,;\ T^4_{.3}=\ \ \overline{T}^{\bar 1}_{.\bar 2}\,;\ T^4_{.4}=-\overline{T}^{\bar 1}_{.\bar 1} \end{cases}$$

in one E_4 and just the same equations with accents in the other E_4.

After the introduction of the invariance of Ω there are *15* real parameters.

§ 10. *The spin spaces for $n=2\nu-1$.*

In R_n we consider the orthogonal transformations leaving a real R_{n-1} invariant. If i^k is a unitvector perpendicular to R_{n-1}, the transformations in $\mathfrak{S}_N$ corresponding to these orthogonal transformations are the transformations of G_s leaving $i^A_{\cdot B}$ invariant. $i^A_{\cdot B}$ is always connecting. If ν is odd we know from (6.22) that $C\,i$ is a reducible bivector for $\nu\equiv 3$ mod *4* and a reducible tensor for $\nu\equiv 5$ mod *4*. That proves that for ν odd and >2 the rotations in R_{n-1} correspond to those transformations of G_s^- in one E_M that leave invariant a bivector ($\nu\equiv 3$, mod *4*) or a tensor ($\nu\equiv 5$, mod *4*) and the transformations coupled with them in the other E_M. From this it follows that the proper real rotations in R_{n-1} correspond to those transformations of G_s^+ in one E_M that leave invariant a bivector or a tensor and the part of Ω (for s even) or Π (for s odd) and the transformations coupled with them in the other E_M. But really there is no difference between s even and s odd because by means of (4.18, 22, 28) it is easily proved that $\overline{C}\,\bar i\,\Pi$ is reducible hermitian and $\bar i\,\overline{C}^{-1}\,\Omega$ reducible invertible. Hence in both cases there are three invariant

quantities in the reduced spinspace $\mathfrak{S}_M$, a bivector and two hybrid quantities, one hermitian and the other invertible, and from these three every one can be expressed to within the sign into the two others. For $v = 3$ the invariance of these quantities is n.a.s. for transformations of $\mathfrak{S}_M$ to correspond to proper real rotations in R_{n-1} [4]) but for $v > 3$ there are additional conditions.

As an example we take $v = 3$ and $i = \overset{6}{i}$. Then we have for $---++$ for the parts in one E_4

$$\begin{aligned} &\text{part } C\overset{6}{i} &&:: -14+23-32+41\\ &\text{part } \Omega &&:: \ \bar{1}2-\bar{2}1-\bar{3}4+\bar{4}3\\ &\text{part } \bar{i}\,\bar{C}^{-1}\Omega &&:: \ {}^{\bar{1}}{}_3+{}^{\bar{2}}{}_4-{}^{\bar{3}}{}_1-{}^{\bar{4}}{}_2 \end{aligned}$$

and for $-----+$

$$\begin{aligned} &\text{part } C\overset{6}{i} &&:: -14+23-32+41\\ &\text{part } \Pi &&:: \ {}^{\bar{1}}{}_4-{}^{\bar{2}}{}_3+{}^{\bar{3}}{}_2-{}^{\bar{4}}{}_1\\ &\text{part } \bar{C}\,\bar{i}\,\Pi &&:: \ \bar{1}1+\bar{2}2+\bar{3}3+\bar{4}4 \end{aligned}$$

For v even $C\,i$ is connecting and the invariance of i gives in each E_M an additional set of conditions reducing the number of complex. parameters for G_s^+ and the number of real parameters for G_s^* to $\binom{n-1}{2}$. That proves that in all cases, except $v = 2$, there is a reduced spin space $\mathfrak{S}_M$ for rotations in R_{2v-1}. For $v = 2$, s even, there was no reduced spin space for rotations in R_4. But $i^A_{\cdot B}$ being connecting it fixes a coupling by itself so that for rotations in R_3 there is always a reduced spin space $\mathfrak{S}_2$. In fact rotations in R_3 can be dealt with by means of quaternions (Pauli matrices).

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[4]) This was found by E. Cartan, Les groupes simples, finis et continus, Ann. de l'école norm. sup. 31, 263–355 (1914), p. 354 for $v = 3$, $s = 3$, 4, 5 and for $v = 3$ and all values of s by J. A. Schouten und J. Haantjes, Konforme Feldtheorie II, p. 11.

ERRATA:

p. (598) 179 line 2 below eqn. (1.7): *for* E^N *read* E_N.
p. (600) 181 line 1 below eqn. (2.5): *for* M_{2s} *read* $M_2{}^s$; line 2 below eqn. (2.5): *for* $\frac{1}{2}\,N$ *read* $^2\log N$.
p. (601) 182 last line: *for* $i^{(p_3)}$ *read* $i^{\binom{p}{2}}$.
p. (603) 184 eqn. (3.2): *for* $\overset{k}{i}{}^B_{\cdot D}$ *read* $\overset{k}{i}{}^C_{\cdot B}$.
p. (688) 218 eqn. (4.9): *for* S *read* S^{-1}.
p. (689) 219 eqn. (4.20): *for* $\binom{p}{2}$ *read* $(\frac{p}{2})$.
p. (690) 220 eqn. (4.26): *for* $\binom{p}{2}$ *read* $(\frac{p}{2})$.
p. (944) 342 line 6 below eqn. (6.21): *for* I *read* I.
p. (945) table (9.22): *for* $\diagup$ *under* $\overset{ik}{i}$ *read* $\diagdown$; table (9.22) line 2 from below: *for* $v \equiv 3$ *read* $v \equiv 4$.

DE VERWANTSCHAP DER CYCLOSTOMEN

DOOR

J. F. VAN BEMMELEN

(Communicated at the meeting of January 28, 1950)

In het hoofdstuk Phylogenie der Tiere in het Handbuch der Vererbungswissenschaft van E. Baur und M. Hartmann, III Bd. 1931 verklaart NAEF, dat de Cyclostomen zich voordoen als de laagste echte Gewervelde Dieren, en ons dus een beeld opleveren van den oorspronkelijken typischen bouw der Vertebrata. Als kenmerken daarvan somt hij op:

1. Een nog zeer wormvormige habitus,
2. gelijkmatige ontwikkeling der lichaamssegmenten in de drie streken des lichaams,
3. een kaaklooze mond,
4. zeer naar elkaar toegeschoven neusgroeven,
5. oogen, die eerst in den loop van het larvenleven hun volle ontwikkeling bereiken,
6. aanleg van een hooger aantal kieuwspleten, afwisselend van 8 tot 14 paar,
7. aanleg van een ononderbroken vinplooi in de dorsale middellijn,
8. een in het tweede kopsegment gelegen mond, die zich aansluit bij de reeks der kieuwspleten,
9. aanwezigheid in het eerste kopsegment van een ongepaarde voorste divertikel van den monddarm, die naar buiten doorbreekt, waardoor de spits van den oerdarm in verbinding treedt met de ectodermale instulping van den aanleg der hypophyse,
10. volstrekte afwezigheid van gepaarde ledematen.

In tegenstelling met deze beweringen van NAEF meen ik den bouw der Cyclostomen te mogen toeschrijven aan verregaande afwijkingen van het oorspronkelijke type der Gewervelde Dieren, onder den invloed van gewijzigde levensomstandigheden. Weliswaar maken zij, wat hun lichaamsbouw aangaat, in velerlei opzichten den indruk van oorspronkelijkheid, maar dit kan verklaard worden uit vereenvoudiging, die juist een gevolg is van aanpassing aan beperkte levensvoorwaarden. Ten bewijze dezer opvatting moge elk der door NAEF in 't midden gebrachte beweringen afzonderlijk besproken worden.

1. *De wormvormige gedaante.*

NAEF noemt hem „noch stark wurmartig", en stelt zich dus blijkbaar voor dat de gemeenschappelijke voorouders van alle Chordaten langgestrekte pootlooze wormen waren, wat hij ook nog nader bevestigt door de reproductie eener photographie van een pas uit het ei gekropen Ammocoeteslarve van Petromyzon van 5 mm lengte, en verder door een reeks schematische figuren, die de ontwikkeling van het „Chordatentype" uit het „Helmintöïdetype" moeten verduidelijken. Op deze laatste wil ik in 't vervolg nog terugkomen, maar omtrent de genoemde photo moet opgemerkt worden, dat van al de details, die NAEF in zijn er bijgevoegde beschrijving opsomt, zoo goed als niets te zien is, zoodat zij den indruk eener mystificatie maakt, en in allen gevalle niets tot de bewijsvoering kan bijdragen.

Gaat men in het gebied der Gnathostomen de vormen met wormvormigen habitus na, dan ziet men dat zij betrekkelijk weinige in aantal zijn, maar daarentegen in allerlei afdeelingen aangetroffen worden naast verwanten van andere gestalte, en overal den indruk maken van secundaire verlenging der lichaamsas, gepaard met gedeeltelijke of zelfs geheele reductie der ledematen. De alen onder de visschen, de Blindwoelers (Gymnophionen) en grottensalamanders. (Proteus) onder de Amphibien, de Slangen, Hazelwormen Amphisbaenen, Pygopiden, Mosasauridae onder de Reptielen, beantwoorden in meerdere of mindere mate aan dit type. Onder de Zoogdieren zijn het alleen de Walvisschen en Zeekoeien, die in sommige opzichten eenigszins met het rolronde en uitgerekte type, gepaard met rudimentaire ledematen, overeenstemmen. Men zou zelfs kunnen veronderstellen, dat bij de pinguins, alken en koeten vervormingen in dezelfde richtingen en onder den invloed van overeenkomstige levenswijze hebben plaatsgevonden. Bij allen is de afkomst uit minder verlengde en van normale ledematen voorziene voorouders niet twijfelachtig. Waarom zouden dan bij de Cyclostomen diezelfde lichaamsgestalte en de afwezigheid der ledematen kenmerken van oorspronkelijke beteekenis zijn?

Juist zulke gevallen als de gedaante der Walvisschen leveren ons het onloochenbare bewijs van de verregaande vervorming, die tot het verdrijven der uitwendige grenzen tusschen kop, hals, romp en staart heeft geleid.

Aan den anderen kant vertoonen de oudste Vertebraten, waarvan ons overblijfselen bekend zijn: de Placodermen uit het Siluur, juist het omgekeerde van een wormvormigen habitus. Weliswaar wekken de zonderlinge gedaanten dezer raadselachtige wezens, die bovendien ook onderling zoozeer uiteenloopen, allerminst den indruk van oorspronkelijkheid, maar wanneer wij beproeven hen te ontdoen van alle bijzonderheden, die aan specificatie toegeschreven kunnen worden, komt wel een meer vischachtig type voor den dag, doch volstrekt geen slang- of wormvormig organisme. Men vergelijke b.v. Cephalaspis, Pterichthus, Astero-

lepis, Coccosteus, zoowel onder elkaar als met de Ganoïde visschen uit het Devoon, welke laatste in tegenstelling met de eerste weliswaar volledig voldoen aan het vischtype, maar evenmin als de Placodermen doen denken aan Cyclostomen of andere aalachtige waterbewoners, men zie hieromtrent de handboeken en tijdschriftartikelen, zooals:

GREGORY p. 341: Ostraderma: had geen buikvinnen
Birkenia: was fusiform, vrijzwemmend, zonder gepaarde vinnen
Thelodus en Lamarkia: waren afgeplat rogvormig en paddelden met hun zijranden
Cephalaspiden: hadden een paar vleeschachtige beschubde vinnen achter het cephalothoracale schild.
Antiarctii: droegen een paar zijdelingsche aanhangsels achter den kop, beschermd door een meerlagig beenig pantser, dat geleed was en buigen kon, vaag doende denken aan ledematen van Arthropoden en aan de vinnen van Zeeschildpadden. Geen buikvinnen, evenmin als Arthrodira.

Het is dus veel waarschijnlijker, dat al deze vreemdsoortige diervormen zich uit viervoetige chordata hebben ontwikkeld, dan uit pootlooze wormen.

2. *Gelijkmatige ontwikkeling der Segmenten in de drie lichaamsstreken.*

Dat NAEF hier bepaaldelijk de driedeeling van het gesegmenteerde en dus tweezijdig symmetrische lichaam vermeldt, wijst er wel op, dat ook hij de samenstelling uit kop, romp en staart als de oorspronkelijkste differentiatie der dierlijke organismen beschouwt. Maar met die opvatting geraakt de voorstelling van een gelijkmatige ontwikkeling der segmenten over de geheele lengte juist in volstrekte tegenspraak. Immers het kopeind moet van den aanvang af zich onderscheiden hebben van het overige lichaam, en hetzelfde moet schoon in mindere mate het geval geweest zijn met het staarteinde. Hoe men zich die nog niet gesegmenteerde oerwezens ook mag voorstellen, de indeeling in drieën zoowel als de later optredende segmentatie in metameren vereischt noodzakelijk het aannemen eener bipolariteit, die dan ook door de geheele wereld der levende wezens, bij planten evengoed als bij dieren, is waar te nemen. Een duidelijk gedifferentieerde kop met zintuigapparaten en monddeelen is dus oorspronkelijker dan een vooreind waaraan deze organen niet of nauwelijks te onderscheiden zijn, wat derhalve aan achteruitgang of ten minste aan aanpassing aan speciale levensvoorwaarden b.v. parasitisme mag toegeschreven worden. Alleen op deze wijze laat zich o.a. de toestand bij Amphioxus verklaren, en hier zien wij dan ook de segmentatie, tezamen met het vooreind der chorda zich tot in het voorste uiteinde van het dier voortzetten, of beter gezegd handhaven. Nog overtuigender is de embryologie der Ascidien, want bij hen zien wij het reductieproces van de kopstreek zich voor onze oogen voltrekken: de hersenblaas met den aanleg van één oog en één

gehoororgaan (evenwichtsorgaan) wordt geresorbeerd en vervangen door een ganglionknoop, terwijl tevens het gesegmenteerde achtergedeelte der larve met chorda eveneens wordt gereduceerd. In tegenstelling met Amphioxus vermag bij de Ascidienlarve de segmentatie van het kopgedeelte zich niet te handhaven, in onderscheiding met alle overige Chordaten, in wier kopgedeelte de sporen der segmentatie altijd min of meer duidelijk zijn weer te vinden. Voor het chordale gedeelte van den schedel behoeft de aanwezigheid der segmentatie geen nader bewijs, maar ook het praechordale vertoont duidelijke teekenen van metamerie, bepaaldelijk wat betreft de oogspieren en hunne zenuwen. In dit opzicht staan de Cyclostomen niet op oorspronkelijker ontwikkelingstrap dan de Gnathostomen. Over de ontwikkeling hunner oogspieren zegt KOLTZOFF:

„Bij de Negenoogen zijn Mesoderm, Visceraalzakken, hersenbuis en peripheer zenuwstelsel gelijkelijk gesegmenteerd. Het eerste somiet in de reeks (het praemandibulaire) maakt zich later dan al de andere van den oerdarm los; het verdeelt zich wel in een myotoom en een sklerotoom, maar mist de bijbehoorende zijplaten; uit zijn myotoom ontstaan die oogspieren, die door de oculomotorius geïnnerveerd worden. Het is homoloog met de kopholten van Amphioxus.

„Ook het tweede somiet blijft lang in open verbinding met den oerdarm. Na zijn afsnoering van het ventrale mesoderm (de mandibulairboog) verdeelt het zich in een sklerotoom (dat den aanleg van het gehemelte vormt) en een myotoom (waaruit de bovenste schuine oogspier, geïnnerveerd door den nervus trochlearis) ontstaat.

„Het derde somiet lost zich spoedig op tot een groep van afzonderlijke mesenchymatische cellen, die gedeeltelijk aan het sklerotoom beantwoorden, en voor een ander deel tot de buitenste rechte oogspier (geïnnerveerd door den nervus abducens) worden.

„Het vierde somiet (het eerste achter de oorkapsel: postotisch) deelt zich in een buitenste en een binnenste stuk, waartusschen de aanleg van de glossopharyngeus ligt. Het buitenste stuk krijgt twee uitloopers naar voren: een onder de oorkapsel, waaruit de onderste voorste schuine kopspier ontstaat, en een boven de oorkapsel, die tezamen met dergelijke uitloopers van eenige volgende myotomen den musc. cap. ant. sup. levert. De binnenste onderdeelen van het vierde en de volgende myotomen verdwijnen.

„De musculatuur onder de Kieuwkorf (subbranchiaal) ontstaat uit de ventrale stukken van meerdere myotomen achter de kieuwkorf, te rekenen van het tiende postotische af; de oorspronkelijke segmentatie wordt later door een secundaire vervangen, in aansluiting aan de kraakbeenige kieuwbogen."

3. *Een kaaklooze mond.*

De onderscheiding der Cyclostomen van de Gnathostomen berust, zooals de namen bewijzen, in de eerste plaats op de al of niet aanwezig-

heid van verhard steunweefsel in de mondranden. Maar dit kaakskelet is niet bij alle Gnathostomen hetzelfde: bij de Selachiï wordt het gevormd door palatoquadratum en onderkaak, bij de Teleostomi treden vóór het palatoquadratum de tusschenkaaks- en de bovenkaaksbeenderen op, en op het onderkaakskraakbeen het Denticulare met bijbehoorend complex. Ofschoon al de laatste skeletstukken dekbeenderen zijn, en dus moeten beschouwd worden als afkomstig van het huidskelet, zijn zij waarschijnlijk toch te vergelijken met de lipkraakbeentjes der Selachiï, welke laatsten een groote overeenkomst met kieuwbogen vertoonen. Daar wij nu de zekerheid hebben, dat de oudste ons bekende Selachiï een huidpantser bezaten, mogen wij aannemen, dat bij de recente de homologa der kaakvormende dekbeenderen door achteruitgang weer verdwenen zijn. Bij Cyclostomen is die achteruitgang nog verder gegaan, palatoquadratum en onderkaak zijn wel niet geheel geobliteereerd, maar hebben zich uit de mondranden teruggetrokken en bij het schedelskelet aangesloten.

Van der Klaauw noemt op blz. 224 van het Handboek der Vergelijkende Anatomie der Vertebraten deze verbinding van neuro- en splachnocranium bij de Cyclostomen „primair", zonder daarvoor argumenten aan te voeren, en stelt ze daarmee tegenover de „primitieve Gnathostomen", bij welke „deze elementen geheel los van elkaar zijn". Op blz. 253 werkt hij deze tegenstelling nader uit, maar vermeldt tevens dat ten eerste „over de homologisatie van den toestand bij de Petromyzonten met dien bij de in vele opzichten anders gebouwde Myxinoïdea groot verschil van meening heerscht, en evenzoo van deze toestanden bij de Gnathostomata", en ten tweede dat „men nogtans een homologon van den kaakboog heeft vastgesteld, dat door vele auteurs vrij ver naar achteren wordt gezocht, terwijl rostraal daarvan eenige praemandibulaire bogen en een ingewikkeld lipskelet ligt."

Ook wijst van der Klaauw er op, „dat het slijmachtig kraakbeen van de kieuwkorf der larven van de prikken bij de fossiele verwanten verbeend is, en boogvormige verdikkingen aan het binnenvlak draagt". Is het dan niet veel waarschijnlijker, dat de palaeozoïsche voorouders der Cyclostomen het bijten afleerden en zich het zuigen aanwenden, en dat in verband daarmee hun kaken met het neurocranium vergroeiden en hun lipskelet zich vervormde: dit laatste dan nog op verschillende wijze bij Petromyzonten en Myxinoïden.

Daar komt bij, dat de mondtrechter van Petromyzon niet werkelijk kegelvormig is, maar aan zijn binnentop duidelijk in een boven- en een onderlip is gedifferentieerd, die beide een rijtje van hoorntanden dragen. Bij Ammocoetes en Myxine zijn de mondranden in 't geheel niet cirkelvormig, maar vormen zij een boven- en een onderlip, waarvan de eerste in een rechter en een linker helft is uitgeplooid. De naam Cyclostomen is dus misleidend: de vorm van hun mondtrechter mag volstrekt niet als een hoofdonderscheid tusschen hen en de Gnathostomen beschouwd worden.

4. *Zeer naar elkaar toegeschoven neusgroeven.*

Dat NAEF de ligging van het enkelvoudige neusgat niet kortweg als ongepaard en mediaan omschrijft, zooals de wetenschappelijke term monorhin dit aangeeft, is wel een bewijs, dat ook hij moet toegeven, dat de enkelvoudigheid van het neusgat niet primair is, maar door samensmelting van twee paramediane neusgangen is ontstaan. Daarmee verliest dus het onderscheid tusschen Mono- en Diplorhinen zijn principieele beteekenis en behoeft het geen nadere overweging. Alleen vraagt men zich af, op welke wijze die versmelting tot stand is gekomen, en wat er de oorzaak van is geweest. Zij moet al lang geleden hebben plaatsgevonden, want bij de gepantserde vormen, zooals de Anaspiden uit het Siluur, is zij reeds aanwezig. Zoeken wij naar analoga onder de Gnathostomen, dan treffen wij eigenlijk alleen de Tanddragende Walvisschen aan, waarbij de enkelvoudigheid van het neusgat door dichtdrukking van een der beide neusgangen onder verplaatsing van de andere naar de middellijn van den rug is teweeggebracht, terwijl zij veroorzaakt is door de behoefte om onder het zwemmen aan de oppervlakte van het water adem te halen.

Daarnaast kan nog opgemerkt worden, dat de neusgaten in vele orden van Zoogdieren een neiging vertoonen, om naar elkaar toe te schuiven, zoodat ten slotte het schot tusschen beide neusgangen tot een dunne scheidingswand slinkt. Bij de haaien en roggen daarentegen verplaatsen de neusgaten zich naar de buikzijde en treden met de mondhoeken in verbinding, mogelijk in verband met de ontwikkeling van het rostrum, ofschoon de configuratie bij de Steuren met hun groote rostrum daarmee in tegenspraak schijnt, evenals de verplaatsing der reukgroeven van de Zaagvisschen naar het vooreind van de zaag.

Ook moeten wij niet uit het oog verliezen, dat bij de Myxinoïden de enkelvoudige neusopening niet dorsaal en achterwaarts maar integendeel geheel vooraan vlak bij den bovenrand van den mondtrechter is gelegen, en door een lange gang, die onder de hypophyse doorstrijkt, met den kieuwdarm communiceert, terwijl bij Petromyzon de veel kortere neusgang voorbij de hypophyse blind eindigt. Deze verschillen wijzen allen op groote vervormingen van den oorspronkelijken aanleg der reukorganen van de Cyclostomen, in stede van het primaire karakter dat NAEF daaraan toekent.

De verplaatsing der neusgaten van de zijkanten naar het midden toe zien wij bij de embryonen der Gnathostomen voor onze oogen geschieden, het duidelijkst bij de zoogdierfoeten, waar zij aanvankelijk door de geheele breedte van het aangezicht van elkaar gescheiden zijn, om betrekkelijk snel elkaar te naderen en slechts door het van boven af ingroeiende dorsale neusschot van elkaar verwijderd te worden gehouden. De breedte van dit schot neemt in den loop der ontwikkeling niet toe, maar integendeel (relatief) af; de beide zeefbeenplaten door wier vergroeiïng het gevormd wordt, slinken tot een dun schot, dat door de naar elkaar toeklappende ploegschaarbeenderen wordt omvangen.

Waarom zou trouwens een ongepaard mediaan orgaan oorspronkelijker zijn dan een stel gepaarde organen? Voor het reukorgaan wordt naar 't mij voorkomt de quaestie definitief opgelost door het feit, dat de reukzenuwen ook bij de Cyclostomen gepaard zijn, en uit een rechter en een linkerkolf van de voorhersenen ontspringen.

Dicht achter de neusporie ligt bij Petromyzon een ander orgaan: de epiphyse of het pineale oog, dat eveneens ongepaard is, maar waarvan zich waarschijnlijk laat bewijzen, dat het die ongepaardheid dankt aan den achteruitgang van zijn wederhelft de paraphyse. Bij de overgroote meerderheid der Gnathostomen is ook de epiphyse tot een rudiment geworden, in zoodanige mate, dat zijn oorspronkelijke beteekenis als gezichtszintuig slechts door zijn betere ontwikkeling bij enkele hagedisachtige Reptielen (Sphenodon) wordt kenbaar gemaakt. Maar tevens blijkt uit het feit, dat een volledig pineaal oog ook nog in een geheel andere groep van Gewervelde Dieren dan de Cyclostomen voorkomt, dat deze laatste niet door het bezit ervan als bijzonder oorspronkelijk worden gekenmerkt.

Bij overweging van de vraag, wat als meer oorspronkelijk moet beschouwd worden: gepaardheid of ongepaardheid van organen, moet men zich niet tot een enkel orgaan bij één of sommige diersoorten bepalen, maar zijn beschouwingen uitstrekken tot het geheele gebied der tweezijdig gebouwde organismen. Doet men dit ten opzichte van de oogen, (gepaarde zoowel als ongepaarde) dan trekt het de aandacht, dat bij de Arthropoden beide vormen van lichtpercipieerende organen naast elkaar voorkomen. Bij de Bijen en vele andere insecten, waar wij tweeërlei soort van oogen aantreffen: samengestelde en enkelvoudige, zijn de eerste altijd gepaard, terwijl de laatste zoowel gepaard ter weerszij van de middellijn, als daarin alleenstaand kunnen zijn. De gepaarde samengestelde oogen naderen elkaar in vele gevallen (vooral bij de Vliegen) zoodanig, dat zij in de middellijn aaneensluiten. De enkelvoudige staan dikwijls in een driehoek geschaard. Bij de Spinnen zijn slechts enkelvoudige oogen aanwezig. Bijzonder opmerkelijk is de Molukkenkrab (Limulus polyphemus) die eigenlijk een Spinachtig dier is, en twee stel oogen heeft: het eene paar vlak bij elkander ter weerszij van de middellijn, het andere ver uiteen aan de zijranden van het sterk verbreede kopschild, zoodat zij met de middelste in een driehoek geschaard staan.

Bij de Tienpootige Kreeften en Krabben vindt men alleen een paar samengestelde oogen op stelen, maar hun larven (Nauplius) hebben slechts een ongepaard enkelvoudig oog voor op den kop, evenals die van andere Crustaceën. Bij de Slakken (Gastropoden) worden alleen enkelvoudige gepaarde oogen aangetroffen, bij sommige Longslakken staan zij op den top van instulpbare tentakels, bij andere aan den voet daarvan. Bij de Echinodermen komen enkelvoudige oogen op den top van watervaatsvoetjes voor, die bij Zeesterren aan het uiteind der armen staan, bij de Zeeëgels in een kring rondom het topschild. Deze voorbeelden, die met

talrijke andere vermeerderd konden worden, wijzen er op, dat gepaarde oogen evengoed oorspronkelijk kunnen zijn als ongepaarde en dat beide vormingen in allerlei afdeelingen in regressie kunnen geraken.

5. *Oogen, die eerst tegen het eind van 't larvenleven hunne volle ontwikkeling bereiken.*

Wat de redenen kunnen geweest zijn, die NAEF tot de voorstelling hebben geleid, dat vertraagde ontwikkeling van een orgaan een kenteeken van primitiviteit zou wezen, is mij onbegrijpelijk. Verschijnselen van retardatie vinden wij overal, bij dieren zoowel als bij planten, maar nergens schijnen zij te wijzen op oorspronkelijkheid, d.w.z. dat zij ons een beeld zouden geven van de wijze waarop phylogenetisch het orgaan zich heeft ontwikkeld. Om bij het oog te blijven: een orgaan voor de waarneming der omgeving moet in eersten aanleg ontstaan zijn uit een plek der uitwendige oppervlakte, die bijzonder gevoelig werd voor lichtprikkels. Die lichtgevoeligheid moet dus het primaire verschijnsel gweest zijn; om van haar te komen tot de waarneming van een beeld moet zich aan den buitenkant der lichtgevoelige plek een of ander lichtbrekend apparaat, b.v. een vochtblaas, hebben ontwikkeld, dat een nierbuisje (segmentaalorgaan) kan geweest zijn, waaruit later de lens ontstond. Bij den aanleg van het oog der Vertebraten zien wij dan ook dat de lens als een afzonderlijk orgaan, onafhankelijk van de oogblaas der tusschenhersenen, zich uit een instulping van het huidblad vormt en eerst secundair met de hersenblaas in contact treedt, terwijl uit drie ter plaatse aanwezige somieten de oogspieren, de oogrokken en het glaslichaam ontstaan.

Komt nu in het verdere beloop dier ontwikkeling een tijdelijke vertraging of zelfs algeheele stilstand, zoo mag hierin niet anders gezien worden dan secundaire remming van het ontwikkelingsproces, waardoor dit in zijn normale evolutie gestuit wordt, tot tijd en wijle dat de uitwendige omstandigheden zijn voltooiing vereischen. Het is er mede als de stilstand der ledematenstompjes bij de donderpadden van de Kikvorschen, of de instulping der vleugelzakjes van de rupsen en andere insectenlarven. Blijft de hervatting der ontwikkeling uit, dan kan het onvoltooide orgaan tot den toestand der rudimentaire organen vervallen, of zelfs geheel geresorbeerd worden.

In het hier besproken geval mag de tijdelijke stilstand in de ontwikkeling der oogen van de Ammocoetes-larve van Petromyzon wellicht toegeschreven worden aan het verblijf in den modderbodem der rivieren, waaruit zij eerst na enkele jaren te voorschijn komt, om zich tot het geslachtsrijpe dier te vervormen.

6. *Aanleg van een hooger aantal Kieuwspleten, afwisselend van acht tot zeventien paar.*

In de schematische figuurtjes, waarmede NAEF zijn opvattingen toelicht, geeft hij van den hypothetischen voorouderlijken vorm der Cyclos-

tomen, zooals hij die zich voorstelt, een schets, die niet 17 maar zelfs 25 kieuwporiën vertoont. Als reden, waarom hij de vormen met een lager aantal kieuwopeningen ontstaan acht uit die met een hooger, wijst hij op de verwantschap der Cyclostomen met de Acrania, de Ascidia en de Enteropneusta. Nu kan men evenwel de zaak evengoed omkeeren, en de vormen met talrijke kieuwopeningen afleiden uit die met een beperkter aantal, en wel bepaaldelijk uit die met vijf paar (eigenlijk met zes, waarvan echter het voorste zich tot spuitgat vernauwt en zelfs weer geheel kan sluiten). Daar bij de overgroote meerderheid der Gewervelde Dieren vijf paar volledige kieuwspleten worden aangelegd, mag men dit als het grondtal aannemen. Weliswaar komen onder de Kraakbeenige Visschen enkele vormen (de Notidani) met zes en zelfs met zeven kieuwspleten voor, maar zooals Versluys opmerkte mag men dit verklaren uit een tusschenschakeling van één of twee paren, op dezelfde wijze als in de wervelkolom een plaatselijke vermeerdering van het normale aantal wervels veelvuldig wordt aangetroffen (zie Bateson: On variation). Aan dezelfde oorzaak zou men de aanzienlijke vermeerdering van kieuwporiën bij Bdellostoma stouti mogen toeschrijven, en evenzoo de nog veel hoogere toename der openingen in de kieuwkorf der Ascidien en van Amphioxus en Balanoglossus, die daarbij nog gepaard gaat met bijzondere differentiatie van het kieuwskelet in tong- en vorkbogen, onderling verbonden door synapticula.

Als hoofdgrond voor de waarschijnlijkheid dezer laatste onderstelling meen ik te mogen aanvoeren, dat in de ontogenie der Chordata de kieuwuitstulpingen van den slokdarm wel achtereenvolgens van kop- naar staartzijde worden aangelegd, maar daarentegen van een algemeene regelmatige reductie in omgekeerden zin niet kan gesproken worden, zelfs bij de Longademhalers, waar de kieuwzakken ten slotte tot op enkele sporen na worden gereduceerd.

De buitensporige vermeerdering van het aantal spleten in de kieuwkorf der Ascidien staat ongetwijfeld in verband met de vastzittende levenswijze, en dezelfde oorzaak zal zich wel hebben doen gevoelen in de phylogenetische ontwikkeling der Acrania, al hebben deze zich ook weder daaruit weten los te maken, zoodat zij het vermogen hebben herkregen of behouden om vanuit het zand, waarin zij zich begraven, vrij te gaan rondzwemmen.

Dat wij bij de Myxinoïden te maken hebben met secundaire vermeerdering hunner kieuwzakparen lijkt mij daarom des te waarschijnlijker, omdat het aantal daarvan bij de verschillende soorten dezer onderorde van de Cyclostomen zoozeer uiteenloopt. Terwijl toch Bdellostoma stouti niet minder dan 17 paar heeft, telt Bd. burgeri er slechts 6. Daarbij komt, zooals Stockard heeft aangetoond, dat bij de Myxinoïden het geheele kieuwapparaat op de voorste vier kieuwzakken na, blijkbaar in verband met de overmatige ontwikkeling van de tongmusculatuur, in staartwaartsche richting is verschoven. Ook dit wijst op groote secundaire

wijzigingen in het grondplan en draagt dus volstrekt niet het karakter van oorspronkelijkheid.

Evengoed als in de metamerie der rompsegmenten kan in de branchiomerie der kieuwstreek bij sommige vormen een verhoogde toename plaats hebben gevonden. Op dezelfde wijze als de Slangen zich uit Hagedisachtige voorouders met een geringer aantal rompwervels ontwikkeld hebben, of de Coecilien en de Proteusachtige Amphibien uit gewone Salamanders, kunnen de Bdellostoma's met een groot aantal kieuwzakken in alle drie de gevallen ontstaan zijn uit vormen met een geringer aantal, onder achteruitgang der ledematen (zie hiervoor onder No. 10).

Beslissend in dit opzicht lijkt mij de loop der hersenzenuwen die de kieuwkorf innerveeren, dus der rami branchiales vagi, en daarbij aansluitend die van den nervus hypoglossus. Beide zenuwen slingeren zich om het caudale eind van den kieuwkorf heen (zijn recurrent) en vormen dus een lis, die in de tong uitstraalt. Bij de uitbreiding van den kieuwkorf worden de voorste ruggemergszenuwen staartwaarts opgedrongen, evenals het hart in zijn hartezakje, zoodat dit in de lichaamsholte van de rompstreek uitpuilt. (Zie hieromtrent de figuur der kopzenuwen van Johnson in Bronn's Klassen und Ordnungen, Vol. Cyclostomen.)

Voor de opvatting, dat het groot aantal kieuwopeningen bij Bdellostoma mag toegeschreven worden aan secundaire vermeerdering pleit ook, dat Petromyzon constant zeven kieuwgaten heeft, dus evenveel als Hexanchus onder de Gnathostomen.

Dit wijst er op, dat de hoogere getallen bij Bdellostoma en zijn fossiele verwanten, die bovendien van 17 tot 26 paar wisselen, aan secundaire vermeerdering mogen toegeschreven worden. Daarbij lijkt mij de afwezigheid van een onderscheid tusschen de eerste kieuwopeningen en de volgenden, dus het ontbreken van een gedifferentieerd spuitgat zeer goed verklaarbaar als een rechtstreeksch gevolg van de vervorming der mondholte van een met kaken bewapende tot een zuigmond. Wat toch is het geval bij de Gnathostomen. De eerste kieuwboog (palatoquadratum + onderkaak) drukt met zijn kaakgewricht tegen den tweeden (hyomandibulare + hyoïd) en duwt daarbij de eerste kieuwspleet tot op zijn dorsaal gedeelte dicht, zoodat het lumen tot op een kleine opening (het spuitgat) toegroeit.

Waar bij de Cyclostomen de kaakvorming uitblijft (beter gezegd is teruggegaan) geschiedt dit ook met het dichtdrukken van de spuitgatkieuwspleet, die dus gelijk aan de overige blijft.

7. *Aanleg van een doorloopende vinplooi langs de dorsale middellijn.*

Bij alle Visschen, ook de sterkst gedifferentieerde, mogen de ongepaarde vinnen beschouwd worden als onderdeelen van een ononderbroken huidplooi, die zich niet alleen langs de geheele rugzij, maar ook om het staarteind heen langs de buikzij van den staart tot aan den anus uitstrekte en zelfs om de aarsopening heen als een dubbelplooi zich in kop-

waartsche richting kon voortzetten, zooals Amphioxus nog laat zien.

Dat bij de Ammocoetes-larve van Petromyzon de mediane vinplooi nog onverdeeld is, mag wel is waar als een oorspronkelijk kenmerk worden beschouwd, maar verleent daaraan volstrekt geen bijzondere beteekenis voor de afstamming van de Gnathostomen uit de Cyclostomen. Het wil niet anders zeggen, dan dat een embryonale aanleg zich nog gedurende het larvenleven handhaaft, zoodat de differentiatie in afzonderlijke rug- en staartvinnen eerst bij den overgang tot den geslachtsrijpen toestand intreedt: weder een voorbeeld van vertraagde ontwikkeling.

Overeenkomstige verschijnselen doen zich voor bij tal van Gnathostome visschen, zelfs bij sterk gedifferentieerde, zzoals de Platvisschen, waarbij haast de geheele omtrek van het lichaam, zelfs een gedeelte van den schedel door een ononderbroken vinplooi wordt omzoomd. In verband daarmede baant het zich verplaatsende oog zich een weg door het vooreind van de rugplooi, wat er op zou kunnen wijzen, dat de verwringing van het oog oorspronkelijk door actieve pogingen van het jonge dier om naar boven te kijken werd tot stand gebracht, waarbij tevens de nog plastische schedelaanleg werd verwrongen. In hoeverre deze vervorming erfelijk is geworden en dus thans zonder bewuste medewerking van de aanvankelijk-symmetrische larve zich ontwikkelt, ware misschien door proefnemingen na te gaan.

8. *Een in het tweede kopsegment gelegen mond, die zich aansluit bij de reeks der Kieuwspleten.*

Het vraagstuk van den aard en de ontwikkeling van den mond der Chordaten lijkt mij nog steeds niet geheel opgelost, ofschoon men als bewezen mag aannemen, dat zoowel bij de Cyclostomen als bij de Gnathostomen de definitieve mond ontstaan is uit een mediane ongepaarde ectoderminstulping: het stomodaeum, en twee gepaarde uitstulpingen van het vooreind van het archenteron, die als eerste paar kieuwspleten kunnen opgevat worden. Dat dit paar nog voor de spuitgaten gelegen was, lijkt mij duidelijk. Waar nu bij de Cyclostomen de voorste twee kieuwbogen in ontwikkeling terugblijven en niet tot functioneerende kaken worden, is het niet vreemd, dat de oorspronkelijk drieledig aangelegde mondopening zonder verdere complicaties in den definitieven mond overgaat. Maar blijft ook al de mondopening zelf eenvoudig, de randen ervan differentieeren zich des te meer, en daarbij nog op verschillende wijze bij de onderscheidene genera en species. Vergelijkt men b.v. de afbeeldingen der mondtrechters van drie genera der Petromyzonten: Petromyzon, Lampretia en Geotria in Bronn's Klassen und Ordnungen, Vol. VI, Bl. 678, 680 en 682, dan ziet men dat de hoorntanden slechts schijnbaar in radiaire rangschikking geordend zijn, maar in 't wezen der zaak tweezijdig symmetrisch geschaard staan, zoodat men een bovenste (antero-dorsale) en een onderste (postero-ventrale) rij onderscheiden kan: dat wil dus zeggen een bovenkaaksche en een onderkaaksche tandgroep.

Rondom de centrale mondopening met deze twee primaire rijen van hoorntanden hebben zich een aantal secundaire rijen van kleinere ontwikkeld, die in verband met den kegelvorm van de mondtrechter een radiaire scharing hebben verkregen.

De hoorntanden zelf mogen opgevat worden als secundaire vervangers van echte beentanden, op dezelfde wijze als bij zoovele Gnathostomen: b.v. vorsch-larven, Schildpadden, Vogels, Monotremen, Edentaten, Baarddragende Walvisschen. Bij alle staat deze vervanging in verband met veranderde wijze van voedselopname, bij Cyclostomen met den overgang van bijten tot zuigen, waarbij ook de mondopening zich vernauwd heeft.

9. *Aanwezigheid in het eerste kopsegment van een ongepaard divertikel van den monddarm, die naar buiten doorbreekt, waardoor de spits van den oerdarm in verbinding treedt met de ectodermale instulping van den aanleg der hypophyse.*

Van deze praeorale voortzetting van het entoderm zegt Naef:

„Daarentegen komt bij Gnathostomen ook een gepaarde open verbinding van het praeorale entoderm met de buitenwereld op een vroegtijdig ontwikkelingsstadium voor, in het bereik van den aanleg der hypophyse, het duidelijkst bij ouderwetsche inrichtingen in den vorm van vroeglarvale hechtorganen" (sic).

Men vraagt zich af, wat Naef mag bewogen hebben om aan dergelijke zuig- en tastapparaten aan de mondranden van visschen, die op of in den bodem van het water leven een primaire beteekenis toe te kennen. Zij komen niet slechts bij larven van Cyclostomen, maar ook bij volwassen Gnathostomen zooals de Siluriden voor en behoeven niet als kenmerken van verwantschap tusschen verschillende, van dergelijke tentakels voorziene zwemmende dieren beschouwd te worden. Maar ook als men ze zou willen opvatten als atavismen uit een archaïschen toestand, zou men eerder geneigd zijn, aan de cirri der Acrania te denken dan aan voorste kieuwspleten der voorouders van de Chordaten.

10. *Afwezigheid van voorste en achterste ledematen.*

In alle afdeelingen van Vertebrata vindt men enkele vormen bij welke hetzij beide paren ledematen ontbreken, dan wel een der beide. Bij de Gnathostomen is zonder uitzondering het bewijs te leveren, dat de afwezigheid van ledematen mag toegeschreven worden aan achteruitgang, zoodat de pootlooze dieren moeten afstammen van voorouders, die in het bezit van twee paar ledematen waren. In de meerderheid der gevallen worden er nog rudimenten van teruggevonden, b.v. bij de Hazelwormen onder de Hagedissen, de Boa's en Python's onder de Slangen. Maar ook wanneer dit niet het geval is, zooals bij de Amphisbaenen onder de Saurii, de Coecilia onder de Amphibia, wijst de nauwe verwantschap dezer pootlooze groepen met de normaal gebouwde, van pooten voorziene afdeelingen op de hooge waarschijnlijkheid, dat de eerste uit de

laatste afstamden, terwijl tusschen de onderscheidene vormen van pootlooze Vertebraten onderling geen sporen van genetisch verband te ontdekken zijn.

Reeds onder de oudere palaeozoïsche Stegocephale Amphibien treden pootlooze vormen, zooals de Aistopoda naast de van ledematen voorziene overige Lepidospondile op. Geheel in overeenstemming hiermee is de opvatting van Stensiö, dat bij de uitgestorven palaeozoïsche verwanten der Cyclostomen resten van ledematen in den vorm van stekels bestonden.

Terwijl men zich nu zeer goed kan voorstellen, dat reeds in het Cambrium de voorouders der recente Cyclostomen uit van ledematen voorziene waterbewoners zijn ontstaan, door aanpassing aan het leven op of in den bodem, dan wel door parasitische vastzetting aan andere visschen, lijkt het daarentegen wel heel onwaarschijnlijk, dat pootlooze primitieve vischachtige dieren, die overigens in hun geheele bouwplan met de overige visschen overeenstemden, achteraf nog ledematen verkregen zouden hebben, die dan op de grenzen tusschen hals en romp, en tusschen romp en staart tusschengeschoven zouden zijn.

In zijn beschouwingen geeft Naef geen oplossing voor deze moeilijkheid, wat voor hem des te bezwaarlijker zou zijn, omdat hij aan de primitieve Vertebraten een groot aantal kiemspleten toeschrijft, waardoor voor de ontwikkeling van ledematen geen plaats zou overschieten. Immers hij zegt: „De oudste Gewervelde dieren stellen wij ons (tenminste op jeugdig stadium) zoo voor: als Zoetwaterdieren met overwegend spoelende (strudelnde) voedingswijze op de manier van de jonge Ammocoetes."

Hoezeer Naef de twijfelachtigheid zijner beweringen zelf heeft gevoeld, blijkt wel uit hetgeen hij er onmiddellijk op laat volgen: „Daarbij mogen wij hun (de Cyclostomen) weliswaar geen zoo in bepaalden zin gespecialiseerden mond, geen met het hypophyse-kanaal verbonden neusgroeven, en geen volledig ontwikkelde Vertebratenoogen toeschrijven."

Dan keert hij echter weer op zijn standpunt terug, daar hij zegt:

„Het oorspronkelijk aantal kieuwspleten moet in allen gevalle grooter gedacht worden als bij de levende Petromyzonten, welke immers ook in andere opzichten zich tegenover de Myxinoïden als atypisch veranderd voordoen: (secundaire differentiatie der (ongepaarde) vinnen, verlenging van den staart). Bij deze" (t.w. de Myxinoïden) „worden tot 17 spleten aangelegd, en ook de aan hen voorafgaande stadiën der Chordaten en Pharyngotremen verlangen het aannemen van een grooter aantal kieuwopeningen".

Hoe hij zich die opvolgende ontwikkelingstrappen der Chordaten voorstelt, tracht Naef te verduidelijken door een reeks schematische figuurtjes, die hij achtereenvolgens betitelt met de namen: blastula, progastrea, gastrula, metagastrea, prohelminth, helminthoid, pharyngotrema. Aan dit laatste stadium kent hij 25 kieuwspleten toe. Maar bovendien kantelt hij het om, zoodat de buikzij tot rugvlakte wordt. Aangezien

in de gehele embryologie der Chordaten geen enkel spoor van zulk een omkeering te vinden is, kan men een dergelijke voorstelling niet anders dan goochelen met schematische figuurtjes noemen, waarmee men naar willekeur alles bewijzen kan.

Uit de figuurtjes, die het prohelminthoïde en helminthoïde stadium moeten voorstellen, is het duidelijk, dat NAEF een zekere overeenkomst met de vrijzwemmende larven der Polychaete Anneliden aanneemt. In werkelijkheid doet de jonge Ammocoeteslarve slechts weinig denken aan een Annelide-larve, maar in zooverre bestaat overeenkomst, dat rug- en buikzijde van beide gelijk georiënteerd zijn ten opzichte van de plaats der mondopening. Daaruit volgt echter noodwendigerwijze, dat bij de larve van Ammocoetes geen omkeering kan plaatsgegrepen hebben.

Summary.

In contradiction to the views of NAEF (and other recent Zoologists) that Cyclostomes should be the few surviving representatives of primitive Vertebrates, which had not yet obtained jaws and limbs, and from which the Gnathostomes had evolved, I suggest that their organisation must be explained as the consequence of far reaching modification and reduction of a gnathostomic type of Fish, by the influence of their abode in the muddy bottom of rivers and seas, and their more or less parasitistic way of nutrition. By these external influences they lost their jaws and their limbs, while their nasal apertures conjoined to one dorsal median pore. The original difference between the spiracle and the other gill-clefts disappeared, while (but only in some Myxinoïds) the gill-clefts considerably increased in number. In this latter instance Acrania and Ascidia reached a still higher grade of modification, in consequence of temporary or everlasting fixation, or of immersion in the sand.

To the same causes may be ascribed the cylindrical bodyform of Cyclostomes, giving them a superficial similarity to eals and serpents and even to worms, which however has nothing in common with phylogenetic origin, but is simply a feature of paralellism.

Judging after the name Helminthoïd, which NAEF accords to some (hypothetical) stages in their ontogeny, he attributes a far higher value to this resemblance with worms. In order to prove this relation he even does not hesitate to assume, that in the course of their development this helminthoïd stage executed a complete turning upside-down, by which the dorsal and ventral side changed in position, though no trace whatever of such a reversion can be observed in their ontogeny.

By his views NAEF is led to the conclusion that the Myxinoïds are more primitive than the Petromyzontidae, especially on account of their higher number of gill-slits, making them more nearly related to Amphioxus. There can be no doubt that this conclusion is erronious.

Résumé.

Selon l'opinion de NAEF et plusieurs autres zoologistes modernes les Cyclostomes devraient être considérés comme des êtres primitifs et ancestraux des autres Vertébrés. Au contraire le présent auteur prétend que leur organisation est le résultat d'une profonde modification, causée par leur mode de vie dans la boue des rivières et de la mer, et encore plus par l'influence de leur nutrition plus ou moins parasitique. Ces causes ont fait disparaître leurs nageoires et leurs machoires, et réduit leurs ouvertures nasales à un seul trou situé dans la ligne médiane du dos. La forme cylindrique de leur corps, qui leur donne une ressemblance superficielle avec les anguilles et les serpents (en outre sans aucune importance génétique) peut être attribuée à ces mêmes causes, aussi bien que l'augmentation du nombre des fentes branchiales, qui de plus est extrêmement variable, et ne se retrouve que chez quelques genres des Myxinoides, en contraste des Petromyzontidae, où le nombre est fixe, et correspond avec celui des Notidanides.

Pour prouver une relation génétique des Cyclostomes avec les Vers, NAEF introduit un stade „helmintoïde" dans leur ontogénie, qui n'existe pas en réalité, mais par là il se voit contraint de concevoir une complête réversion dans le cours de leur développement, qui fait changer les surfaces dorsale et ventrale de position. Ce revirement est absolument imaginaire, puisqu'on n'en trouve aucune trace dans leur ontogénie, de sorte que cette hypothèse ne prouve rien en faveur d'une origine directe des Cyclostomes d'ancêtres vermiformes.

Zusammenfassung.

In Erwiderung der Behauptungen NAEF's, wonach die Cyclostomen als ursprünglich organisirte Vorfahren der Gnathostomen zu betrachten seien, halte ich sie für weitgehend umgeändert, in Folge ihrer Lebensweise im Schlammboden der Gewässer, und ihrer saugenden Ernährungsart. Durch die erstere haben sie die wurmförmige Körpergestalt bekommen und ihre Gliedmassen verloren, durch die letztere haben sich ihre Kiefer rückgebildet, und hat sich ihre Mundhöhle zu einem Saugtrichter umgestaltet, der ausserdem bei Petromyzonten und Myxinoïden noch von gänzlich verschiedener Form ist, und dabei nur scheinbar kreisförmig, in Wirklichkeit aber zweiseitig symmetrisch. Durch dieselben Ursachen haben sich die Nasenlöcher nach der Rückenseite verlagert und zu einer unpaaren Oeffnung vereinigt.

NAEF sieht auch in der höheren Zahl der Kiemenöffnungen bei einigen Gattungen der Myxinoïden ein ursprüngliches Merkmal, das nebenbei noch auf nähere Verwandtschaft mit den Acraniern und Ascidien deuten soll. Von diesem Gesichtspunkt aus kommt er zu der Schlussfolgerung, dass die Petromyzonten mehr von dem ursprünglichen Bildungstypus der Cyclostomen abgewichen seien als die Myxinoïden. Es ist aber klar,

dass diese beiden Abteilungen sich gerade umgekehrt zu einander verhalten, worauf auch die grosse Variationsbreite in der Kiemenspaltenzahl bei den Myxinoïden, in Gegensatz zu der Konstanz bei den Petromyzonten hinweist. Die letzteren stimmen in dieser Hinsicht mit den Notidani unter den Selachiern überein.

In Bezug auf die wurmförmige Körperform sei bemerkt, dass eine solche in allen Abteilungen der Vertebraten bei einigen wenigen Formen vorkommt, und überall als eine secundäre Abrundung der ursprünglich nicht drehrunden Gestalt gedeutet wird. Es liegt also kein Grund vor, bei Cyclostomen an die Wurmförmigkeit einen primären Charakter zuzuschreiben, der auf nähere Verwandtschaft mit den Anneliden hinweisen sollte. Naef nimmt eine solche an, indem er in ihre Ontogenie einige von ihm abgebildete, in Wirklichkeit jedoch nicht bestehende „helminthoïde" Stadien hineininterpretrit. Er sieht sich aber dadurch genötigt anzunehmen, dass die fertig ausgebildete annelidenartige Larve sich schliesslich um 90° um ihre Längsachse umwälzt, obwohl in der Entwicklungsgeschichte von Petromyzon von einer solchen Rotirung keine Spur zu entdecken ist.

ABSORPTION OF TOOTH TISSUE IN THE SPERM WHALE

BY

H. BOSCHMA

(Communicated at the meeting of February 25, 1950)

According to COLYER (1936, p. 638) in chronic inflammatory processes of the parodontal tissues there is often some absorption of the root of the tooth. As examples of this phenomenon he describes and figures a tooth of a Killer Whale (*Orcinus orca* (L.)) and one of a Sperm Whale (*Physeter macrocephalus* L.). Of the latter COLYER (loc. cit., p. 639) states:

"The tooth of a Sperm Whale, Fig. 864 B, shows a large area of absorption limited to one side of the root. The pathological process causing the loss of tissue was probably associated with pressure from the adjacent tooth. Martin A. C. Hinton, who has a wide knowledge of the Cetacea, informs me that the teeth are set loosely in their sockets and can be moved easily. It is not unusual in these animals for a tooth to be brought into close contact with its neighbour and is therefore possible when this occurs that foreign bodies may become lodged between the teeth and so start pathological processes in the periodontal membrane leading to absorption of the hard tissues. The condition in this tooth of a Sperm Whale is very similar to that seen in human teeth from pressure of one tooth on another, a typical example being the absorption of the root of a mandibular second molar by the crown of a misplaced third molar."

As in other Cetacea the teeth of the Sperm Whale are set loosely in their sockets of the lower jaw, but by no means they can be moved easily. The roots of the teeth of the Sperm Whale are firmly embedded in a very hard and strongly fibrous gum which immovably fixes them to the jaw. Moreover, the individual teeth are implanted in the jaw at distances from each other equalling their thickness, so that it is altogether impossible for two adjacent teeth to come into contact. The firmness of the gum of the Sperm Whale is well illustrated by a case mentioned by NEUVILLE (1932, p. 305, pl. XII), in which the gum was left on the lower jaw to keep the teeth fastened to the jaw in their original position, but after desiccation and contraction of the gum the latter pulled the teeth out of the jaw.

The area of absorption on one side of the root of COLYER's specimen therefore cannot have been caused by pressure of an adjacent tooth. It remains, however, possible that friction by an antagonistic tooth, one

of the more or less rudimentary teeth as they not unfrequently occur in the upper jaw of the Sperm Whale, was the primary cause of the inflammation that gave rise to the described absorption. But COLYER's figure does not show anything pointing to a facet caused by a regular contact of this mandibular tooth with an antagonist in the maxilla.

A Sperm Whale tooth in the Nantucket Museum, described by POUCHET and BEAUREGARD (1889, Pl. VI fig. 3), in nearly every respect is similar to the specimen dealt with by COLYER. On one side of this tooth there is a distinct area of absorption, whilst the inflammation at the basal part of the root has given rise to a thickening of this region. The cited authors describe the tooth as "fortement cariée", without giving the reason why they come to the conclusion that this abnormality was caused by caries. The figure does not show any indications of a former contact of the tooth with another.

The three Sperm Whale teeth described in the present paper are in the collections of the Leiden Museum. They were selected from material brought together by whaling cruises some time before the last war, probably in the antarctic region. The teeth dealt with here show peculiarities of absorption similar to that in the teeth described by COLYER and by POUCHET and BEAUREGARD. In one (fig. 1) the factors influencing the abnormality are not apparent, in each of the two other teeth (figs. 2 and 3) the inflammation obviously was brought about as a result of contact with an antagonistic tooth in the upper jaw.

Judging by its comparatively large size the first tooth (fig. 1) is from a male Sperm Whale. Its total length is 143 mm; in all probability in the living animal about one third of the tooth protruded from the gum. The topmost part is evenly rounded by wear, so that here a blunt cone of dentine has lost its covering layer of cement. The figure in all probability shows the lingual side of the tooth. With the exception of the top and the lower part of the root this side of the tooth is covered by a dark layer in which some irregular white patches are visible, indicating that here the parodontal tissues have undergone inflammatory processes. At the labial side of the tooth the same abnormal state is present though to a less extensive degree. The pathological process resulted in weakening and partial absorption of the root. Moreover, inside the pulp cavity the absorption of tooth tissue continued upwards, so that the tooth now has a wide central cavity extending for 73 mm from the lingual base of the root (for 113 mm from the labial base of the root).

After the process of absorption of the root had set in, the lower part of the root partly regenerated, resulting in more or less irregular outgrowths of tooth tissue around the area of inflammation.

As stated above the abnormality of this tooth is of a similar kind as that previously described by other authors; the cause of the process is unknown. There is at least not any indication of a contact of this tooth with an antagonist in the upper jaw.

Maxillary teeth of the Sperm Whale have been mentioned or described in a more or less detailed manner by several authors (cf. BOSCHMA, 1938, pp. 207—225). In some papers, moreover, phenomena of wear of mandibular teeth have been described which must have been due to contact of these teeth with their antagonists in the upper jaw. Beautiful examples of this kind were described and figured by NEUVILLE (1932, p. 326, figs. 50—53): the large facets occurring on these teeth must have been brought about by friction of a maxillary tooth against their crown. Similar facets are found on some of the teeth of the lower jaw of a female Sperm Whale in the Leiden Museum (BOSCHMA, 1938, figs. 17 and 18). In the cited paper, moreover, several smaller facets on maxillary as well as mandibular teeth are described: as here the situation of the maxillary dentition in respect to the mandibular toothrow was known it could be definitely proven that the facets are the result of contact of antagonistic teeth (loc. cit., pp. 237 — 239, figs. 14 and 16).

All the hitherto described facets of mandibular teeth of the Sperm Whale were of comparatively small length, so that they were confined to the crown of the teeth only (in this case the crown may be defined as the part of the tooth projecting above the gum). In the two specimens described below, however, the facets run from the top of the tooth to about half way its length, so that here during the contact of the antagonistic teeth the tops of the maxillary teeth must at least have come into contact with the gum of the lower jaw, and probably penetrated into the gum.

Judging by their size and shape the two teeth are from the lower jaws of female Sperm Whales, but as they as yet do not show any signs of a beginning basal occlusion of the pulp cavity it is not out of the question that they might have been taken from the jaws of young males. One (fig. 2) is from a left mandibula, the other (fig. 3) from a right mandibula. The two teeth are of a similar shape, both are slightly curved and have a rather sharp top.

One of these teeth (fig. 2) has a length of 137 mm and a greater transverse diameter of 41 mm: it has a smoothly conical pulp cavity with a height of 76 mm. At the lingual surface from the top to about half way the length of the tooth there are a number of facets running in a longitudinal direction: each of these facets has a breadth of up to 5 mm. This indicates that the tooth during the act of feeding was rubbing against a tooth in the upper jaw, whilst the mutual position of the two antagonistic teeth during this process did not remain absolutely fixed. From time to time the position of at least one of the teeth (probably the maxillary tooth which must have been less strongly fixed in the gum than the mandibular tooth) slightly shifted its position so that the manner of contact of the two teeth took place in a slightly different way. The facets are continued to about half way the length of the tooth, a region that undoubtedly was below the surface of the gum. Apparently here the

parodontal tissue was partially destroyed which gave rise to a pathological development of this part of the tooth. Possibly also foreign matter became imbedded in the opening between the tooth and the gum, which may have had its influence on the process of inflammation starting here. Absorption of a small part of the tooth tissue has taken place, and as a reaction against this process some regeneration has occurred, so that at the lower margin of the region of absorption there is a small exostosis of tooth material extending downwards from this region. The root of the tooth is entirely normal except in the middle of the lingual surface where there is a longitudinal narrow opening extending from the area of absorption to the base of the root. The margins of this slit have slightly grown inwards. With the exception of this narrow open space the pulp cavity does not show any irregularities; as yet the process causing absorption of tooth tissue had not penetrated into the interior of the pulp cavity.

The other tooth (fig. 3) has a length of 135 mm and a greater transverse diameter of 35 mm. From the top to about half way the length of the tooth there is, in the middle of the lingual surface, a facet of a breadth of about 10 mm. As a matter of fact this facet ended downwards below the surface of the gum, so that here also the action of the maxillary tooth must have destroyed part of the parodontal membrane and of the adjacent gum, bringing about a distinct herd of inflammation. Here again introduction of foreign matter may have hastened the process of the pathological conditions. Just below the lower end of the facet there is a rather wide cavity which has a basal transverse diameter of 25 mm and extends upwards into the central part of the tooth for a distance of 35 mm. The conical pulp cavity has a depth of 55 mm, it shows a large cleft in the middle of the lingual surface of the tooth. Here again the margins of this opening are distinctly curved inwards. At each side of the large opening, just below the region of absorption of the tooth tissue, some regeneration of tooth tissue has occurred, so that two scaly exostoses are to be seen here.

COLYER's explanation for the causes of absorption of tooth tissue in the Sperm Whale cited above cannot be upheld for the specimen described and figured by him. This explanation, however, exactly holds for the two teeth of figs. 2 and 3 in the present paper, in so far that not pressure of adjacent teeth is the factor causing the abnormality, but prolonged contact of antagonistic teeth. The maxillary teeth causing the deformities of the mandibular teeth of figs. 2 and 3 must have been of fairly large size in contradistinction to the rudimentary condition of the teeth generally occurring in the upper jaw of the Sperm Whale.

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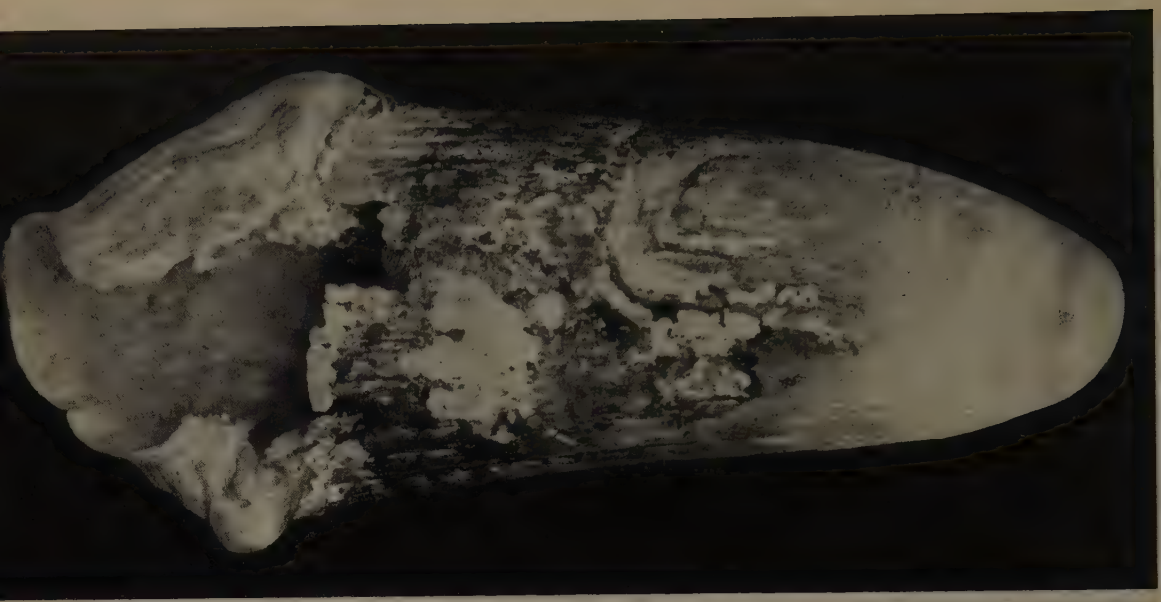

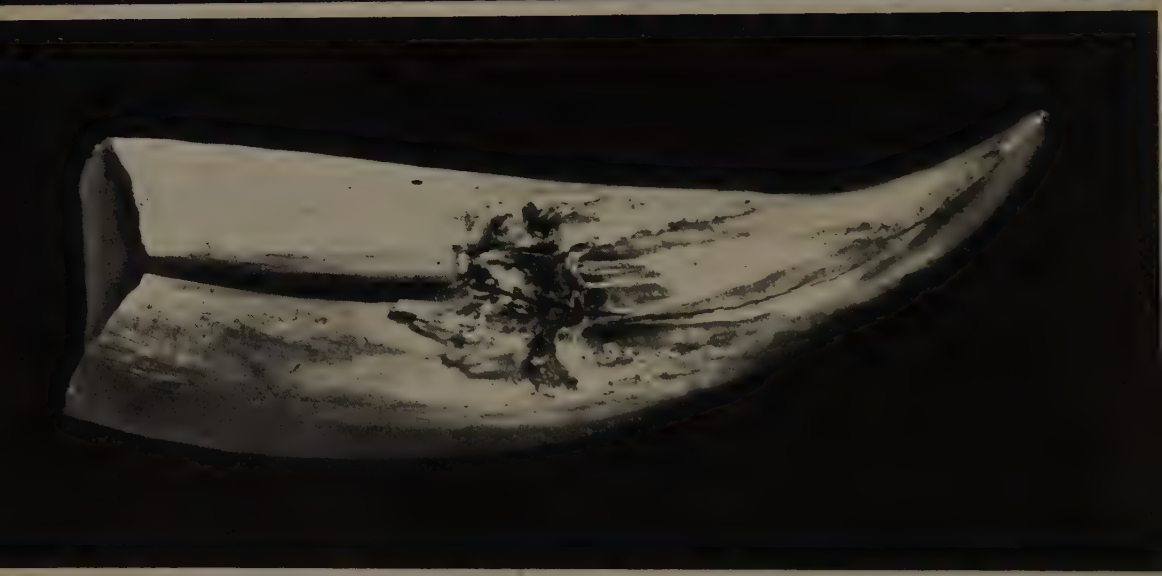

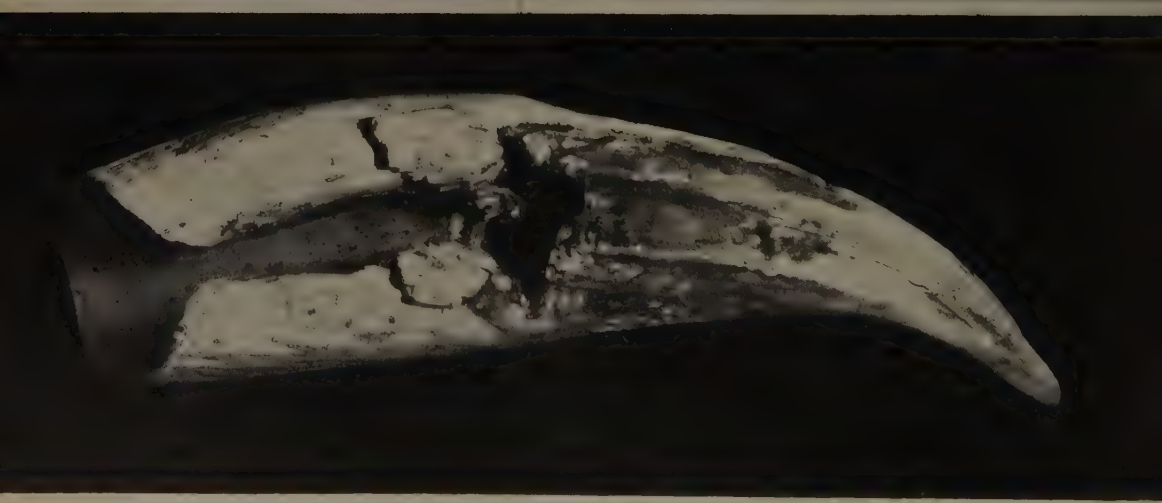

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Fig. 1. *Physeter macrocephalus* L. Right (?) mandibular tooth of male specimen, probably lingual surface.

Fig. 2. *Physeter macrocephalus* L. Left mandibular tooth of female (?) specimen, lingual surface.

Fig. 3. *Physeter macrocephalus* L. Right mandibular tooth of female (?) specimen, lingual surface.

All figures approximately natural size.

NOTES ON THE CORAL REEFS NEAR SUVA IN THE FIJI ISLANDS

BY

H. BOSCHMA

(Communicated at the meeting of February 25, 1950)

When in March 1949 I stayed for nineteen days in the Fiji Islands I had the opportunity to visit the coral reefs of Mbengga, Makuluva, Kandavu and neighbouring islands, Suva, and Tomberua. Of these the Suva reefs are the least flourishing as far as regards the abundance of living corals. As, however, these reefs possess some peculiarities of special interest, notably as far as concerns the distribution of two species of *Millepora*, some notes on these reefs may be given here as an introduction to further notes on Fijian corals. I am under great obligations to Mr and Mrs HAROLD GATTY of Suva for their hospitality and for the assistance to enable me to visit the various reefs mentioned above. Moreover I want to express my thanks to the Public Relations Office at Suva for the photographs of the plates in the present paper, taken at my request by the official photographer Mr R. WRIGHT.

Of the Suva reefs I repeatedly visited the two reef flats bordering the entrance to Suva Harbour, indicated on fig. 1 with the numbers 1 and 2. The dot near the number 1 gives the position of the wreck marked on the Admiralty Charts. The northern parts of these reef flats have hardly any living corals, but towards the outer opening of the entrance to Suva Harbour the flats show a rather broad marginal belt of living corals, almost entirely consisting of the species *Acropora millepora* (Ehrenberg) [1]). Plate 1 represents a photograph of this marginal area of the reef, taken at the locality near the number 2 in fig. 1, looking towards the southwest. Besides *Acropora millepora* in this region there are several stunted colonies of a knobby yellow *Porites* and some scattered colonies of Astraeids, obviously living under rather unfavourable conditions. Moreover in the shallow pools in the *Acropora*-zone and next to this zone there are numerous colonies of an incrusting species of *Montipora* of a grayish or brownish or greenish colour, apparently also living under more or less adverse conditions.

Towards the edge of the reef (the region of the first line of breakers

[1]) I owe this identification to Dr J. VERWEY, who added a note explaining that in literature this species is also known as *Acropora prostrata* (Dana) and *Acropora squamosa* (Brook).

in Plate I) the living corals become of a more luxuriant growth. A part of the edge of the reef is seen on Plate II. Among Madreporarian corals a species of *Pocillopora* is common here, consisting of fairly large colonies of normal appearance. Moreover Astraeid corals obtain here a rather

Fig. 1. Suva Harbour. Shaded parts: land and reefs or mud flats, dry at low ebb tide; heavy line: coast line. Scale 1 : 48.600. After British Admiralty Chart no. 1660.

luxuriant development. In many parts of the edge of the reef there are numerous colonies of *Millepora tenera* [2]), which often grow out to a comparatively large size.

Plate II clearly shows that *Millepora tenera* on this part of the reef is living under favourable conditions. The colonies are fully exposed to the rather heavy surf, and, apparently as a result of the constant action

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[2]) In a revision of the genus *Millepora* (Boschma, 1948) for this species I used the name *Millepora tenella* Ortmann. As this name proved to be preoccupied it was changed to *Millepora tenera* (cf. Boschma, 1949).

of the waves, the colonies consist of rather broad more or less flabelliform plates which on their upper margins only are divided into a row of short branches. The colonies are of a brown colour with the exception of the tips of the branches which have a light yellow colour. The more or less plate-like larger branches of the colonies have a similar arrangement as those of previously described specimens of *Millepora tenera* (cf. BOSCHMA, 1948, Pl. XIII figs. 1 and 2, Pl. XIV fig. 1); the colonies do not distinctly show a tendency of exposing their broadest side towards the direction of the breakers (in the area photographed in Plate II the breakers come from the right side). This does not confirm ABE's (1937) statements, who found that in the Palao Islands "*Millepora alcicornis*" (judging by the figures the colonies must have been *M. tenera*) under the influence of the prevailing current develops a system of branches "vertical to the current direction". Here the only apparent result of the heavy surf is a tendency for strength of the branches, whilst as a whole the colonies grow out in a more or less semiglobular shape so as to offer the least possible resistence to the waves.

Millepora tenera is not the only species of the genus occurring on the edge of the Suva reefs. In this region also a few colonies of *Millepora platyphylla* Hemprich and Ehrenberg were found. Plate III, taken from the side of a small outlet on the edge of the reef, shows two colonies of the latter species. On a more reduced scale the corals of this plate are drawn in outline in fig. 2. In the central part there is a colony of an Astraeid,

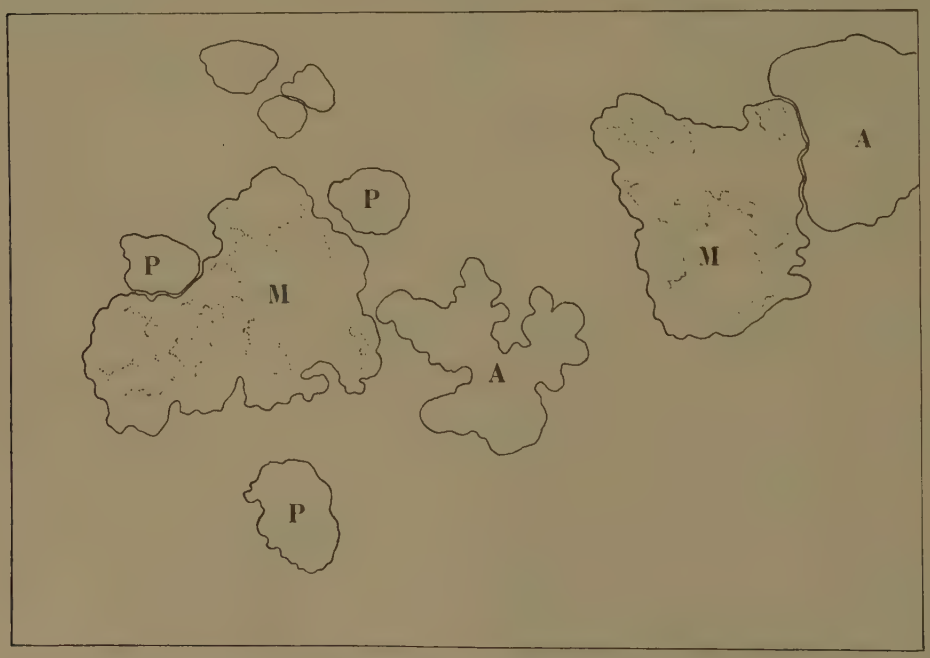

Fig. 2. The mutual position of the larger colonies of Astraeids (A), of *Pocillopora* (P), and of *Millepora* (M), as they are photographed on Plate III.

which at each side is flanked by an encrusting colony of *Millepora platyphylla*, one of which is surrounded by a number of small colonies of *Pocillopora*. On the encrusting bases of the colonies of *Millepora* there are numerous ridges of a slightly lighter colour than the flatter parts of the colonies: these ridges are the beginnings of what under more favourable conditions would grow out as more or less vertical upstanding plates.

In the region dealt with here *Millepora platyphylla* occurs in dwarfed condition only. This proves that the circumstances prevailing in this part of the reef are not favourable for a luxuriant development of the species. Several authors (DARWIN, 1842; FORBES, 1885; CROSSLAND, 1927, 1928; SEWELL, 1935, 1936) have commented upon the fact that *Millepora platyphylla* (often mentioned under another name, but easily recognizable as such) is common on the surf-swept edge of the reef; in agitated water it seems to find the conditions for a characteristic development into a honeycombed mass of upstanding plates. Another instance of luxuriant growth of *Millepora platyphylla*, on the edge of a reef in the Bikini group, is shown on Pl. I fig. 2 of the paper by TRACEY, LADD and HOFFMEISTER (1948). Now the locality indicated with 2 on fig. 1 of the present paper indeed is under the influence of the full surf from the south, but this locality and that indicated with 1 in fig. 1 are parts of the reef flanking the outlet of Suva Harbour. As shown in fig. 1 there are several small rivers discharging their water in the Suva Harbour region, and as this water is passing the outlet between the localities 1 and 2 it is not improbable that this causes distinctly unfavourable conditions for a vigorous development of *Millepora platyphylla*. In this connexion it may be noted that the reef surrounding Viti Levu, the main island of the Fiji group, clearly shows a wide passage in front of every river of any importance. GARDINER (1898) pointed to the fact that growth of the reefs everywhere seems greatest, where exposed to the heaviest seas, but this may be counteracted by other factors.

On the edge of the reef *Millepora platyphylla* is not confined to the outlets only, as it is also found in localities as the one represented on Plate II, among colonies of *Millepora tenera*. But whilst the latter species here appears to find ideal conditions for luxuriant growth, *M. platyphylla* always remains stunted.

The remarks noted above again point to the fact that in the genus *Millepora* there are distinct species, each with its own preference of habitat. In a locality in which *M. tenera* is flourishing, *M. platyphylla* may just find the means for existence without a possibility for normal development.

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EXPLANATION OF THE PLATES

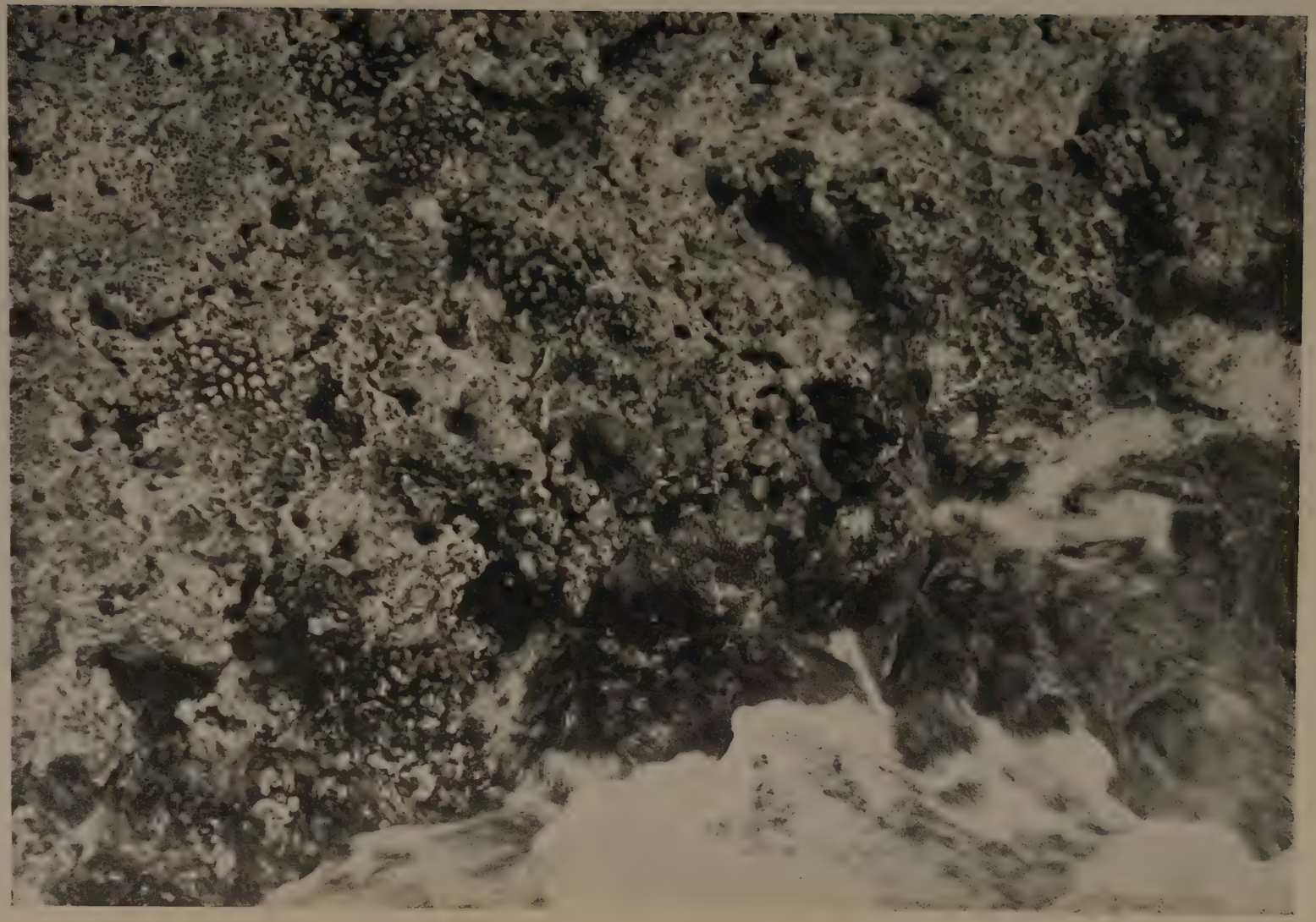
Plate 1. Marginal zone of the reef, chiefly covered with *Acropora millepora* (Ehrenberg).

Plate 2. Edge of the reef with colonies of Astraeids, of *Pocillopora*, and of *Millepora tenera* Boschma.

Plate 3. Side of erosional outlet of the reef with colonies of Astraeids, of *Pocillopora*, and of *Millepora platyphylla* Hemprich & Ehrenberg. In the right hand lower corner a wave is rushing back to the sea.

Courtesy Public Relations Office, Suva, Fiji.

Courtesy Public Relations Office, Suva, Fiji.



Courtesy Public Relations Office, Suva, Fiji.

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP- AND PHOSPHATIDE COACERVATES — XI [1])

THE ACTION OF FATTY ACIDS AND ALCOHOLS ON ALKYLSULPHATE COACERVATES

BY

H. L. BOOIJ AND D. VREUGDENHIL

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of Jan. 28, 1950)

1. *Introduction.*

It has been observed (BOOIJ and BUNGENBERG DE JONG) that fatty acids exert an opening action on an oleate coacervate. In these experiments the p_H was always chosen very high (in the neighbourhood of 12 — 13), so that we were certain that we measured the influence of the anions. When the p_H of an oleate coacervate is lowered, one can observe a considerable shrinking of the volume and the only possible explanation seems to be that, by lowering the p_H, free oleic acid is formed, which undissociated acid exerts a condensing action. Yet this is rather peculiar, as it would mean that at relatively high p_H already (10 — 11, consequently far above the p_K of the carboxyl group) undissociated oleic acid would be present within the soap micelles. Confirmation of this hypothesis would be of importance for biology too, as it would mean that fatty acids (when present in a suitable lipoid layer — e.g. the protoplasmic membrane) might show an "apparent p_K" much higher than the real p_K of the carboxyl group.

PETERS has drawn the attention to the fact that in interfaces too the ionisation of the COOH group takes place at a more alkaline reaction than in true solution. The apparent p_K of palmitic acid at the interface benzene/aqueous phosphate solutions was approximately 7.8 according to his experiments. We think that these phenomena have much in common with the facts reported by us.

Then the acids might have a condensing activity at a biological p_H, and an opening action at slightly higher p_H. Therefore we decided to investigate the influence of fatty acids on coacervates of low p_H. It was not easy to find a soap solution which would coacervate at low p_H. An oleate solution will flocculate when the p_H is lowered below a certain value (approximately 9). It is evident that one must look for a soap molecule with a

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[1]) Publication no. X of this series will be found in Proc. Kon. Ned. Acad. Wetensch. Amst. **53**, 59 (1950).

stronger acidic group in order to be able to perform these experiments. We investigated two synthetic soaps, dodecylsulphate and cetylsulphate. [2])

2. *The action of fatty acids on a coacervate of dodecylsulphate.*

Two difficulties arose when we tried to make a coacervate of this synthetic soap. Firstly a solution of the compound crystallizes at room temperature and secondly it was not possible to get a coacervate with any of a whole series of salts. At very high concentration and at relatively high temperature only NH_4Cl produced a coacervation in the solution. This could not be used, however, but it provided the basis of an adequate method. We added to the soap solution a certain amount of menthol which — as a condensing substance — made coacervation possible at much lower concentration of NH_4Cl. This salt is especially favourable for the solution of the question we put ourselves as it serves not only as the coacervating agent, but forms — together with a small amount of base — a buffer of practically neutral p_H as well. This made it possible for us to dissolve the fatty acids to be investigated in a small amount of KOH, for we were certain that the base added would be buffered by the large amount of ammonium chloride (approximately 1.5 mol/l). We made the following solution:

$$\left\{\begin{array}{l} 5 \text{ g sodiumdodecylsulphate} \\ 200 \text{ ml water} \end{array}\right.$$
$$\left\{\begin{array}{l} 4 \text{ ml KOH } 2\,n \\ 100 \text{ mg menthol} \end{array}\right.$$

This solution must be made fresh every day and during the experiments it must be kept at 35° C, as at room temperature crystallisation will eventually set in. To 25 ml of the "blank" solution we added a weighed portion of a fatty acid, and measured the amount of NH_4Cl needed to produce a coacervate layer of 50 % in the usual manner (see Booij and Bungenberg de Jong).

When plotting the activity (expressed in mol/l NH_4Cl) of 1 mmol/l fatty acid in relation to the number of carbon atoms (fig. 1) it is evident that this influence is no longer the same as it was at high p_H when working with oleate. Most fatty acids have a condensing action now, with a maximum of activity at 14 C atoms. The resulting curve resembles much more that of the action of alcohols on an oleate coacervate (Booij, Vogelsang and Lycklama) than that of the influence of fatty anions on the same substratum (Booij and Bungenberg de Jong). The lower fatty acids exert an opening activity and the picture may be explained best by assuming that the smaller fatty acids act in this system as anions, while the longer

[2]) We want to express our sincere thanks to the Combinatie N.V. en Amsterdamsche, Bandoengsche en Nederlandsche Kininefabrieken for providing us with both compounds.

acids act in the undissociated form. That would mean that the "apparent" p_K of the fatty acids is dependent on the chain length of the substance. This aspect was studied separately and will be published later. It is clear that once more there is a maximum of activity when the chain length of the added fatty acid resembles that of the substratum. Here it must be noted that dodecylsulphate (I) is somewhat longer than the fatty acid with the same number of carbon atoms (II; laurate):

I $CH_3CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2—OSO_3K$

II $CH_3CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2CH_2COOK$

Therefore it is logical to expect the maximum of activity not at 12 C atoms, but somewhat higher.

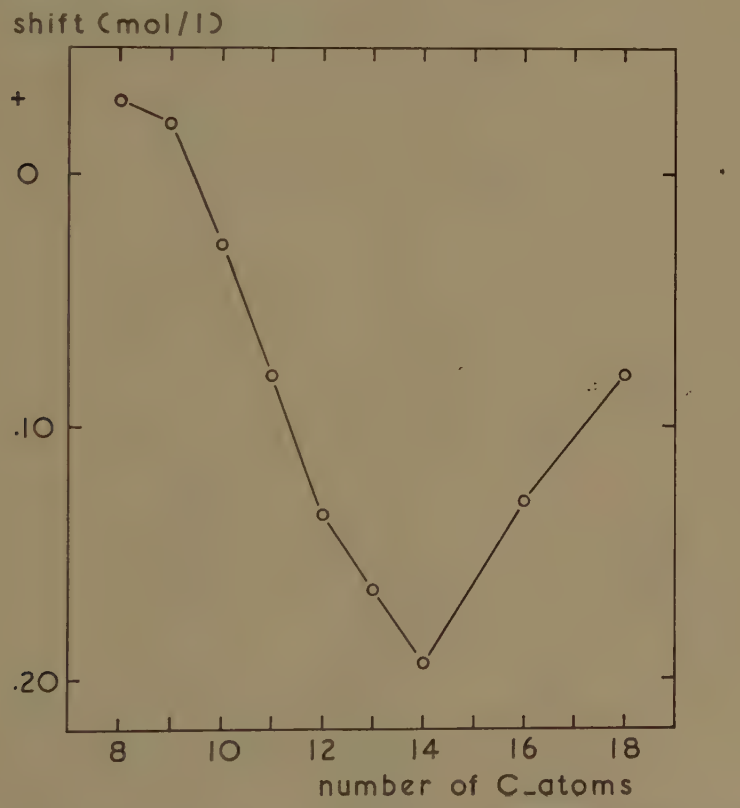

Fig. 1. Influence of fatty acids on a coacervate of dodecylsulphate. The shift of the coacervation curves (at a coacervate volume of 50 %) is expressed in mol/l NH_4Cl (+ = opening, — = condensing activity). In every case 1 mmol/l of the fatty acids was added. Mean of two sets of experiments.

3. *The influence of fatty acids on a cetylsulphate coacervate.*

According to previous investigations it was expected that the maximum of activity would shift to a longer fatty acid if we chose a substratum

with more carbon atoms. This proved to be the case indeed. The crystallisation, however, is a still more serious handicap when working with cetylsulphate instead of dodecylsulphate. Even at 60° C crystallisation begins to appear, and it was necessary to perform the experiments in a thermostate of 66° C. This thermostate must be closed as otherwise

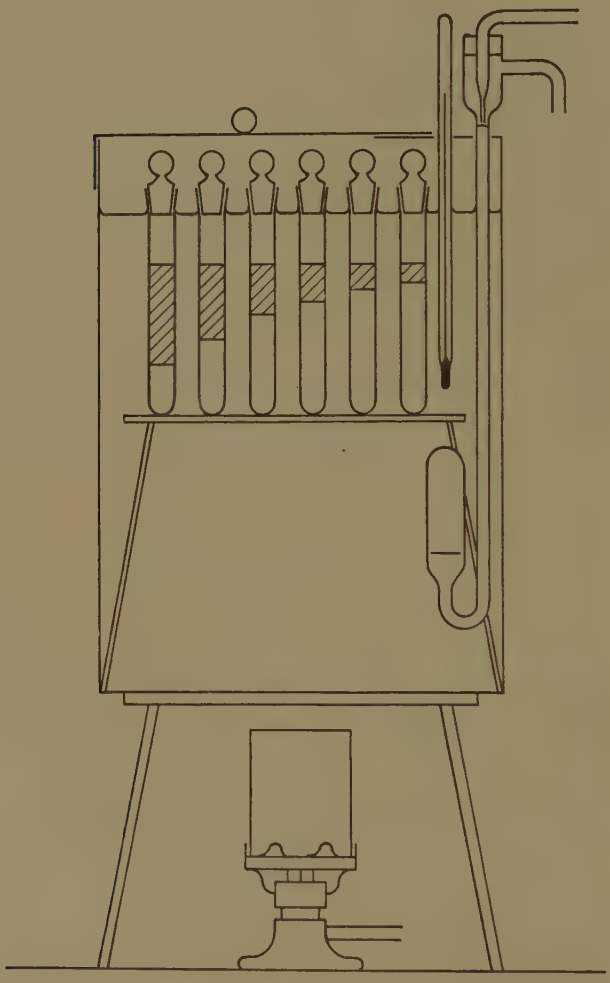

Fig. 2. Thermostate used for experiments with cetylsulphate.

the vaporization of the water interferes seriously, especially because the coacervate forms relatively slowly and the tubes must be left in the thermostate overnight. See fig. 2 for the construction of the thermostate.

Preliminary experiments showed that here coacervation is possible with the aid of ammoniumchloride (where the coacervate was lighter than the equilibrium liquid) and potassium citrate (there the coacervate was found at the bottom of the tubes). It is of importance that the chain

length of cetylsulphate is evidently long enough to give coacervation without the addition of a condensing substance (menthol).

Thus the mixture from which we started (the blank) had the following composition:

5 g sodium cetylsulphate
200 ml water
8 ml KOH 2 n

and the method used was exactly the same as for dodecylsulphate. Fig. 3 demonstrates that here too most fatty acids act as condensing substances (so presumably as undissociated molecules), and that once more a pronounced maximum of activity has been found. The chain length of the substratum is four carbon atoms longer than in the former case and it is in agreement with our hypothesis that the maximum of activity shifts from 14 to 18 carbon atoms.

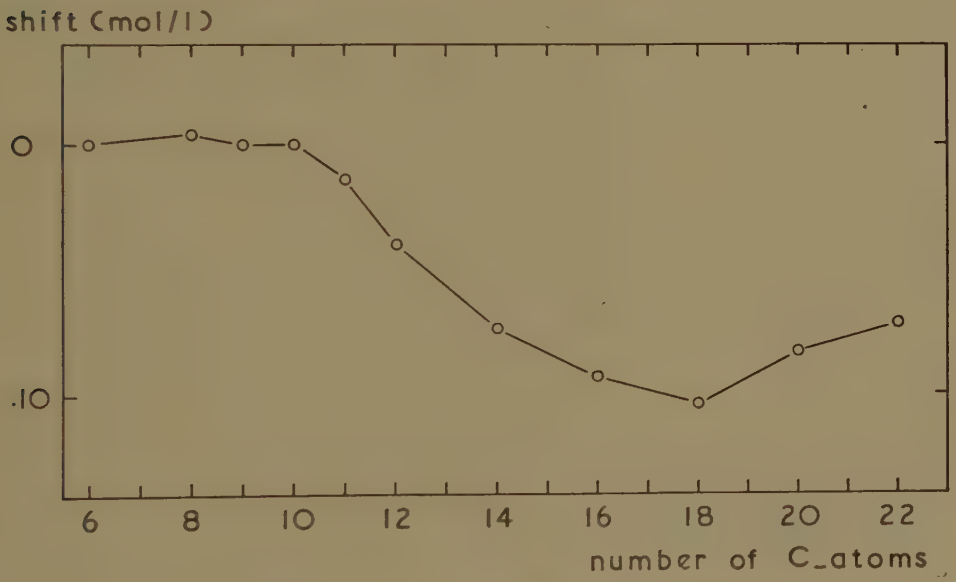

Fig. 3. Influence of fatty acids (1 mmol/l) on a coacervate of cetylsulphate.

4. *The homologous series of the normal alcohols.*

Normal alcohols will have a stronger condensing action on a soap coacervate than the corresponding undissocoated acids, as the polar group in the former is much less hydrophilic than in the latter. Moreover the maximum of activity is expected to lie at a chain lenghth of one carbon atom less, as in the case of the fatty acids the carbon atom is more or less incorporated in the polar group. Our experiments with dodecylsulphate were in agreement with these expectations. When comparing figs. 4 and 1, one observes that a concentration of 0.5 mmol/l of an alcohol has approximately the same activity as 1 mmol/l of the corresponding acid. The

maximum of activity lies at 12 (perhaps 13?) carbon atoms, thus we may conclude that these experiments too may be fitted well into our hypothesis.

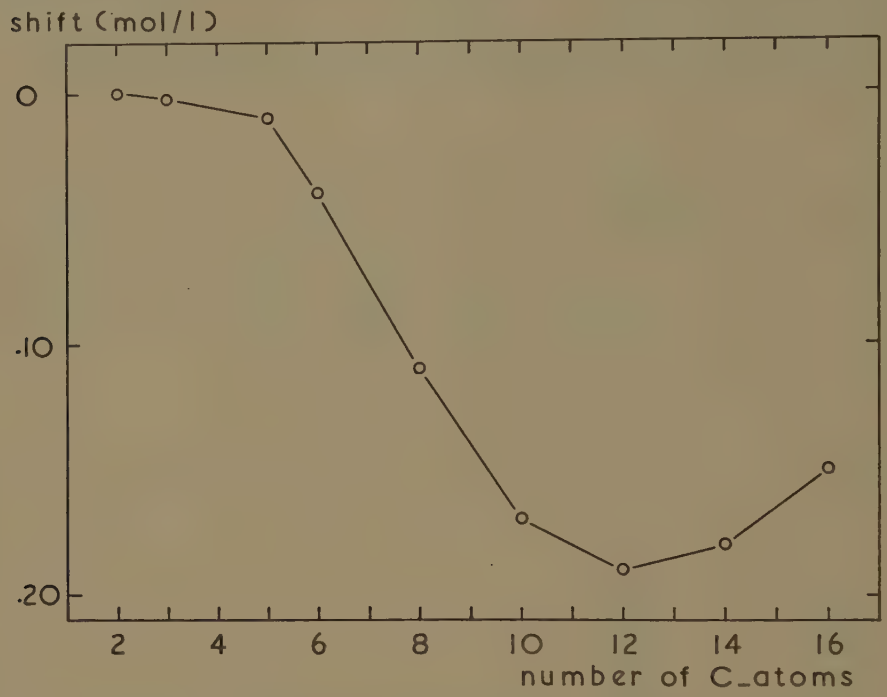

Fig. 4. Action of the series of normal primary alcohols on a coacervate of dodecylsulphate. The concentration of the alcohols was 0.5 mmol/l.

5. *Summary.*

1. In a coacervate of dodecylsulphate at relatively low p_H many fatty acids act as condensing substances. This must mean that at that p_H (higher than the p_K of the carboxyl group) the acids are present in the undissociated form within the micelles.

2. A maximal condensing activity on dodecylsulphate is observed with myristic acid, just as would be expected from our hypothesis.

3. If the substratum has a chain lenght of four carbon atoms more (cetylsulphate), the maximum of activity shifts to stearic acid in accordance with the theory.

4. The normal alcohols exert a stronger condensing action, while the maximal activity lies at 12 (13?) C atoms (with dodecylsulphate as substratum).

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SOME THEOREMS IN THE THEORY OF UNIFORM DISTRIBUTION

BY

L. KUIPERS AND B. MEULENBELD

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of Febr. 25, 1950)

§ 1. In a preceding paper [1] we deduced the following

Theorem. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $f'(t)$ is bounded with*

$$t f'(t) \to 0 \text{ for } t \to \infty,$$

then $f(t)$ is not C_1-uniformly distributed (mod 1). [1] [2]

In the present paper we prove the following generalisations:

Theorem I. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $t f'(t)$ is bounded with*

$$t f'(t) \to A \text{ for } t \to \infty,$$

where A is a fixed number, then $f(t)$ is not C_1 uniformly distributed (mod 1).

This Theorem is a special case of the following

Theorem II. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, if $t f'(t)$ is bounded, and if there exists a fixed $T^* \geqslant 0$ such that for $t > T^*$*

$$|t f'(t) - A| < B < (1/2\pi),$$

where A and B are fixed numbers, then $f(t)$ is not C^1 uniformly distributed (mod 1).

Proof of Theorem II.

We apply the C-test. We shall prove that

$$I = (1/T) \int_0^T e^{2\pi i h f(t)}\, dt$$

with $h = 1$ does not tend to zero for $T \to \infty$.

By integration by parts we have

$$I = e^{2\pi i f(T)} - (2\pi i/T) \int_0^T t f'(t)\, e^{2\pi i f(t)}\, dt.$$

---

[1] For the definitions of C^{I}, C^{II} and C^{III} uniform distribution (mod 1) we refer to [2].

[2] The condition, $f(t) \to \infty$ for $t \to \infty$, which occurs in Theorem 3 of [1], can be omitted.

Hence

$$(1+2\pi i A)\, I = e^{2\pi i f(T)} - (2\pi i/T) \int_0^T \{t f'(t) - A\}\, e^{2\pi i f(t)}\, dt$$

$$= e^{2\pi i f(T)} - (2\pi i/T) \int_0^{T^*} - (2\pi i/T) \int_{T^*}^T$$

$$= e^{2\pi i f(T)} - I_1 - I_2.$$

From this it follows:

$$\sqrt{1+4\pi^2 A^2}\, |I| \geqslant 1 - |I_1| - |I_2|. \tag{1}$$

For an arbitrarily chosen positive number ε there exists a $T^{**} \geqslant T^*$ such that for $T > T^{**}$

$$|I_1| < \varepsilon, \tag{2}$$

since $t f'(t)$ is bounded. For all $T > T^*$ we have furthermore

$$|I_2| < \frac{2\pi (T - T^*)}{T} B < 2\pi B. \tag{3}$$

From (1), (2) and (3) it follows that

$$\sqrt{1+4\pi^2 A^2}\, |I| > 1 - 2\pi B - \varepsilon \text{ for } T > T^{**}. \tag{4}$$

Since the positive number ε may be chosen arbitrarily small, it follows from (4) and $2\pi B < 1$ that $|I|$ does not tend to zero for $T \to \infty$. This completes the proof.

Examples. The following functions, of which the behaviour with regard to the C-uniform distribution (mod 1) could not yet ascertained by the Theorems in [1], satisfy the assumptions of Theorem II:

$$(a) \quad f(t) \equiv \log t + c \int_0^t (\sin u/u)\, du,$$

where c is a fixed number with $|c| < 1/2\pi$.

For $c = 0$ we meet again the function $\log t$.

$$(b) \quad f(t) = A \log t \sin (\log \log t),$$

where A is a constant with $|A| < 1/2\pi\sqrt{2}$.

Hence these functions are not C^1-uniformly distributed (mod 1).

§ 2. In [1] we proved the following

Theorem. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $f'(t) > 0$ and monotonically non-decreasing for $t \geqslant 0$, then $f(t)$ is C^{III}-uniformly distributed* (mod 1).

N. H. Kuiper (Delft, Netherlands) reported us, that, if $f'(t)$ tends to a constant $c \neq 0$ for $t \to \infty$, the condition of the monotony of $f'(t)$ is not necessary for $f(t)$ being C^1 uniformly distributed (mod 1). In order

to prove this statement he made use of a method similar to that developed in [3].

For the case that $f'(t) \to c \neq 0$ for $t \to \infty$ we shall prove the following

Theorem III. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if*

$$f'(t) \to c \text{ for } t \to \infty, \tag{5}$$

where c is a fixed number $\neq 0$, then $f(t)$ is C^{III}-uniformly distributed (mod 1).

Proof. Without loss of generality we assume that c is positive.

It follows from (5) that for T sufficiently large $f(t)$ possesses an inverse function $t = F(u)$. Then from (5) it follows that

$$F'(u) \to (1/c) > 0 \text{ for } u \to \infty.$$

Hence, for $u > U^* = U^*(\varepsilon) = f(T^*)$ we have

$$|F'(u) - (1/c)| < (\varepsilon/c),$$

where ε is an arbitrarily small positive number. Now, applying the C-test, we have for every fixed h, integer and $\neq 0$, and $T > T^*$

$$I = (1/T)\int_0^T e^{2\pi i h f(t)}\,dt = (1/T)\int_0^{T^*} + (1/T)\int_{T^*}^T =$$

$$(1/T)\int_0^{T^*} e^{2\pi i h f(t)}\,dt + (1/T)\int_{f(T^*)}^{f(T)} e^{2\pi i h u} F'(u)\,du =$$

$$(1/T)\int_0^{T^*} e^{2\pi i h f(t)}\,dt + (1/T)\int_{f(T^*)}^{f(T)} \{F'(u) - (1/c)\}\, e^{2\pi i h u}\,du +$$

$$(1/cT)\int_{f(T^*)}^{f(T)} e^{2\pi i h u}\,du = I_1 + I_2 + I_3.$$

It is obvious that $I_1 \to 0$ for $T \to \infty$. Furthermore we have

$$|I_2| < \frac{\varepsilon}{cT}\{f(T) - f(T^*)\} < \frac{\varepsilon f(T)}{cT}.$$

It follows from (5) that

$$\frac{f(T)}{T} \to c \text{ for } T \to \infty.$$

Hence

$$I_2 \to 0 \text{ for } T \to \infty.$$

Finally we have

$$|I_3| \leqslant \frac{1}{\pi |h| cT} \text{ and so } I_3 \to 0 \text{ for } T \to \infty.$$

Thus, for $T \to \infty$, the expression I tends to zero.

Hence: $f(t)$ is C^{I}-uniformly distributed (mod 1). Furthermore it follows from the *Theorem:*

If $f(t)$ $(t \geqslant 0)$ *is a differentiable function with* $f(t)/t$ *bounded, if* $f'(t) \geqslant \lambda > 0$ *with fixed* λ, *and if* $f(t)$ *is* C^{I}*-uniformly distributed* (mod 1), *then* $f(t)$ *is also* C^{III}*-uniformly distributed* (mod 1),

proved in [4], that the function of Theorem III is also C^{III}-uniformly distributed (mod 1).

Example. The function

$$f(t) \equiv t + \frac{\sin t}{t}$$

satisfies the assumptions of Theorem III with $c = 1$, so that $f(t)$ is C^{III}-uniformly distributed (mod 1).

University of Indonesia.

Bandung, January 1950.

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PRIME-REPRESENTING FUNCTIONS

BY

L. KUIPERS

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of February 25, 1950)

W. H. MILLS [1] proved the following

Theorem. There exists a real number A, such that $[A^{3^x}]$, where $[R]$ denotes the greatest integer less than or equal to R, is a prime number for every positive integer x.

In order to prove this statement he makes use of the inequality, deduced by A. E. INGHAM [2]

$$p_{n+1} - p_n < K\, p_n^{5/8},$$

where p_n denotes the n^{th} prime number and K is a positive integer.

We generalise the theorem mentioned above as follows:

Theorem.[1] For each integer $c \geqq 3$ a real number A depending only on c can be found such that $[A^{c^x}]$ is a prime number for all positive integers x.

The proof runs as follows. The integers $a = (c-1)\ w - 1$ and $b = cw - 1$ satisfy the relations

$$(1) \qquad (a,\ b) = 1;\ a < b - 1;\ a/b \geqq 5/8;\ ca + 1 \equiv 0 \pmod{b}.$$

Then

$$(2) \qquad p_{n+1} - p_n < K\, p^{a/b},$$

where K has the same value as before.

The lemma used by MILLS can be established here in the following form:

If a, b and c are positive integers satisfying (1) and (2), and if N is an integer with

$$(3) \qquad N > K^b,$$

then there exists a prime p such that

$$N^c < p < (N+1)^c - 1.$$

Proof. If p_n is the greatest prime less than N^c, then we have, using (1), (2) and (3)

$$p_n < N^c < p_{n+1} < p_n + K p_n^{a/b} < N^c + K N^{ca/b} < N^c + N^{(ca+1)/b} < (N+1)^c - 1.$$

[1] We are indepted to Prof. J. G. VAN DER CORPUT for a generalisation of the Theorem as it was originally stated by us.

If now P_0 is a prime greater than K^b, then we are able to construct a sequence of primes $P_0, P_1, \ldots$ such that

$$P_n^c < P_{n+1} < (P_n+1)^c - 1 \qquad (n=0,1,2,\ldots). \tag{4}$$

In the same way, as it has been done by MILLS for the case $a=5,\ b=8,\ c=3$, we can prove by (4), that the sequence $P_n^{c^{-n}}$ is a bounded increasing sequence, and that therefore $P_n^{c^{-n}}$ tends to a limit, say A.

The boundedness of $P_n^{c^{-n}}$ follows from the inequality $P_n^{c^{-n}} < (P_n+1)^{c^{-n}}$, where the sequence $(P_n+1)^{c^{-n}}$ is a decreasing sequence, which also follows from (4).

So we have

$$P_n^{c^{-n}} < A < (P_n+1)^{c^{-n}}$$

or

$$P_n < A^{c^n} < P_n + 1.$$

This means that $[A^{c^x}]$ $(x=1, 2, \ldots)$ is a prime-representing function.

For two different values of c, say c and γ, one of the corresponding sequences $P_n(c)$ and $p_n(\gamma)$ cannot be a subsequence of the other. For instance, if we assume, that the sequence $p_n(\gamma)$ is a subsequence of $P_n(c)$, then there would be a subsequence $Q_n(c)$ of $P_n(c)$, such that $Q_n^{\gamma^{-n}}$ and $Q_n^{c^{-n}}$ would tend to the same limit, and this is impossible on account of $c \neq \gamma$.

University of Indonesia

Bandung, February 7, 1950.

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ON THE UNIFORM DISTRIBUTION OF THE VALUES OF FUNCTIONS OF n VARIABLES

BY

B. MEULENBELD

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of Jan. 28, 1950)

§ 1. *Introduction.*

In a previous paper KUIPERS and the author of this paper [1] introduced the notion of a (mod. 1) continuously- (or C-)uniformly distributed function $f(t)$ of one variable t. This notion is an analogue of that of a (mod. 1) discretely- (or D-) uniformly distributed sequence of numbers $f(n)$ $(n = 1, 2, \ldots)$.

In the present paper I shall give an extension of the theory in [1], by considering instead of a function $f(t)$ of one variable now a function $f(t_1, \ldots, t_n)$ of the variables $t_1, \ldots, t_n$. For the D-case we recall the definition of D-uniform distribution (mod 1) of a system of m functions each depending on n variables given by WEIJL [2] and VAN DER CORPUT [3]. We mention this definition [4].

D-Definition. *Let m and n be given positive integers, and let F be a sequence of n-dimensional intervals*

$$Q: \quad a_\mu \leqq x_\mu < b_\mu \quad (a_\mu \text{ and } b_\mu \text{ integer}, \ \mu = 1, \ldots, n),$$

where the number $N = N(Q)$ of lattice-points $(x) = (x_1, \ldots, x_n)$ of Q tends to infinity if Q runs through the sequence F. In each lattice-point of Q a system of m real functions $f_\nu(x) = f_\nu(x_1, \ldots, x_n)$ $(\nu = 1, \ldots, m)$ is defined. This system of functions f_ν is said to be D-uniformly distributed (mod 1) in the intervals Q of F, if for each system of m numbers $\gamma_1, \ldots, \gamma_m$ with $0 \leqq \gamma_\nu \leqq 1$ the number $N' = N'(Q) = N'(Q, \gamma_1, \ldots, \gamma_m)$ of lattice-points (x) of Q with

$$0 \leqq f_\nu < \gamma_\nu \pmod 1 \qquad (\nu = 1, \ldots, m)$$

satisfies the relation:

$$\lim \frac{N'(Q)}{N(Q)} = \gamma_1 \gamma_2 \cdots \gamma_m,$$

if Q runs through the sequence F.

VAN DER CORPUT [3] proved the following theorem on uniform distribution.

D-test. *It is necessary and sufficient for the D-uniform distribution (mod 1) of the system of functions $f_\nu(x_1, \ldots, x_n)$ $(\nu = 1, \ldots, m)$ that,*

for any lattice-point $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$ *this system satisfies the relation:*

$$\lim \frac{1}{N(Q)} \sum_{(x) \text{ in } Q} e^{2\pi i \{h_1 f_1(x) + \ldots + h_m f_m(x)\}} = 0,$$

if Q *runs through the sequence* F.

In § 2 I shall give an analogous definition and an analogous theorem for the case of C-uniform distribution (mod 1). In §§ 3 and 4 I shall apply this C-test in the special case of one function of n variables. § 3 deals with a class of functions which are not C-uniformly distributed. The functions considered in § 4 are shown to be C-uniformly distributed. § 5 gives some examples.

§ 2. *C-definition and C-test.*

An n-dimensional extension of the definition of C-uniform distribution may be formulated as follows.

C-Definition. *Let* m *and* n *be given positive integers and let* F *be a sequence of* n*-dimensional intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q *tends to infinity, if* Q *runs through* F. *For all points* $(t) = (t_1, \ldots, t_n)$ *of all* Q *a system of* n *real measurable functions* $f_\nu(t_1, \ldots, t_n) \quad (\nu = 1, \ldots, m)$ *is defined.*

Let $\gamma_1, \ldots, \gamma_m$ *be* m *numbers with* $0 \leqq \gamma_\nu < 1 \quad (\nu = 1, \ldots, m)$. *We denote by*

$$\theta_\nu(0, \gamma_\nu, f_\nu(t_1, \ldots, t_n)) \qquad (\nu = 1, \ldots, m)$$

the following characteristic functions:

$$\theta_\nu = 1 \text{ for } 0 \leqq f_\nu(t_1, \ldots, t_n) < \gamma_\nu \pmod 1,$$

$$\theta_\nu = 0 \text{ elsewhere.}$$

The measure of the set of points $(t_1, \ldots, t_n)$ *in* Q *satisfying:*

$$0 \leqq f_\nu(t_1, \ldots t_n) < \gamma_\nu \pmod 1 \qquad (\nu = 1, \ldots, m)$$

is the Lebesgue-integral:

$$I(Q) = \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} \theta_1(0, \gamma_1, f_1) \ldots \theta(0, \gamma_m, f_m)\, dt_1 \ldots dt_n.$$

Now we define:

The system of functions $f_\nu(t_1, \ldots, t_n)$ *is said to be* C*-uniformly distributed* (*mod* 1) *in the intervals* Q *of* F, *if for any system of fixed* γ_ν *the system* f_ν *satisfies:*

$$\lim \frac{I(Q)}{(T_1 - S_1) \ldots (T_n - S_n)} = \gamma_1 \ldots \gamma_m,$$

if Q *runs through* F.

Analogous to the D-test we have now:

C-test. *It is necessary and sufficient for the C-uniform distribution (mod 1) of the system of functions $f_\nu(t_1, \ldots, t_n)$ $(\nu = 1, \ldots, m)$ in the intervals Q of F that, for any lattice-point $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$ the system f_ν satisfies the relation:*

$$\lim \frac{1}{(T_1 - S_1) \ldots (T_n - S_n)} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i \{h_1 f_1(t) + \ldots + h_m f_m(t)\}} dt_1 \ldots dt_n = 0,$$

if Q runs through F.

Remark. The proof of this test is quite analogous to that of the D-test given by VAN DER CORPUT [3]. For the case of one function with one variable, and a system of n functions of one variable KUIPERS gave the C-proof in his dissertation [5].

§ 3. *Functions which are not C-uniformly distributed.*

Theorem 1. *Let F be a sequence of intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity, if Q runs through F. Let $f(t_1, \ldots, t_n)$ be a function, defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess for these points (t) first partial derivatives with respect to each variable. Furthermore we suppose that $f(t)$ has the following properties:

(1⁰)*) $\left| t_n \frac{\partial f}{\partial t_n} \right| < M$ *for all $t = (t_1, \ldots, t_n)$ in all Q, where M is a fixed positive number.*

(2⁰) $$\lim_{t_n \to \infty} t_\nu \frac{\partial f}{\partial t_\nu} = 0 \qquad (\nu = 1, \ldots, n)$$

uniformly in $(t_1, \ldots, t_{n-1})$.

Then $f(t_1, \ldots, t_n)$ is not C-uniformly distributed (mod 1) in the intervals Q of F.

Proof. Putting

$$P_l = (T_1 - S_1) \ldots (T_l - S_l),$$

we have for each fixed integer $h \neq 0$:

$$I = \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)} dt_1 \ldots dt_n =$$

$$= \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \left\{ T_n e^{2\pi i h f(t_1, \ldots, t_{n-1}, T_n)} - S_n e^{2\pi i h f(t_1, \ldots, t_{n-1}, S_n)} + \right.$$

$$\left. - 2\pi i h \int_{S_n}^{T_n} t_n \frac{\partial f(t_1, \ldots, t_n)}{\partial t_n} e^{2\pi i h f(t_1, \ldots, t_n)} dt_n \right\} dt_1 \ldots dt_{n-1}.$$

*) *Note added while correcting proofsheets:*
This assumption (1⁰) can be omitted without considerable alternation in the proof.

From assumption (2^0) it follows that, given a positive number ε, there exists a T_n^* such that for $t_n \geqq T_n^* = T_n(\varepsilon)$ (independent of $t_1, \ldots, t_{n-1}$):

$$(1) \qquad \left| t_n \frac{\partial f(t_1,\ldots,t_n)}{\partial t_n} \right| < \frac{\varepsilon}{4\pi|h|}.$$

Hence, if $T_n^* > S_n$, and S_n is bounded:

$$(2) \quad \left\{ \begin{array}{l} \left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} t_n \frac{\partial f(t_1,\ldots,t_n)}{\partial t_n} e^{2\pi i h f(t_1,\ldots,t_n)} \, dt_1 \ldots dt_n \right| \\ \leqq \left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n^*} \right| + \left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{T_n^*}^{T_n} \right| < \frac{2\pi|h|(T_n^* - S_n) M}{T_n - S_n} + \\ + \frac{T_n - T_n^*}{T_n - S_n} \frac{\varepsilon}{2} \end{array} \right.$$

(by assumption (1^0) and (1)), and so it is obvious that (2) tends to zero, if $T_n - S_n \to \infty$.

If S_n is not bounded, then for S_n large enough the expression (2) is $< \varepsilon/2$, so that also in this case (2) tends to zero if $T_n - S_n \to \infty$.

Now we consider

$$\frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \{T_n e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} - S_n e^{2\pi i h f(t_1,\ldots,t_{n-1},S_n)}\} \, dt_1 \ldots dt_{n-1} =$$

$$= \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \Big\{ (T_n - S_n) e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} + S_n \{ e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} +$$

$$- e^{2\pi i h f(t_1,\ldots,t_{n-1},S_n)} \} \Big\} \, dt_1 \ldots dt_{n-1} =$$

$$= \frac{1}{P_{n-1}} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} \, dt_1 \ldots dt_{n-1} + I_n.$$

If S_n is bounded, then

$$|I_n| \leqq \frac{2S_n}{T_n - S_n}, \text{ and so } I_n \to 0 \text{ if } T_n - S_n \to \infty.$$

If S_n is not bounded, then, using the inequality:

$$|e^{2\pi i u} - e^{2\pi i v}| \leqq 2\pi |u - v|,$$

we have:

$$|I_n| \leqq \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} S_n 2\pi |h| \, |f(t_1,\ldots,t_{n-1},T_n) - f(t_1,\ldots,t_{n-1},S_n)| \, dt_1 \ldots dt_{n-1}$$

$$= \frac{2\pi|h|}{P_{n-1}} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} S_n \left| \frac{\partial f(t_1,\ldots,t_{n-1},\xi)}{\partial t_n} \right| dt_1 \ldots dt_{n-1}$$

(where $S_n < \xi < T_n$)

$$< \frac{2\pi|h|}{P_{n-1}} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \left| \frac{\partial f(t_1, \ldots, t_{n-1}, \xi)}{\partial t_n} \right| dt_1 \ldots dt_{n-1}.$$

From assumption (2^0) it follows that this expression tends to zero, if $T_n - S_n \to \infty$, hence $I_n \to 0$ also in this case.

Now we apply the same argument to the integrals:

$$\frac{1}{P_{n-1}} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} e^{2\pi i h f(t_1, \ldots, t_{n-1}, T_n)} dt_1 \ldots dt_{n-1},$$

$$\frac{1}{P_{n-2}} \int_{S_1}^{T_1} \cdots \int_{S_{n-2}}^{T_{n-2}} e^{2\pi i h f(t_1, \ldots, t_{n-2}, T_{n-1}, T_n)} dt_1 \ldots dt_{n-2},$$

and so on, so that I can be reduced to the expression

$$e^{2\pi i h f(T_1, \ldots, T_n)},$$

apart from a finite sum of terms, of which each tends to zero, if Q runs through F. Hence, by the C-test the function $f(t_1, \ldots, t_n)$ is not uniformly distributed (mod 1).

§ 4. *C-distributed functions.*

Theorem 2. *Let F be a sequence of intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity if Q runs through F. Let $f(t) = f(t_1, \ldots, t_n)$ be a function defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess a partial derivative with respect to t_n with the properties:

$\frac{\partial f}{\partial t_n}$ is monotonically non-decreasing in t_n for each fixed $(t_1, \ldots, t_{n-1})$;

$\frac{\partial f}{\partial t_n} \geqq \lambda > 0$ for λ fixed, independent of $t_1, \ldots, t_{n-1}$. Then $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod 1) in the intervals Q of F.

Proof. Putting $P = (T_1 - S_1) \ldots (T_n - S_n)$ we have for every integer $h \neq 0$:

$$I = \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_n}^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)} dt_1 \ldots dt_n$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \left\{ \int_{f(t_1, \ldots, t_{n-1}, S_n)}^{f(t_1, \ldots, t_{n-1}, T_n)} \frac{e^{2\pi i h u} \, du}{\frac{\partial f(t_1, \ldots, t_n)}{\partial t_n}} \right\} dt_1 \ldots dt_{n-1}$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \left\{ \frac{1}{\frac{\partial f(t_1, \ldots, t_{n-1}, S_n)}{\partial t_n}} \int_{f(t_1, \ldots, t_{n-1}, S_n)}^{f(t_1, \ldots, t_{n-1}, \xi)} e^{2\pi i h u} \, du \right\} dt_1 \ldots dt_{n-1}$$

where $S_n < \xi < T_n$ (by the mean-value-theorem). Hence by our assumptions:

$$|I| \leqq \frac{1}{P} \int_{S_1}^{T_1} \int_{S_{n-1}}^{T_{n-1}} \frac{dt_1 \ldots dt_{n-1}}{\pi |h| \lambda} = \frac{1}{(T_n - S_n)\pi |h| \lambda},$$

and this expression tends to zero if $T_n - S_n \to \infty$. So $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod. 1).

Theorem 3. *Let F be a sequence of intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity if Q runs through F.

Let $f(t) = f(t_1, \ldots, t_n)$ be a function, defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess a partial derivative with respect to t_n with the properties:

$\frac{\partial f}{\partial t_n} < 0$, *monotonically non-increasing, and continuous in t_n, uniformly in $t_1, \ldots, t_{n-1}$;*

$(T_n - S_n) \frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n} \to \infty$, *uniformly in $t_1, \ldots . t_{n-1}$, if $T_n - S_n \to \infty$.*

Then $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod 1) in the intervals Q of F.

Proof. Putting $P = (T_1 - S_1) \ldots (T_n - S_n)$ we have for every integer $h \neq 0$:

$$I = \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)}\, dt_1 \ldots dt_n$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \left\{ \int_{f(t_1, \ldots, t_{n-1}, S_n)}^{f(t_1, \ldots, t_{n-1}, T_n)} \frac{e^{2\pi i h u}\, du}{\frac{\partial f(t_1, \ldots, t_n)}{\partial t_n}} \right\} dt_1 \ldots dt_{n-1}$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \left\{ \frac{1}{\frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n}} \int_{f(t_1, \ldots, t_{n-1}, \xi_1)}^{f(t_1, \ldots, t_{n-1}, T_n)} e^{2\pi i h u}\, du \right\} dt_1 \ldots dt_{n-1}$$

(where $S_n < \xi_1 < T_n$).

Hence

$$|I| \leqq \frac{1}{(T_1 - S_1) \ldots (T_{n-1} - S_{n-1}) \pi |h|} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \frac{dt_1 \ldots dt_{n-1}}{(T_n - S_n) \frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n}}.$$

From our assumptions it follows that the last expression tends to zero if $T_n - S_n \to \infty$, so that also $I \to 0$. So $f(t_1, \ldots, t_n)$ is uniformly distributed (mod 1).

§ 5. *Examples.*

In the present paragraph we consider the sequence F of intervals

$$Q: \quad 2 < t_\mu < T \qquad (\mu = 1, \ldots, n)$$

with $T \to \infty$.

It is easily seen that for the following functions $f(t) = f(t_1, \ldots, t_n)$ the assumptions of Theorem 1 are satisfied, so that these functions are *not* C-uniformly distributed (mod 1) in the intervals Q of F:

$$f(t) = \lg \lg (t_1 \ldots t_n);$$
$$f(t) = \lg \lg (t_1 + \ldots + t_n);$$
$$f(t) = \sqrt[p]{\lg (t_1 \ldots t_n)} \qquad (p > 1);$$
$$f(t) = \sqrt[p]{\lg (t_1 + \ldots + t_n)} \qquad (p > 1).$$

From Theorem 2 it follows that the functions

$$f(t) = t_1 \ldots t_n,$$
$$f(t) = t_1 + \ldots + t_n,$$
$$f(t) = e^{t_1 \ldots t_n},$$
$$f(t) = e^{t_1 + \ldots + t_n}$$

are uniformly distributed (mod 1) in the intervals Q of F, and from Theorem 3 it follows that the functions

$$f(t) = \sqrt[p]{t_1 \ldots t_n} \qquad (p > 1),$$
$$f(t) = \sqrt[p]{t_1 + \ldots + t_n} \qquad (p > 1),$$
$$f(t) = \{\lg (t_1 \ldots t_n)\}^p \qquad (p > 1),$$
$$f(t) = \{\lg (t_1 + \ldots + t_n)\}^p \qquad (p > 1)$$

are C-uniformly distributed (mod 1) in the intervals Q of F.

Bandung, Jan. 1950 *University of Indonesia*

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PROJECTIVE GEOMETRIZATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS, I: NORMAL POINTS

BY

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Synopsis. In a sequence of four papers we shall deal with a projective geometrization of the system (2,1*a*) of partial differential equations. This geometrization leads to a projective embedding theory of a X_m in a projective linear space P_n, $(1 \leqslant m < n, n \geqslant 2)$. [1]) The results are based partly on the affine embedding theory as developed in our previous papers "Affine embedding theory I, II, III" (Kon. Ned. Akad. van Wet., Proceedings Vol. LII, no. 5, 7 and 9) which will be referred to as A I, A II and A III. Throughout these papers we use the *homogeneous* projective parameters for our X_m. [2])

In this first paper some fundamental transformation laws are found, which enable us to make a construction of a connection [which stamps our X_m to a projective curved space $\mathfrak{P}_m$] as well as of higher connections which lead to a well-defined set of "projective normal points" of the $\mathfrak{P}_m$. The device used here may easily be generalized for a X_m in $\mathfrak{P}_n$.

1. *Introduction.*

a. Let $\mathbf{P}$ be a generic point of a projective linear space P_n and let

$$\mathbf{P} = \mathbf{P}(\eta^1, \ldots, \eta^m) \tag{1, 1a}$$

be the parametric equations of a X_m in P_n, $(1 \leqslant m < n)$. Let $\varphi^1(y^0, \ldots y^m), \ldots, \varphi^m(y^0, \ldots, y^m)$ be m functions of projective homogeneous coordinates in X_m, homogeneous of degree zero in these arguments. Put

$$\eta^1 = \varphi^1, \ldots, \eta^m = \varphi^m$$

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[1]) The embedding theory of a curve ($m = 1$) in a $\mathfrak{P}_n$ was dealt with in my previous paper "Système complet des invariants d'une courbe dans un espace projectif courbe" (Institut des Math. à l'Université d'état de Moscou, Séminaire pour le calcul vectoriel et tensoriel, "Mémoires", Livraison II-III, 1935). The case of a hypersurface ($m = n - 1$) in a $\mathfrak{P}_n$ and its embedding theory was dealt with in my paper "Hypersurfaces in a projective curved space" (Annals of Math., **39**, 725—761 (1938)). In both cases non-homogeneous parameters were used.

[2]) Homogeneous projective coordinates in curved spaces were introduced by D. VAN DANTZIG: "Theorie des projektiven Zusammenhangs n-dimensionaler Raume" (Math. Ann. **106** 400—454) and used by J. A. SCHOUTEN, and J. HAANTJES: "Zur allgemeinen projektiven Differentialgeometrie" (Compositio Math. **3** 1—51) where also the case $m = n - 1$ is dealt with.

in (1,1*a*)

$$\text{(1, 1}b\text{)} \qquad \mathbf{P} = \mathbf{P}(\varphi^1, \ldots, \varphi^m).$$

Then $\mathbf{P}$ is homogeneous of degree zero in the arguments $y^0, \ldots, y^m$. Let $f(y) \neq 0$ be a homogeneous function of the y's of degree $N+1$ ($N>1$ being an integer whose geometrical significance will be explained later on). Then obviously the subspace (1,1) is defined also by the equations

$$\text{(1, 1}c\text{)} \qquad \mathbf{x} = f\mathbf{P}.$$

Hence $\mathbf{x}$ is homogeneous of degree $N+1$ in the y's. Consequently the following formulae must hold [3])

$$\text{(1, 2)} \quad \begin{cases} a) \quad \mathbf{x}_{a_1 \ldots a_{s-1} \ldots a_r} y^{a_s} \ldots y^{a_r} = \dfrac{(N-s+2)!}{(N-r+1)!} \mathbf{x}_{a_1 \ldots a_{s-1}}, \\ \qquad\qquad\qquad\qquad (r = 2, 3, \ldots, N+1; s = 2, 3, \ldots r) \\ b) \quad \mathbf{x}_{a_1 \ldots a_r} y^{a_1} \ldots y^{a_r} = \dfrac{(N+1)!}{(N+1-r)!} \mathbf{x}, \qquad (r = 1, 2, \ldots, N+1) \end{cases}$$

where $0! \equiv 1$ and

$$\mathbf{x}_{a_1 \ldots a_r} \equiv \frac{\partial^r \mathbf{x}}{\partial y^{a_1} \ldots \partial y^{a_r}}.$$

Throughout this paper we shall assume that the matrix $\left(\left(\dfrac{\partial \mathbf{x}}{\partial y^a}\right)\right)$ is of rank $m+1$. Hence the points $\mathbf{x}_a$ are linearly independent.

b. Let $g = g(y) \neq 0$ be an arbitrary function homogeneous of degree zero. Then obviously the subspace (1,1) is defined also by

$$\text{(1, 3}a\text{)} \qquad \overset{*}{\mathbf{x}} = \mathbf{x}(y)\, g^{N+1}(y)$$

where again $\overset{*}{\mathbf{x}}$ is homogeneous of degree $N+1$. On the other hand y^a and

$$\text{(1, 3}b\text{)} \qquad \overset{*}{y}{}^a = y^a\, g(y)$$

define the same point of our X_m. Because of

$$\mathbf{x}(\overset{*}{y}) = g^{N+1}(y)\, \mathbf{x}(y)$$

we have

$$\text{(1, 3}c\text{)} \qquad \overset{*}{\mathbf{x}} = \mathbf{x}(\overset{*}{y}) = \mathbf{x}(y)\, g^{N+1}(y).$$

Definition (1,1). *Any geometrical object or property invariant with respect to the change* (1,3) *of the factor of proportionality will be termed g-invariant.*

Remark. If we introduce the abbreviations

$$\overset{*}{\mathbf{x}}_{a_1 \ldots a_r} \equiv \frac{\partial^r \mathbf{x}(\overset{*}{y})}{\partial \overset{*}{y}{}^{a_1} \ldots \partial \overset{*}{y}{}^{a_r}}, \qquad r = 1, 2, \ldots$$

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[3]) It is understood that Latin indices a, b, c, d, e run from 0 to m.

then we have obviously

$$(1,4)\qquad \overset{*}{\mathbf{x}}_{a_1\ldots a_r} = \mathbf{x}_{a_1\ldots a_r}\, g^{N+1-r}, \qquad r = 1, 2, \ldots$$

Hence the space spanned by the points $\mathbf{x}_{a_1\ldots a_r}$ $(r = 1, 2, \ldots)$ is g-invariant.

c. The parameters y^a are allowed to be transformed into $y^{a'}$ by means of

$$(1,5)\qquad y^{a'} = f^{a'}(y^o,\ldots,y^m) \text{ and vice versa } y^a = f^a(y^{o'},\ldots,y^{m'})$$

where f are homogeneous functions of degree one in the arguments. Hence if we introduce the abbreviations

$$(1,6a)\qquad P_{a'_1\ldots a'_r}^{\ \ a} \equiv \frac{\partial^r f^a}{\partial y^{a'_1}\ldots\partial y^{a'_r}}, \quad P_{a_1\ldots a_r}^{\ \ a'} \equiv \frac{\partial^r f^{a'}}{\partial y^{a_1}\ldots\partial y^{a_r}}$$
$$r = 1, 2, \ldots$$

we have

$$(1,6b)\qquad \begin{cases} P_{a'}^{a}\, y^{a'} = y^a & \\ P_{a'_1\ldots a'_r}^{\ \ a}\, y^{a'_2}\ldots y^{a'_r} = 0 & r = 2, 3,\ldots \\ P_{a'_1\ldots a'_r}^{\ \ a}\, y^{a'_s}\ldots y^{a'_r} = (-1)^{r-s+1}\dfrac{(r-2)!}{(s-3)!}\, P_{a'_1\ldots a'_{s-1}}^{\ \ a_1}, & \begin{pmatrix} r = 3, 4,\ldots \\ s = 3, 4, 5,\ldots r \end{pmatrix} \end{cases}$$

(with $0! \equiv 1$) and similar equations for $P_{a_1\ldots}^{\ \ a'}$.

Definition (1,2). *Any property* (*any geometrical object*) *which does not depend on the choice of the coordinate of* P_n (*of the parameters of* X_m) *will be termed x-invariant* (*y-invariant*). *Any property* (*any geometrical object*) *which is x-invariant as well as y-invariant will be referred to as* (*x, y*)*-invariant. Any* (*x, y*)*-invariant which is also g-invariant will be called* (*x, y, g*)*-invariant.*

The transformation law with respect to (1,5) of $\mathbf{x}_{a_r\ldots a_1}$ is the following one

$$(1,7a)\qquad \mathbf{x}_{a'} = P_{a'}^{a}\, x_a$$

$$(1,7b)\qquad \begin{cases} \mathbf{x}_{a'_r\ldots a'_1} = P_{a'_r\ldots a'_1}^{\ \ a_r\ldots a_1}\, \mathbf{x}_{a_r\ldots a_1} & \\ \quad + \sum_{s=2}^{r-1} \left[\left\{ P_{a'_r\ldots a'_s}^{\ \ a_s}\, P_{a'_{s-1}\ldots a'_1}^{\ \ a_{s-1}\ldots a_1} \right\} + p_{a'_r\ldots a'_s\ldots a'_1}^{\ \ a_s\ldots a_1} \right] \mathbf{x}_{a_s\ldots a_1} & \\ \quad + P_{a'_r\ldots a'_1}^{\ \ a}\, \mathbf{x}_a, \qquad r = 2, 3,\ldots & \end{cases}$$

where $P_{a'_r\ldots a'_1}^{\ \ a_r\ldots a_1} \equiv P_{a'_r}^{a_r}\ldots P_{a_1}^{a'_1}$ and the symbols

$$\left\{ P_{a'_r\ldots a'_s}^{\ \ a'_s}\, P_{a'_{s-1}\ldots a_1}^{\ \ a_{s-1}\ldots a'_1} \right\} \text{ and } p_{a'_r\ldots a'_s\ldots a'_1}^{\ \ a_s\ldots a_1}$$

are explained (and computed) in A I p. 167 and 168. Because of $\mathbf{x}_{a_s\ldots a_1} = \mathbf{x}_{(a_s\ldots a_1)}$ the symbols $\left\{ P_{a'_r\ldots a'_s}^{\ \ a_s}\, \delta_{a'_{s-1}\ldots a'_1}^{\ a_{s-1}\ldots a_1} \right\}$, obviously symmetric in

the subscripts may be thought of here as symmetric in the superscripts. The same holds for the symbols $p_{a'_r \ldots a'_s \ldots a'_1}^{a_s \ldots a_1}$.

d. Because n is a finite integer a least integer N may be found for which any of the points $\mathbf{x}_{a_M \ldots a_1}$ for $M > N$ is a linear combination of the points $\mathbf{x}$ and $\mathbf{x}_{a_r \ldots a_1}$ $(r = 1, \ldots, N)$. *Throughout this paper we assume* $N > 1$. N is obviously (x, y, g)-invariant.

The linear projective space $\dot{P}_{m_z}$ spanned at $\mathbf{x}$ by the points

$$\mathbf{x}, \ \mathbf{x}_{a_1}, \ldots, \mathbf{x}_{a_z \ldots a_1}, \qquad z = 1, \ldots, N$$

will be referred to as the *z-th osculating space* of X_m at $\mathbf{x}$. According to (1,4) and (1,7) $\dot{P}_{m_z}$, which is obviously x-invariant is also y-invariant as well as g-invariant. Hence it is (x, y, g)-invariant. The matrix $\left(\left(\frac{\partial \mathbf{x}}{\partial y}\right)\right)$ being of rank $m - 1$ we have

$$m_1 = m.$$

On the other hand the space spanned by the points $\mathbf{x}_{a_1 \ldots a_r}$ $(r = 1, \ldots N)$ contains the points $\mathbf{x}$, $\mathbf{x}_{a_s \ldots a_1}$, $s = 1, 2, \ldots r - 1$ (cf. the eq. (1,2)). Hence because $\mathbf{x}_{a_1 \ldots a_r}$ are symmetric in all indices we have

$$1 + m_r \leqq \binom{m+r}{r} \equiv \frac{(m+r)!}{m!\, r!}.$$

If in particular we have

$$m_r = \binom{m+r}{r} - 1, \qquad (r = 2, \ldots N)$$

then our X_m will be called *a maximal* X_m, the remaining case being termed *a restricted* X_m. Hence in the maximal case the points $\mathbf{x}_{(a_r \ldots a_1)}$ are linearly independent.

2. *The functions* Φ.

Our starting point will be the analytic expression of the statement (contained in *d*) of the previous section) on the integer N, namely

$$\mathbf{x}_{a_{N+1} \ldots a_1} = \Psi_{a_{N+1} \ldots a_1} \mathbf{x} + \sum_1^N {}_r \Psi_{a_{N+1} \ldots a_r \ldots a_1}^{b_r \ldots b} \mathbf{x}_{b_r \ldots b_1}$$

Because of (1,2*b*) this equation may be reduced to

$$(2, 1a) \qquad \mathbf{F}_{a_{N+1} \ldots a_1} \equiv \mathbf{x}_{a_{N+1} \ldots a_1} - \sum_1^N {}_r \Phi_{a_{N+1} \ldots a_r \ldots a_1}^{b_r \ldots b_1} \mathbf{x}_{b_r \ldots b_1} = 0$$

where the Φ's are symmetric in all their superscripts as well as in all their subscripts.

Theorem (2,1). *The equations* (2,1*a*) *may be written*

$$\text{(2, 1}b\text{)} \qquad \mathbf{F}_{a_{N+1}\dots a_1} \equiv \mathbf{x}_{a_{N+1}\dots h_1} - \Lambda_{a_{N+1}}{}^{b_N\dots b_1}{}_{a_N\dots a_1}\,\mathbf{x}_{b_N\dots h_1} = 0.$$

If dealing with the maximal case, then this equation admits only one solution $\Lambda \neq 0$. *In any case*

$$\text{(2, 2}a\text{)} \quad \Lambda_{a_{N+1}}{}^{b_N\dots b_1}{}_{a_N\dots a_1} = \Phi_{a_{N+1}}{}^{b_N\dots b_1}{}_{a_N\dots a_1} + \sum_1^{N-1} \frac{1}{(N-r+1)!}\, y^{(b_N}\dots y^{b_{r+1}}\, \Phi_{a_{N+1}\dots a_r}{}^{b_r\dots b_1)}{}_{\dots a_1}.$$

Proof. The equations (2,1*b*) and (2,2*a*) follow at once from (2,1*a*) by virtue of (1,2). Since $\mathbf{x}_{a_N\dots a_1} \neq 0$ is homogeneous of degree 1, it is not constant and consequently $\mathbf{x}_{a_{N+1}\dots a_1} \neq 0$. Hence the equations (2,1*b*) are not homogeneous in the Λ's, and admit in the maximal case (where the points $\mathbf{x}_{(a_N\dots a_1)}$ are linearly independent) only one solution $\Lambda \neq 0$.

Note I. We have from (2,1*b*)

$$\mathbf{x}_{[a_{N+2}a_{N+1}]\dots a_1} = \Omega_{a_{N+2}\dots a_N\dots a_1}{}^{b_N\dots b_1}\,\mathbf{x}_{b_N\dots b_1} = 0$$

where

$$\Omega_{a_{N+2}\dots a_N\dots a_1}{}^{b_N\dots b_1} \equiv \partial_{[a_{N+2}}\Lambda_{a_{N+1}]}{}^{b_N\dots b_1}{}_{a_N\dots a_1} + \delta^{e}_{[a_{N+2}}\delta^{f}_{a_{N+1}]}\Lambda_{fa_N\dots a_1}{}^{c_N\dots c_1}\Lambda_{ec_N\dots c_1}{}^{b_N\dots b_1}$$

and consequently in the maximal case

$$\text{(2, 2}b\text{)} \qquad \Omega_{a_{N+2}\dots a_N\dots a_1}{}^{b_N\dots b_1} = 0.$$

Note II. In the maximal case the Φ's have to satisfy (2,2*a*) (where the left hand member is the solution of (2,1*b*) and satisfies (2,2*b*)). In the restricted case even the Λ's are not uniquely determined. In this case we consider (2,1*a*) as an à priori given set of partial differential equations *with uniquely given coefficients* Φ. These coefficients have to satisfy (2,2*b*) where Λ is defined by (2,2*a*).

Theorem (2,2). $\Phi_{a_{N+1}\dots a_r\dots a_1}{}^{b_r\dots b_1}$ *is homogeneous of degree* $-(N+1-r)$

$$\text{(2, 3)} \qquad \dot{\Phi}_{a_{N+1}\dots a_r\dots a_1}{}^{b_r\dots b_1} = g^{-(N+1-r)}\,\Phi_{a_{N+r}\dots a_r\dots a_1}{}^{b_r\dots b_1}.$$

Proof. The equation (2,1*b*) has to be g-invariant. Because $\mathbf{x}_{a_{N+1}\dots a}$ is homogeneous of degree zero, we must have

$$\dot{\mathbf{F}}_{a_{N+1}\dots a_1} = \mathbf{F}_{a_{N+1}\dots a_1}.$$

This equation together with (1,3) and (1,4) leads to (2,3).

Theorem (2,3). *The transformation law*

$$
(2,4)\quad
\begin{cases}
(a)\quad \Phi_{a_{N+1}a_N\dots a_1}^{\quad b_N\dots b_1} = P^{a'_{N+1}\dots a'_1}_{a_{N+1}\dots a_1}\left[P^{b_N\dots b_1}_{b'_N\dots b'_1}\,\Phi_{a'_{N+1}a'_N\dots a'_1}^{\quad b'_N\dots b'_1} - \left\{P_{a'_{N+1}a'_N}^{\quad b_N}\,P^{b_{N-1}\dots b_1}_{a'_{N-1}\dots a'_1}\right\}\right],\\
(b)\quad \Phi_{a_{N+1}\dots a_r\dots a_1}^{\quad b_r\dots b_1} = P^{a'_{N+1}\dots a'_1}_{a_{N+1}\dots a_1}\Big[P^{b_r\dots b_1}_{b'_r\dots b'_1}\,\Phi_{a'_{N+1}\dots a'_r\dots a'_1}^{\quad b'_r\dots b'_1} +\\
\quad + \sum_{1}^{N}{}_q\,\Phi_{a'_{N+1}\dots a'_q\dots a'_1}^{\quad b'_q\dots b'_1}\left(\left\{P_{b'_q\dots b'_r}^{\quad b_r}\,P^{b_{r-1}\dots b_1}_{b'_{r-1}\dots b'_1}\right\} + p_{b'_q\dots b'_r}^{\quad b_r\dots b_1}\right)\\
\quad - \left\{P_{a'_{N-r}\dots a'_r}^{\quad b_r}\,P^{b_{r-1}\dots b_1}_{a'_{r-1}\dots a'_1}\right\} - p_{a'_{N+1}\dots a'_r\dots a'_1}^{\quad b_r\dots b_1}\Big], \qquad r=2,\dots,N-1\\
(c)\quad \Phi_{a_{N-1}\dots a_1}^{\quad b_1} = P^{a'_{N+1}\dots a'_1}_{a_{N-1}\dots a_1}\left[P^{b_1}_{b'_1}\,\Phi_{a'_{N+1}\dots a'_1}^{\quad b'_1} + \sum_{2}^{N}{}_q\,\Phi_{a'\dots a'_q\dots a'_1}^{\quad b'_q\dots b'_1}\,P_{b'_q\dots b'_1}^{\quad b_1} - P_{a'_{N+1}\dots a'_1}^{\quad b_1}\right].
\end{cases}
$$

makes (2,1*a*) *y-invariant. According to* (2,4) *the* $\Phi_{a_{N+1}\dots a_r\dots a_1}^{\quad b_r\dots b_1}$ *is of the y-order* [4]) $N+2-r$, $r=1,\dots,N$.

Proof. The equation (2,1*a*) has to be y-invariant. Hence we must have

$$\mathbf{F}_{a'_{N-1}\dots a'_1} = \varrho\,P^{a_{N+1}\dots a_1}_{a'_{N+1}\dots a'_1}\,\mathbf{F}_{a_{N+1}\dots a_1}. \tag{2,5a}$$

Comparing (2,5*a*) with (1,7*b*) for $r=N+1$ we get

$$\varrho = 1. \tag{2,5b}$$

Substituting from (1,7) in (2,5) we obtain

$$\sum_{1}^{N}{}_r\,\mathbf{x}_{b_r\dots b_1}\,X_{a_{N+1}\dots a_r\dots a_1}^{\quad b_r\dots b_1} = 0 \tag{2,6a}$$

which may be satisfied by

$$X_{a_{N+1}\dots a_r\dots a_1}^{\quad b_r\dots b_1} = 0. \tag{2,6b}$$

The equation (2,6*b*) is equivalent to (2,4).

Note. It is easily seen that all the Φ's are of the same x-order $N+1$ (0) [5])

3. *The connections* $\Gamma_{b_r\dots b_1}^{\quad a}$, $(r=2,3,\dots N)$.

Let us introduce the functions

$$\Lambda^a_{bc} = \Lambda^a_{cb} \equiv \Phi_{ca_N\dots a_2b}^{\quad a_N\dots a_2a},\ \Lambda_b \equiv \Lambda^a_{ab}$$

---

[4]) An object is said to be of the y-order q if its construction in any parameter system $y^{a'}$ requires the existence of the derivatives of the $y^{a'}$ with respect to the y^a up to the q-th order only.

[5]) An object is said to be of the x-order q, if its construction requires the derivatives of $\mathbf{x}$ with respect to the y^a up to the q-th order only. The first number (here $N+1$) refers to the maximal case, the number in parenthesis (here 0) refers to the restricted case.

which appear in the following theorem:

Theorem (3,1). *The functions*

$$(3,1) \qquad \Gamma^{a}_{bc} = \Gamma^{a}_{cb} \equiv C_1\left[\Lambda^{a}_{cb} - C_2\left(\delta^{a}_{b}\Lambda_c + \delta^{a}_{c}\Lambda_b\right)\right]$$

(*where* C_1 C_2 *are conveniently chosen numbers different from zero*) *are coefficients of a projective connection in* X_m, *which stamps our* X_m *to a* $\mathfrak{P}_m$:

$$(3,2a) \qquad \Gamma^{a}_{cb}\, P^{c\,b}_{c'b'} = P^{a}_{a'}\, \Gamma^{a'}_{c'b'} - P^{a}_{c'b'}.$$

This connection is of the $\left.\begin{matrix} x- \\ y- \end{matrix}\right\}$ *order* $\left\{\begin{matrix} N+1\,(0) \\ 2 \end{matrix}\right.$.

The proof of this theorem is the same as the proof of Theorem (3,1) in A I.

Theorem (3,2). *The equations* (2,4*b*) for $r = 2, 3, 4, \ldots, N-1$ *admit only one solution.*

$$(3,2b) \qquad P^{\ a}_{b'_q\ldots b'_1} = P^{a}_{a'}\,\Omega^{\ a'}_{b'_q\ldots b'_1} + P^{a}_{a'}\,\Xi^{\ a'}_{b'_q\ldots b'_1} - P^{b_q\ldots b_1}_{b'_q\ldots b'_1}\,\Gamma^{\ a}_{b_q\ldots b_1}, \quad q = 3,4,\ldots N$$

where Ω, Ξ *and* Γ *are symmetric in their lower indices and*

a) $\Gamma^{\ a}_{b_q\ldots b_1}\left(\Omega^{\ a'}_{b'_q\ldots b'_1}\right)$ *is a function of the* $\Phi^{b_s\ldots b_1}_{a_{N+1}\ldots a_s\ldots a_1}\left(\Phi^{b'_s\ldots b'_1}_{a'_{N+1}\ldots a'_s\ldots a'_1}\right)$ *only,* $(s = N+2-q,\ldots,N)$ *and does not contain* $P^{a}_{a'}$, $P^{b'}_{b}$.

b) $\Xi^{\ a'}_{b'_q\ldots b'_1}$ *represents a sum of products of at least one* $\Gamma^{\ a}_{b_u\ldots b_1}$. $(u = 2,\ldots,q-1)$ *and some* $\Phi^{b'_p\ldots b'_1}_{a'_{N+1}\ldots a'_p\ldots a'_1}$, $(p = N+3-q,\ldots,N)$ *and eventually of some* $P^{a'}_{a}$, $P^{b}_{b'}$.

c) The functions $\Gamma^{\ a}_{b_q\ldots b_1}$ *are of the* $\left.\begin{matrix} x- \\ y- \end{matrix}\right\}$ *order* $\left\{\begin{matrix} N+1\,(0) \\ q \end{matrix}\right.$.

d) No derivatives of $\Gamma^{\ a}_{b_u\ldots b_1}$, $\Omega^{\ a}_{b'_u\ldots b'_1}$, $\Xi^{\ a'}_{b'_u\ldots b'_1}$, $(u = 2, 3,\ldots, q-1)$ *occur in* (3, 2*b*).

The proof of this theorem is the same as the proof of Theorems (4,1) and (4, 2) in A I.

Note. For the same reasons as exposed in the Note of A I p. 173, the $\Gamma^{\ a}_{b_q\ldots b_1}$ involved in (3,2*b*) may be thought of as coefficients of a "*higher degree connection*": Eliminating the $P^{\ a}_{b'_r\ldots b'_1}$ $(r = 2, 3, \ldots N)$ from (3,2) and from the transformation law for the $(r-1)$-th derivative of a projective tensor (T) one obtains a new projective tensor which may be thought of as a "generalized" $(r-1)$-th covariant derivative of (T). As far as the Γ are concerned this new tensor is of the x-order $N+1$ (0),

while the ordinary $(r-1)$-th covariant derivative of (T) contains Γ^a_{bc} and its first $r-2$ derivatives (with respect to the y's) and consequently is of the x-order $N+r-1$ $(r-2)$ as far as Γ^a_{bc} are concerned.

Theorem (3,3) $\Gamma_{b_r\ldots b_1}^{\ \ \ b}$ *is homogeneous of degree* $1-r$

$$\Gamma_{b_r\ldots b_1}^{\ \ \ a}(\dot{y}) \equiv \dot{\Gamma}_{b_r\ldots b_1}^{\ \ \ a} = g^{1-r}\,\Gamma_{b_r\ldots b_1}^{\ \ \ a}(y), \quad r=2,\ldots,N. \tag{3,3}$$

Proof. The equations (3,2) are obviously g-invariant, while $P_{b'_s\ldots b'_1}^{\ \ \ a}$ is homogeneous of degree $(s-1)$, $(s=1,2,\ldots)$. Hence (3,3) follows at once from (3,2) [but may be derived also from (2,3)].

In the last theorem of this section we introduce a projective tensor built up from the Γ^a_{bc}:

Theorem (3,4). *The components*

$$Q^c_b = y^a\,\Gamma_{a\ b}^{\ \ c} \tag{3,4}$$

are components of a projective tensor of the $\left.\begin{matrix}x-\\y-\end{matrix}\right\}$ *order* $\left\{\begin{matrix}N+1\\2\end{matrix}\right.$ $^{(0)}$ *homogeneous of degree zero*

$$Q^c_b(\dot{y}) \equiv \dot{Q}^c_b = Q^c_b(y). \tag{3,5}$$

The very simple proof is based on (3,2a) as well as on (1,6b).

4. *Projective normal points.*

In the next theorem we shall make use of the symbols $\gamma_{a_r\ldots a_s\ldots a_1}^{\ \ \ \ \ \ b_s\ldots b_1}$ which represent a sum of products of at least two of the $\Gamma_{a_p\ldots a_1}^{\ \ \ b}$, $p=2,\ldots,r-s$ only and do not contain $P^a_{a'}$, $P^{a'}_a$ (ct. the footnore 7) in A I):

Theorem (5,1a). *If we eliminate from* (1,7) *(for* $r=2,\ldots N$*) and* (3,2) *the* $P_{b'\ldots}^{\ \ \ a}$ *we obtain the transformation law*

$$\mathbf{n}_{a'_r\ldots a'_1} = P^{a_r\ldots a_1}_{a'_r\ldots a'_1}\,\mathbf{n}_{a_r\ldots a_1}, \quad (r=2,3,\ldots,N) \tag{4,1}$$

where [6])

$$\begin{cases} a) & \mathbf{n}_{ab} \equiv \mathbf{x}_{ab} - \Gamma_{ab}^{\ \ c}\,\mathbf{x}_c \\ b) & \mathbf{n}_{a_r\ldots a_1} \equiv \mathbf{x}_{a_r\ldots a_1} - \sum_2^{r-1} \mathbf{x}_{b_s\ldots b_1}\left[\left\{\begin{matrix}\Gamma_{a_r\ldots a_s}^{\ \ \ \ b_s}\,\delta^{b_{s-1}\ldots b_1}_{a_{s-1}\ldots a_1}\end{matrix}\right\} + \gamma_{a_r\ldots a_s\ldots a_1}^{\ \ \ \ \ \ b_s\ldots b_1}\right] - \Gamma_{a_r\ldots a_1}^{\ \ \ b}\,\mathbf{x}_b \\ & \qquad\qquad r=3,4,\ldots,N. \end{cases} \tag{4,2}$$

[6]) In (4, 2b) we use the symbol $\delta^{b_q\ldots b_1}_{a_q\ldots a_1} \equiv \delta^{b_q}_{a_q}\ldots\delta^{b_1}_{a_1}$.

The proof of this theorem is the same as the proof of the Theorem (5,1) in "A I.

Theorem (4,1*b*). *The points* $\mathbf{n}_{a_r\ldots a_1}$ *may be expressed also in the following way*

$$\mathbf{n}_{a_r\ldots a_1} \equiv \Gamma^{b_r\ldots b_1}_{a_r\ldots a_1}\,\mathbf{x}_{b_r\ldots b_1} \qquad r=2,3,\ldots,N \tag{4,3a}$$

where

$$\left\{\begin{array}{l} a)\quad \Gamma^{b_2b_1}_{a_2a_1} \equiv \delta^{(b_2b_1)}_{(a_2a_1)} - \frac{1}{N}\,\Gamma^{(b_2}_{a_2a_1}\,y^{b_1)} \\[2ex] b)\quad \Gamma^{b_r\ldots b_1}_{a_r\ldots a_1} \equiv \delta^{(b_r\ldots b_1)}_{(a_r\ldots a_1)} - (N-r+1)!\times \\[2ex] \left[\sum_2^{r-1}\frac{1}{(N-s+1)!}\,y^{(b_r\ldots b_{s+1}}\left[\left\{\Gamma_{a_r\ldots a_s}^{b_s}\,\delta^{b_{s-1}\ldots b_1)}_{a_{s-1}\ldots a_1}\right\} + \gamma_{a_r\ldots a_s\ldots a_1}^{b_s\ldots b_1)}\right] + \frac{1}{N!}\,\Gamma_{a_r\ldots a_1}^{(b_1}\,y^{b_2}\ldots y^{b_r)}\right] \\[2ex] \qquad\qquad r=3,4,\ldots N. \end{array}\right. \tag{4,4}$$

The proof follows at once from (4,2) and (1,2).

Theorem (4,2). *The points* $\mathbf{n}_{a_r\ldots a_1}$ $(r=2,3,\ldots N)$ *have the following properties*

a. They are (x,y)*-invariant*

b. homogeneous of degree $N+1-r$

$$\mathbf{n}_{a_r\ldots a_1}(\overset{*}{y}) \equiv \overset{*}{\mathbf{n}}_{a_r\ldots a_1} = g^{N+1-r}\,\mathbf{n}_{a_r\ldots a_1}(y). \tag{4.5}$$

c) Their $\left.\begin{matrix}x-\\ y-\end{matrix}\right\}$ *order is* $\left\{\begin{matrix}N+1\\ r\end{matrix}\right.(r)$

d) They are all in $\overset{r}{P}_{m_r}$ *but not in* $\overset{s}{P}_{m_s}$, $(s<r)$.

Proof. The statements *a.* and *d.* follow at once from (4,1) and (4,2). The equations (1,7) and (3,2) being g-invariant the result (4,2) of the elimination mentioned in the theorem (4,1) must also be g-invariant. Hence $\mathbf{n}_{a_r\ldots a_1}$ must be homogeneous and its degree must be the degree of $\mathbf{x}_{a_r\ldots a_1}$ (cf. equations (4,2)) which is $N+1-r$. The statement *c.* follows from (4,2) and the statement *c.* of the theorem (3,2).

Definition (4,1). *The points* $\mathbf{n}_{a_r\ldots a_1}$ $(r=2,\ldots N)$ *are termed* $(r-1)$*-st projective normal points.*

In the next paper we shall establish the Frenet formules for the projective normal points.

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FORMALISTISCHE BETRACHTUNGEN ÜBER INTUITIONISTISCHE UND VERWANDTE LOGISCHE SYSTEME I

BY

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Im Jahre 1879 hat G. FREGE in seiner „Begriffsschrift" den Aussagenkalkül in fast vollendeter Form entwickelt; es war der erste Kalkül mit einer explizit formulierten Umformungsregel (die sogenannte Abtrennungsregel); eine Einsetzungsregel war noch nicht vorhanden.[1]) Damit war der Weg über die Boolesche Algebra („algebra of logic") verlassen; die Logistik hatte einen Anfang genommen. Man kann diese Entwicklung nur als notwendig betrachten; denn, wie JOERGENSEN in seinem „Treatise of formal logic", Vol. 2 (1931), S. 144 bemerkt, "the algebra of logic is a purely formal calculus, the fundamental principles and theorems of which express those formal qualities which are common to different groups of objects such as propositions, classes, relations and areas: . . .; common to all these forms of calculus is the fact that they assume the validity of a series of principles of inference, which are not expressly formulated, but are constantly brought into use in the proofs; this logical imperfection can, however, be removed, . . .; this is provided in Logistics, . . ." Viele Axiomensysteme des Aussagenkalküls wurden nachher gebildet (RUSSEL-WHITEHEAD; NICOD; LUKASIEWICZ; HILBERT-ACKERMANN; HILBERT-BERNAYS; u.s.w.). Dabei wurde auf Anschlusz an die Boolesche Algebra nicht geachtet.

In einer in Proceed. Kon. Ned. Akad. Wet. Amsterdam **49** (1946), **50** (1947) erschienenen Arbeit zeigte ich nun, dasz es sehr wohl möglich ist die in einer Booleschen Algebra vorkommenden, anschaulichen Elementen zu einem Aufbau eines Axiomensystems für den klassischen Aussagenkalkül zu benutzen. Durch Hinzufügung der Axiome $\lambda \to X$ und $X \to \nu$ [2]) zum RUSSELL-WHITEHEADschen System wird dies völlig gleichwertig mit einer Booleschen Algebra, falls man einige der Axiome dieser Algebra als Umformungsregeln deutet, und auszerdem eine Einsetzungsregel hinzufügt.

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[1]) Siehe H. HERMES und A. SCHOLZ, Ein neuer Vollständigkeitsbeweis für das reduzierte Fregesche Axiomensystem des Aussagenkalküls. Forschungen zur Logik. Neue Folge, **1** (1937).

[2]) Zu interpretieren als „aus einer falschen Aussage läszt sich jede Aussage, aus einer jeden Aussage läszt sich eine wahre Aussage folgern".

Man kann sich nun die Frage vorlegen ob nicht auch für den intuitionistischen Aussagenkalkül (in der HEYTINGschen Symbolisierung [3])) ein gleichartiger Aufbau, also ein Aufbau, welcher sich möglichst eng an die Theorie der Strukturen („lattices") anschlieszt, möglich sei. Wie wir sehen werden, läszt sich diese Frage bejahend beantworten. [4]) Das neue Axiomensystem läszt sich zu einem Axiomensystem für den intuitionistischen Prädikatenkalkül [5]) erweitern.

Dual gegenüber diesen Systemen stehen zwei weitere im folgenden untersuchte Systeme, welche u.a. die folgenden bemerkenswerten Eigenschaften haben: 1° in ihnen gilt das principium tertii exclusi, das principium contradictionis dagegen nicht; 2° die bekannten GÖDELschen Deutungen der klassischen Aussagen- und engeren Prädikatenlogik innerhalb der zugehörigen intuitionistischen [6]) korrespondieren mit analogen Deutungen innerhalb der dualen Kalküle.

Zur Erleichterung der Lektüre geben wir hier schematisch die gegenseitigen Verhältnisse der verschiedenen aussagenlogischen Axiomensysteme: $A_1 \supset A_2 \supset M \supset I \supset K \equiv K^* \subset I^* \subset M^* \subset A_2^* \subset A_1^*$; I ist ein intuitionistischer Aussagenkalkül, I^* der zu diesem dualen Kalkül; $K \equiv K^*$ ist der klassische Aussagenkalkül; M deutet einen Minimalkalkül im Sinne von JOHANSSON [7]) an. In A_1 sind die Axiome 1^α, 2^α, $3°$, und die Schluszregeln E, E_0, 1^β, 2^β; in A_2 kommen hinzu Axiom 4^α, Regel 4^β; in M auszerdem Ax. 6^α, Ax. 6^β ($\equiv 7^*$); in I noch Ax. 5°; schlieszlich in K Ax. 7° ($\equiv 6^{\beta *}$). Für die mit einem Asterisk angedeuteten dualen Systeme der vorigen sind Axiome und Regeln: für A_1^*, 1^α, 4^α, 5°; E^*, E_0^*, 1^β, 4^β; für A_2^* daneben 2^α; 2^β; für M^* noch $6^{\alpha *}$, $6^{\beta *}$ ($\equiv 7°$); für I^* noch 3°; für K^* 7^* ($\equiv 6^\beta$). Nur die Regeln E_0, E_0^* und die Axiome 6^α, $6^{\alpha *}$ lassen sich nicht in eine in der Booleschen Algebra geläufigen Form bringen: nur durch diese unterscheiden sich die Kalküle K und K^* (scheinbar) voneinander.

Anwendung der GENTZENschen Schluszweisenkalküle [8]) führt zu einigen

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[3]) Siehe A. HEYTING, Sitz. ber. Preusz. Akad. Wiss., Phys.-math. Kl., 1, 42–56, (1930).

[4]) Die Antwort wird nicht geliefert durch die von GARRETT BIRKHOFF konstruierte „Brouwerian logic" (siehe etwa G. BIRKHOFF, Lattice theory, Rev. Ed. 1948, Chapter 12), von MCKINSEY und TARSKI mehr zutreffend „Brouwerian algebra" genannt (siehe MC KINSEY a. TARSKI, Annals of math. 47, 122–162, (1946), insbes. S. 124); denn als „Logik" betrachtet sind gegen diese Algebra dieselben Einwände zu erheben wie im JOERGENSENschen Zitate gegen die Boolesche Algebra.

[5]) Siehe HEYTING, loc. cit. 3), 57–65.

[6]) Siehe K. GÖDEL, Ergebnisse e. math. Koll. herausgegeben von K. MENGER, Heft 4, 34–38, (1933); auch D. VAN DANTZIG, Proceed. Kon. Ned. Akad. v. Wet. Amsterdam 50, 918–929, (1947), insbes. S. 923, 924.

[7]) Siehe I. JOHANSSON, Comp. math., 4, 119–136, (1936).

[8]) Siehe G. GENTZEN, Math. Ztschr., 39, 176–210, 405–431, (1935).

weiteren Resultaten. Zu jedem der eben genannten Axiomensysteme gehört ein „Kalkül des natürlichen Schlieszens" und ein „logistischer Kalkül". Es wird u.a. gezeigt, dasz es im „logistischen Kalkül" für K schon genügt, in Abweichung von GENTZEN, nur im Schema *NES* mehrgliedrige Sukzedentia (im Sinne von GENTZEN) zuzulassen; diese Bedingung ist dann auch notwendig, da man sonst zum intuitionistischen Kalkül käme. [9])

Eine affirmative [10]) Aussagenlogik A_1.

§ 1. *Elementare Aussagen* (*elementare Kalkülformeln*) sollen durch grosze lateinische Buchstaben angedeutet werden. [11]) Undefinierte *Grundverknüpfungen* sind die *Implikation* $\subset$ und die Verknüpfung der Konjunktion „*Und*", angedeutet durch $\cdot$; die Relationen, welche man zwischen diesen Verknüpfungen annehmen soll, sind in den nachfolgenden Axiomen enthalten.

Einsetzungsregel E (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden groszen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die groszen lateinischen Buchstaben und ν als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} \cdot \mathfrak{S}$, und $\mathfrak{R} \subset \mathfrak{S}$. [12])

Einsetzungsregel E (zweiter Teil). Läszt sich (auf Grund der nachfolgenden Axiome und Schluszregeln) schreiben $\mathfrak{A} \subset \mathfrak{B}$, mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bezw. $\mathfrak{B}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, so läszt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$. [12])

Axiom 1^α. $X \subset X$.

Schluszschema 1^β. $\dfrac{\mathfrak{A} \subset \mathfrak{B} \quad \mathfrak{B} \subset \mathfrak{C}}{\mathfrak{A} \subset \mathfrak{C}}$.

Definition. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von:

$$\mathfrak{A} \subset \mathfrak{B} \qquad \mathfrak{B} \subset \mathfrak{A}.$$

Satz 1. $X = X$ (die Gleichheitsrelation ist *reflexiv*).

Satz 2 (Schluszschema). $\dfrac{\mathfrak{A} = \mathfrak{B}}{\mathfrak{B} = \mathfrak{A}}$ (die Gleichheitsrelation ist *symmetrisch*).

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[9]) Unsere Darstellung von Kalkülen NI, NK, LI und LK zeigt noch einige weitere Abweichungen; so lassen wir u.a. in den Schluszfiguren keine leeren Antezedentia und Sukzedentia zu; die G. Verfahren bleiben dabei doch im wesentlichen beibehalten.

[10]) „Affirmativ", weil die Implikation $X \subset Y$ in A_1 sich interpretieren läszt als „die Affirmation von X zieht die Affirmation von Y nach sich".

[11]) Also durch $A, B, C, \ldots$; doch auch $A_1, B_1, C_1 \ldots$; $A_2, B_2, \ldots$; $A_n, \ldots$ (n eine natürliche Zahl).

[12]) Grosze deutsche Buchstaben werden immer Kalkülformeln andeuten.

Satz 3 (Schluszschemata). $\dfrac{\mathfrak{A}=\mathfrak{B}\ \ \mathfrak{B}=\mathfrak{C}}{\mathfrak{A}=\mathfrak{C}}$ (die Gleichheitsrelation ist *transitiv*).

Axiom 2^{α}. $(X\cdot Y)\subset X$ und $(X\cdot Y)\subset Y$.

Schluszschema 2^{β}. $\dfrac{\mathfrak{C}\subset\mathfrak{A}\ \ \mathfrak{C}\subset\mathfrak{B}}{\mathfrak{C}\subset(\mathfrak{A}\cdot\mathfrak{B})}$.

Satz 4 (Schluszschema). $\dfrac{\mathfrak{A}=\mathfrak{A}_1\ \ \mathfrak{B}=\mathfrak{B}_1}{(\mathfrak{A}\cdot\mathfrak{B})=(\mathfrak{A}_1\cdot\mathfrak{B}_1)}$.

Satz 5. $[(X\cdot Y)\cdot Z]=[X\cdot(Y\cdot Z)]$.

Satz 6. $(X\cdot Y)=(Y\cdot X)$.

Satz 7. $(X\cdot X)=X$.

Satz 8 (Schluszschema). $\dfrac{\mathfrak{A}\subset\mathfrak{B}}{(\mathfrak{A}\cdot\mathfrak{B})=\mathfrak{A}}$ und $\dfrac{(\mathfrak{A}\cdot\mathfrak{B})=\mathfrak{A}}{\mathfrak{A}\subset\mathfrak{B}}$.

Ersetzungsschema (ta) E_0. $\dfrac{\mathfrak{A}\subset[\mathfrak{B}\subset\mathfrak{C}]}{(\mathfrak{A}\cdot\mathfrak{B})\subset\mathfrak{C}}$ und $\dfrac{(\mathfrak{A}\cdot\mathfrak{B})\subset\mathfrak{C}}{\mathfrak{A}\subset[\mathfrak{B}\subset\mathfrak{C}]}$ [13].

Mit Schema 1^{β} korrespondiert der

Satz 9. $[(X\subset Y)\cdot(Y\subset Z)]\subset(X\subset Z)$.

Beweis. (Ax. 1^{α}, Regel E) $(X\subset Y)\subset(X\subset Y)$; (Sch. E_0, Satz 6, Regeln E, 1^{β}) $[X\cdot(X\subset Y)]\subset Y$; (Regel E, Ax. 2^{α}) $[(X\subset Y)\cdot(Y\subset Z)]\subset\subset(Y\subset Z)$; $\{[X\cdot(X\subset Y)]\cdot[(X\subset Y)\cdot(Y\subset Z)]\}\subset[Y\cdot(Y\subset Z)]$; $\{[(X\subset Y)\cdot(Y\subset Z)]\cdot X\}\subset[Y\cdot(Y\subset Z)]$. Daneben: (Sch. E_0) $[(Y\subset Z)\cdot Y]\subset Z$. Also (Sch. 1^{β}) $\{[(X\subset Y)\cdot(Y\subset Z)]\cdot X\}\subset Z$, oder (Sch. E_0) $[(X\subset Y)\cdot(Y\subset Z)]\subset\subset[X\subset Z]$.

Mit Schema 2^{β} korrespondiert der

Satz 10. $[(X\subset Y)\cdot(X\subset Z)]\subset[X\subset(Y\cdot Z)]$.

Beweis. (Ax. 1^{α}, Regeln E, E_0) $[(X\subset Y)\cdot X]\subset Y$, $[(X\subset Z)\cdot X]\subset Z$, also ($2^{\alpha}$, 1^{β}, 2^{β}) $[\{(X\subset Y)\cdot(X\subset Z)\}\cdot X]\subset(Y\cdot Z)$; ($E_0$) Satz 10.

Satz 11 (Abtrennungsregel A). $\dfrac{\nu\subset[\mathfrak{A}\subset\mathfrak{B}]\qquad\nu\subset\mathfrak{A}}{\nu\subset\mathfrak{B}}$.

Beweis.
$$\dfrac{\dfrac{\nu\subset[\mathfrak{A}\subset\mathfrak{B}]}{(\nu\cdot\mathfrak{A})\subset\mathfrak{B}}\ (\text{Sch. } E_0)\qquad\dfrac{\nu\subset\mathfrak{A}\qquad\nu\subset\nu\,(1^{\alpha},E)}{\nu\subset(\mathfrak{A}\cdot\nu)}\ (\text{Sch. } 2^{\beta})}{\nu\subset\mathfrak{B}}\ (\text{Sch. } 1^{\beta})$$

Axiom 3°. $X\subset\nu$ [14].

Definition. Ein Ausdruck $\nu\subset\mathfrak{A}$, oder, was wegen Axiom 3° und Regel E auf dasselbe hinauskommt, $\nu=\mathfrak{A}$, entstanden nach endlichmaliger Anwendung von Axiomen oder (und) Regeln (Schemata), ist *ein (affirmatives) Theorem*.

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[13]) Interpretation: „Aus der Assertion von $\mathfrak{A}$ folgt, dasz die Assertion von $\mathfrak{B}$ die von $\mathfrak{C}$ ergibt" ist aequivalent mit „aus der Assertion von $\mathfrak{A}$ und von $\mathfrak{B}$ folgt die von $\mathfrak{C}$".

[14]) Interpretation: Die Assertion einer willkürlichen Aussage (ob wahr oder falsch) liefert die Assertion einer wahren Aussage ν.

Satz 12 (Schluszschemata). Ist $\mathfrak{A} \subset \mathfrak{B}$ ein Satz, somit durch endlichmalige Anwendung von Axiomen oder (und) Regeln (Schemata) ableitbar, so ist $\nu \subset [\mathfrak{A} \subset \mathfrak{B}]$ ein Theorem, und umgekehrt.

Beweis. $\mathfrak{A} \subset \mathfrak{B}$, $(2^a, E)$ $(\nu \cdot \mathfrak{A}) \subset \mathfrak{A}$, also (1^β) $(\nu \cdot \mathfrak{A}) \subset \mathfrak{B}$, (E_0) $\nu \subset [\mathfrak{A} \subset \mathfrak{B}]$.
Umgekehrt: $\nu \subset [\mathfrak{A} \subset \mathfrak{B}]$, (E_0) $(\nu \cdot \mathfrak{A}) \subset \mathfrak{B}$; $(3^\circ, 1^a, E, 2^\beta)$ $\mathfrak{A} \subset (\nu \cdot \mathfrak{A})$; (1^β) $\mathfrak{A} \subset \mathfrak{B}$.

Mit Schema E_0 korrespondiert

Satz 13. $\{X \subset [Y \subset Z]\} = \{(X \cdot Y) \subset Z\}$.

Beweis. $[\{X \subset [Y \subset Z]\} \cdot X \cdot Y] \subset (E_0)\ \{[Y \subset Z] \cdot Y\} \subset (E_0)\ Z$, oder (E_0) $\{X \subset [Y \subset Z]\} \subset \{(X \cdot Y) \subset Z\}$.
Umgekehrt, $[(X \cdot Y) \subset Z] \subset$ (Satz 9, Sch. E_0) $\{[Y \subset (X \cdot Y)] \subset (Y \subset Z)\}$, also $[\{X \subset [Y \subset (X \cdot Y)]\} \cdot [(X \cdot Y) \subset Z]] \subset [\{X \subset [Y \subset (X \cdot Y)]\} \cdot \{[Y \subset (X \cdot Y)] \subset (Y \subset Z)\}]$, oder (Regeln E_0, A) $[(X \cdot Y) \subset Z] \subset [\{X \subset [Y \subset (X \cdot Y)]\} \cdot \{[Y \subset (X \cdot Y)] \subset (Y \subset Z)\}] \subset$ (S. 9) $[X \subset (Y \subset Z)]$, $[(X \cdot Y) \subset Z] \subset [X \subset (Y \subset Z)]$.

Satz 14 (Schluszschema). $\dfrac{\mathfrak{A} = \mathfrak{A}_1 \qquad \mathfrak{B} = \mathfrak{B}_1}{(\mathfrak{A} \subset \mathfrak{B}) = (\mathfrak{A}_1 \subset \mathfrak{B}_1)}$.

Beweis. Mit S. 9, S. 12, E_0, A und 1^β.

Bemerkung. I Unter Annahme der Axiome 1^a, 2^a, 3° und der Regeln E, 1^β, 2^β läszt sich zwar. nach Satz 11, *die Abtrennungsregel A aus dem Ersetzungsschema E_0 ableiten*; *das Umgekehrte gilt jedoch nicht.*
Das folgt sofort aus einem Beispiel in Hilbert-Bernays, Grundlagen der Mathematik I (1934), S. 77, welches zeigt, dasz mit den Axiomen 1^a, 2^a, 3° [15]) und den Regeln E, A, 1^β und 2^β sich die Kalkülformel $X \subset (Y \subset X)$ nicht ableiten läszt, während sie sofort ableitbar wird, wenn man Regel A durch Regel E_0 ersetzt.

Bemerkung II. Nicht alle Theoreme haben die Form $\nu \subset [\mathfrak{A} \subset \mathfrak{B}]$ (siehe Satz 12). Denn das Schluszschema 2^β gibt unmittelbar

$$(S) \qquad \frac{\nu \subset \mathfrak{A} \qquad \nu \subset \mathfrak{B}}{\nu \subset (\mathfrak{A} \cdot \mathfrak{B})}.$$

Bemerkung III. Obiges Axiomensystem für Implikation und Konjunktion ist, so weit es die affirmativen Theoreme betrifft, gleichwertig mit dem *mit unserem Axiom* 3° $[X \subset \nu$ und $\nu \subset (X \subset \nu)]$ *erweiterten* System der Formeln, die Heyting in seiner Formalisierung der intuitionistischen Logik, loc. cit. 3), S. 45—47 (§§ 1 u. 2) für diese Verknüpfungen aufgestellt hat [15bis]); nur soll man, loc. cit., überall die Zeichen $\vdash$ und $\vdash\vdash$ durch $\nu \subset$ oder $\nu =$ ersetzen; die Ersetzungsregel E_0 und der bei Heyting korrespondierende Satz 2.27 machen die Ableitung dieser Behauptung ganz leicht.

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[15]) Man füge ν immer den Wert a hinzu.

[15bis]) Und dadurch auch gleichwertig mit dem System der Axiome I, 1—3; II, 1—3, Einsetzungsregel und Abtrennungsregel in Hilbert-Bernays, Grundlagen der Math. I, 66 (1934), vermehrt mit dem Axiom $X \subset \nu$ (oder $X \to \nu$).

ν und das zugehörige Axiom 3° übernehmen hier in gewissem Sinne die Rolle der üblichen Assertionszeichen; die Klassen der affirmativen Theoreme, welche ν nur am Anfang enthalten ($\nu \subset$), und der HEYTINGschen Theoreme (somit bewiesen ohne Axiom 3°) fallen zusammen [16]); hier gibt es noch Sätze, welche die Form $\mathfrak{A} \subset \mathfrak{B}$ haben mit $\mathfrak{A}$ von ν verschieden, und Theoreme $\nu \subset \mathfrak{A}$, mit ν auch in $\mathfrak{A}$ vorkommend.

Eine verneinende (negierende) [17]) Aussagenlogik A_1^*.

§ 2. *Elementare Aussagen*: A, B, ...; A_1, ...; A_2, ...; ... Grundverknüpfungen: *Implikation* $\subset$, Disjunktion „*oder*", angedeutet durch $+$.

Einsetzungsregel E^* (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden groszen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die groszen lateinischen Buchstaben und λ als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R} + \mathfrak{S}$, und $\mathfrak{R} \subset \mathfrak{S}$. [12])

Einsetzungsregel E^* (zweiter Teil). Läszt sich (auf Grund der nachfolgenden Axiome und Schluszregeln) schreiben $\mathfrak{A} \subset \mathfrak{B}$, mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E^* (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln ersetzt sind, so läszt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$. [12])

Axiom 1^α. $X \subset X$.

Schluszschema 1^β. $\dfrac{\mathfrak{A} \subset \mathfrak{B} \quad \mathfrak{B} \subset \mathfrak{C}}{\mathfrak{A} \subset \mathfrak{C}}$.

Definition von $\mathfrak{A} = \mathfrak{B}$, und die Sätze 1—3 bleiben ungeändert.

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[16]) Man hat nur zu zeigen, dasz jede Ableitung, welche in der Logik A_1 zu einem affirmativen Theorem, mit ν nur am Anfang, führt, ersetzt werden kann durch eine Ableitung, bei der: 1° am Anfang jeder auftretenden Kalkülformel $\nu \subset$ steht; 2° in jeder Formel $\nu \subset \mathfrak{A}$ $\mathfrak{A}$ ν nicht enthält; 3° Axiom 3° nicht benutzt wird.

ad 1°: Wegen Satz 12 lassen sich die Formeln in den Axiomen 1^α, 2^α und den Regeln E_0, 1^β, 2^β durch $\nu \subset$ vorangehen; sie werden so zu Formeln und Regeln im HEYTINGschen System.

ad 2°: Da, wegen Ax. 3° und Satz 12, $\nu \subset (X \subset X) \subset \nu$ oder $\nu = (X \subset X)$, darf man in jeder in der Ableitung auftretenden Formel ν durch $X_n \subset X_n$ (n eine genügend hohe natürliche Zahl, damit X_n nicht schon in der Ableitung vorkäme) ersetzen; dies ist eine Folge der Sätze 4 und 14.

ad 3°: In der ursprünglichen Ableitung konnte ν nur auf zwei Weisen eingeführt werden: I. mit Regel E allein; dies wird nun geändert in eine Einführung von $X_n \subset X_n$ mittels derselben Regel; II. mit Ax. 3° und Regel E: $\mathfrak{A} \subset \nu$; dies ist zu ersetzen durch: (Ax. 2^α, geändert wie ad 1° angegeben) $\nu \subset [(\mathfrak{A} \cdot X_n) \subset X_n]$; ($E_0$, geändert wie ad 1° angegeben) $\nu \subset [\{\mathfrak{A} \subset (X_n \subset X_n)\}]$.

[17]) „Verneinend", weil die Implikation $X \subset Y$ in A_1^* sich interpretieren läszt als „die Negation von Y kann nur mit der Negation von X koexistieren". Dies führt zu entsprechenden Deutungen für alle hier folgenden Axiome und Schluszregeln. Siehe auch Fuszn. 18 und 19.

Axiom 4^α. $X \subset (X + Y)$, [18]) und $Y \subset (X + Y)$.

Schluszschema 4^β. $\dfrac{\mathfrak{A} \subset \mathfrak{C} \quad \mathfrak{B} \subset \mathfrak{C}}{(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}}$.

Satz 15 (Schluszschema). $\dfrac{\mathfrak{A} = \mathfrak{A}_1 \quad \mathfrak{B} = \mathfrak{B}_1}{(\mathfrak{A} + \mathfrak{B}) = (\mathfrak{A}_1 + \mathfrak{B}_1)}$.

Satz 16. $[(X + Y) + Z] = [X + (Y + Z)]$.

Satz 17. $(X + Y) = (Y + X)$.

Satz 18. $(X + X) = X$.

Satz 19 (Schluszschemata). $\dfrac{\mathfrak{B} \subset \mathfrak{A}}{(\mathfrak{A} + \mathfrak{B}) = \mathfrak{A}}$ und $\dfrac{(\mathfrak{A} + \mathfrak{B}) = \mathfrak{A}}{\mathfrak{B} \subset \mathfrak{A}}$.

Ersetzungsschema(ta) E_0^*. $\dfrac{[\mathfrak{C} \subset \mathfrak{B}] \subset \mathfrak{A}}{\mathfrak{C} \subset (\mathfrak{B} + \mathfrak{A})}$ und $\dfrac{\mathfrak{C} \subset (\mathfrak{B} + \mathfrak{A})}{[\mathfrak{C} \subset \mathfrak{B}] \subset \mathfrak{A}}$ [19]).

Mit Schema 1^β korrespondiert der

Satz 20. $(Z \subset X) \subset [(Z \subset Y) + (Y \subset X)]$.

Beweis. $(Y \subset X) \subset (Y \subset X)$; (Sch. E_0^*, Satz 17, Regeln E^*, 1^β) $Y \subset [(Y \subset X) + X]$; (Regel E^*, Ax. 4^α) $(Z \subset Y) \subset [(Z \subset Y) + (Y \subset X)]$; $[(Z \subset Y) + Y] \subset \{[(Z \subset Y) + (Y \subset X)] + [(Y \subset X) + X]\}$; $[(Z \subset Y) + Y] \subset \subset \{[(Z \subset Y) + (Y \subset X)] + X\}$. Daneben: (Sch. E_0^*) $Z \subset [Y + (Z \subset Y)]$. Also (Sch. 1^β) $Z \subset \{[(Z \subset Y) + (Y \subset X)] + X\}$, oder (Sch. E_0^*) $(Z \subset X) \subset \subset [(Z \subset Y) + (Y \subset X)]$.

Mit Schema 4^β korrespondiert der

Satz 21. $[(Z + Y) \subset X] \subset [(Z \subset X) + (Y \subset X)]$.

Satz 22 (Abtrennungsregel A^*). $\dfrac{(\mathfrak{B} \subset \mathfrak{A}) \subset \lambda \quad \mathfrak{A} \subset \lambda}{\mathfrak{B} \subset \lambda}$ [20]).

Beweis. $\dfrac{\dfrac{(\mathfrak{B} \subset \mathfrak{A}) \subset \lambda}{\mathfrak{B} \subset (\mathfrak{A} + \lambda)}\ (\text{Sch. } E_0^*) \quad \dfrac{\mathfrak{A} \subset \lambda \quad \lambda \subset \lambda\ (1^\alpha, E^*)}{(\mathfrak{A} + \lambda) \subset \lambda}\ (\text{Sch. } 4^\beta)}{\mathfrak{B} \subset \lambda}\ (\text{Sch. } 1^\beta)$.

Axiom 5°. $\lambda \subset X$ [21]).

Definition. Ein Ausdruck $\mathfrak{A} \subset \lambda$, oder, was wegen Axiom 5° und Regel E^* auf dasselbe hinauskommt, $\mathfrak{A} = \lambda$, entstanden nach endlichmaliger Anwendung von Axiomen oder (und) Regeln (Schemata), ist *ein negierendes Theorem.*

Es wird schon deutlich sein, dasz folgendes Dualitätsprinzip besteht: *Die Axiome, Schluszregeln, Sätze und Theoreme der affirmativen Aussagenlogik stehen dual gegenüber denen der verneinenden Aussagenlogik.*

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[18]) Deutung: Die Verneinung von $X + Y$ hat die Verneinung von X zur Folge.

[19]) Deutung: „Die Verneinung von $\mathfrak{A}$ gibt, dasz die Verneinung von $\mathfrak{B}$ die von $\mathfrak{C}$ zur Folge hat" ist aequivalent mit „die Verneinung von $\mathfrak{A} + \mathfrak{B}$ (d.h. die Verneinungen von $\mathfrak{A}$ und von $\mathfrak{B}$) hat die von $\mathfrak{C}$ zur Folge".

[20]) Deutung: Sind $\mathfrak{B} \subset \mathfrak{A}$ und $\mathfrak{A}$ falsche Aussagen, so auch $\mathfrak{B}$. Ausführlicher: $\mathfrak{A} \subset \lambda$ drückt die Negation von $\mathfrak{A}$ aus (denn λ ist eine falsche Aussage); $(\mathfrak{B} \subset \mathfrak{A}) \subset \lambda$ oder $\mathfrak{B} \subset (\mathfrak{A} + \lambda)$ drückt aus (siehe Fuszn. 19), dasz die Negation von $\mathfrak{A}$ die von $\mathfrak{B}$ zur Folge hat. Somit führt dies zur Negation von $\mathfrak{B}$ oder zu $\mathfrak{B} \subset \lambda$.

[21]) Eine falsche Aussage impliziert jede Aussage.

Letztgenannte Logik geht mittels folgender Umsetzungen aus der ersten hervor: α) man ändere in jeder Formel von § 1 · in +, ν in λ; β) $\mathfrak{A} \subset \mathfrak{B}$ werde durch $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$ ersetzt; dabei sollen $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ die gemäsz α) und β) aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehenden Kalkülformeln sein. [22])

Dadurch können wir die Beweise der folgenden Behauptungen unterdrücken.

Satz 23 (Schluszschemata). Läszt sich durch endlichmalige Anwendung von Axiomen und Schluszregeln ableiten $\mathfrak{B} \subset \mathfrak{A}$, so ist $[\mathfrak{B} \subset \mathfrak{A}] \subset \lambda$ ein negierendes Theorem, und umgekehrt.

Mit Schema E_0^* korrespondiert

Satz 24. $\{Z \subset (Y + X)\} = \{(Z \subset Y) \subset X\}$.

Satz 14 (Schluszschema) bleibt ungeändert gültig.

Bemerkung I. Unter Annahme der Axiome 1^α, 4^α, 5° und der Regeln E^*, 1^β, 4^β läszt sich *die Abtrennungsregel A^* aus dem Ersetzungsschema E_0^* ableiten; das Umgekehrte gilt jedoch nicht.*

Bemerkung II. Nicht alle verneinenden Theoreme haben die Form $[\mathfrak{A} \subset \mathfrak{B}] \subset \lambda$. Denn es läszt sich folgendes Schluszschema ableiten:

$$(S^*) \qquad \frac{\mathfrak{A} \subset \lambda \qquad \mathfrak{B} \subset \lambda}{(\mathfrak{A} + \mathfrak{B}) \subset \lambda}.$$

Affirmative Aussagenlogik A_2.

§ 3. In diesem Par. wird das Axiomensystem von § 1 angenommen und fortgesetzt. In der Einsetzungsregel E nehme man $\mathfrak{R} + \mathfrak{S}$ auf (neben $\mathfrak{R} \cdot \mathfrak{S}$ und $\mathfrak{R} \subset \mathfrak{S}$).

Axiom 4^α. $X \subset (X + Y)$ [23]) und $Y \subset (X + Y)$.

Schluszschema 4^β. $\dfrac{\mathfrak{A} \subset \mathfrak{C} \qquad \mathfrak{B} \subset \mathfrak{C}}{(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}}$.

Hier gelten nun auch die Sätze 15—19.

Satz 25. $[(X + Y) \cdot Z] = [(X \cdot Y) + (X \cdot Z)]$ (die Logik A_2 ist eine *distributive* Struktur).

Beweis. $(X \cdot Z) \subset$ (Ax. 4^α) $[(X \cdot Z) + (Y \cdot Z)]$, also (Regel E_0) $X \subset [Z \subset \subset \{(X \cdot Z) + (Y \cdot Z)\}]$; ebenso $Y \subset [Z \subset \{(X \cdot Z) + (Y \cdot Z)\}]$; (Sch. 4^β) $(X + Y) \subset \subset [Z \subset \{(X \cdot Z) + (Y \cdot Z)\}]$; (Regel E_0) $\{(X + Y) \cdot Z\} \subset \{(X \cdot Z) + (Y \cdot Z)\}$.

Umgekehrt, $(X \cdot Y) \subset [X \cdot (Y + Z)]$, und $(X \cdot Z) \subset [X \cdot (Y + Z)]$, also $\{(X \cdot Y) + (X \cdot Z)\} \subset [X \cdot (Y + Z)]$.

Satz 26. $[(X \cdot Y) + Z] = [(X + Z) \cdot (Y + Z)]$ [24]).

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[22]) $\mathfrak{A} = \mathfrak{B}$ geht somit in $\overline{\mathfrak{A}} = \overline{\mathfrak{B}}$ über.

[23]) Deutung: Die Assertion von X hat die von X oder die von Y zur Folge. Vergl. die Deutung in Fuszn. 18.

[24]) Dieser Satz läszt sich ohne Benutzung von Schema E_0 und Axiom 3° aus Satz 25 ableiten. Siehe V. Glivenko, Théorie générale des structures. Act. sci. Paris, S. 32 (1938).

In der Logik A_2 gibt es *eine teilweise Dualität:* Zu jedem mit den Axiomen 1^a, 2^a, 4^a, den Regeln E, 1^β, 2^β, 4^β und den Sätzen 21 und 22 (also ohne Anwendung von Ax. 3° und ohne weitere Anwendung von Schema E_0, welches bei der Ableitung der eben genannten Sätze benutzt wurde) ableitbaren Satz erhält man einen dualen, wenn: α) in jeder vorkommenden Kalkülformel $+$ durch $\cdot$, und umgekehrt, ersetzt wird; β) $\mathfrak{A} \subset \mathfrak{B}$ durch $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$ ersetzt wird: dabei sollen $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ gemäsz α) (ohne β)) aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehende Kalkülformeln sein. [22])

Schluszschema 4^β korrespondiert mit

Satz 27. $[(X \subset Z) \cdot (Y \subset Z)] \subset [(X + Y) \subset Z]$.

Beweis. $[X \cdot (X \subset Z)] \subset$ (Regel E_0) Z; $[Y \cdot (Y \subset Z)] \subset Z$; (Sch. 4^β) $[\{X \cdot (X \subset Z)\} + \{Y \cdot (Y \subset Z)\}] \subset Z$; (Satz 25) $[(X + Y) \cdot (X \subset Z) \cdot (Y \subset Z)] \subset Z$; (Regel E_0) $[(X \subset Z) \cdot (Y \subset Z)] \subset [(X + Y) \subset Z]$.

Bemerkung A. Das Axiomensystem der Logik A_2 ist, so weit es die affirmativen Theoreme betrifft, gleichwertig mit dem *mit unserem Axiom* 3° $[X \subset \nu$ und $\nu \subset (X \subset \nu)]$ *erweiterten* System von Heyting, loc. cit. 3), S. 45—48 (§§ 1, 2 u. 3) [25]); über ν, $\vdash$ und $\vdash\vdash$ ist dabei dasselbe zu sagen wie in Bemerkung III von § 1. Neben der Klasse der mit den Heytingschen Theoremen zusammenfallenden affirmativen Theoreme, welche ν nur am Anfang enthalten ($\nu \subset$), [26]) gibt es hier auch noch Sätze der Form $\mathfrak{A} \subset \mathfrak{B}$ mit von ν verschiedenem $\mathfrak{A}$, und Theoreme $\nu \subset \mathfrak{A}$, wobei ν auch in $\mathfrak{A}$ vorkommt.

Bemerkung B. Ist $\nu \subset \mathfrak{A}$ ein Theorem, so auch $\nu \subset (\mathfrak{A} + \mathfrak{B})$, wobei $\mathfrak{B}$ eine willkürliche Formel.

§ 3^{bis}. Hinzufügung des Axioms 5° (§ 2) würde zu einer Erweiterung der teilweisen Dualität führen; unter α) im vorigen Par. könnte man dann Ersetzung von λ durch ν, und umgekehrt, hinzufügen.

Verneinende Aussagenlogik A_2^*.

§ 4. In diesem Par. wird das System von § 2 angenommen und fortgeführt. Diese Fortführung wird so gewählt, *dasz die Axiome, Schluszregeln, Sätze und Theoreme der assertorischen Logik A_2 dual gegenüber denen der verneinenden Logik A_2^* stehen.* Letztgenannte Logik geht aus der ersten mittels folgender Umsetzungen hervor: α_1) man ändere in den Formeln von §§ 1, 3 $\cdot$ in $+$, und umgekehrt, λ in ν, und umgekehrt; β_1) $\mathfrak{A} \subset \mathfrak{B}$ werde ersetzt durch $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$; dabei sollen $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ die gemäsz α_1) und β_1) aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehende Kalkülformeln sein.

In der Einsetzungsregel E^* (erster Teil) nehme man $\mathfrak{R} \cdot \mathfrak{S}$ auf (neben $\mathfrak{R} + \mathfrak{S}$ und $\mathfrak{R} \subset \mathfrak{S}$).

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[25]) Und ebenso mit dem System der Axiome I, 1—3; II, 1—3; III, 1—3, Einsetzungsregel und Abtrennungsregel in Hilbert-Bernays, loc. cit. 16), S. 66, vermehrt mit dem Axiom $X \subset \nu$ (oder $X \to \nu$).

[26]) Der Beweis der Aequivalenz der hier betrachteten Klassen verläuft wie der Beweis in Fuszn. 16 (ad 2°: hier kommt Satz 15 hinzu).

Axiom 2^{α}. $(X \cdot Y) \subset X$ und $(X \cdot Y) \subset Y$.

Schluszschema 2^{β}. $\dfrac{\mathfrak{C} \subset \mathfrak{A} \quad \mathfrak{C} \subset \mathfrak{B}}{\mathfrak{C} \subset (\mathfrak{A} \cdot \mathfrak{B})}$.

In A_2^* gelten somit auch die Sätze 4—8, 25 und 26; auch A_2^* ist eine *distributive Struktur*.

In der verneinenden Aussagenlogik A_2^* gibt es *eine teilweise Dualität* (vergl. §§ 3, 3^{bis}).

Mit Schluszschema 2^{β} korrespondiert nun der

Satz 28. $[Z \subset (X \cdot Y)] \subset [(Z \subset X) + (Z \subset Y)]$.

Bemerkung. Ist $\mathfrak{A} \subset \lambda$ ein verneinendes Theorem, so auch $(\mathfrak{A} \cdot \mathfrak{B}) \subset \lambda$, mit $\mathfrak{B}$ eine willkürliche Kalkülformel.

PETROLOGY OF THE ST. JEAN DU GARD AREA IN THE SOUTHEASTERN CEVENNES, FRANCE

BY

J. W. C. M. VAN DER SIJP

(Communicated by Prof. H. A. BROUWER at the meeting of February 25, 1950)

1. *Introduction.*

In the summer of 1948 a detailed mapping was carried out of the structural geology and the petrography of the S.E. Cevennes of France. In the neighbourhood of L'Estréchure, 10 km W. of St. Jean du Gard, on the southern part of the geological map of France (1 : 80.000), the Alais sheet (THIERY, 1923), a small body of granite ("gneis granitoide") surrounded by "gneis oeuillé" is shown. The mapping has been done in and around this massif.

During this survey this massif appeared to be augen gneiss, surrounded partly by slaty country rock and partly by granite of the Mont du Liron pluton (fig. 1).

A short petrographic description of the different kinds of rock found in this area, excluding the Mont du Liron granite and its dikes, is given here, whereas DE WAARD (1950) discusses their tectonics.

2. *Outline of the geology.*

The body of augen gneiss in the centre of the mapped area is about 10 km long and 2 km wide. Almost everywhere the augen gneiss is surrounded by a margin of spotted slate, apparently the result of contact metamorphism. The planes of contact show a great variety of types (see DE WAARD, 1950).

The country rock at a distance from the contact consists of slates. These slates are probably of Cambrian age, according to DEMAY (1931, 1948). In general they are dipping to the N.E., and form part of a large complex of slates in the S. Cevennes.

Dikes cut through the slates and the gneiss. It is possible to distinguish older dikes with cleavage and boudinage — associated with the gneiss, from posttectonic dikes, associated with the Mont du Liron pluton.

The geological history in this area may be summarized as follows. During the early Palaeozoicum sedimentation of clay and argillaceous sand took place. Before or at the beginning of the (probably) sudetic phase of the variscan orogeny the intrusion occurred of a small granite pluton with associated dikes. This pluton affected the country rock,

forming a margin of contact metamorphic sediments. During subsequent folding with regional epi metamorphism the slates and the orthogneiss

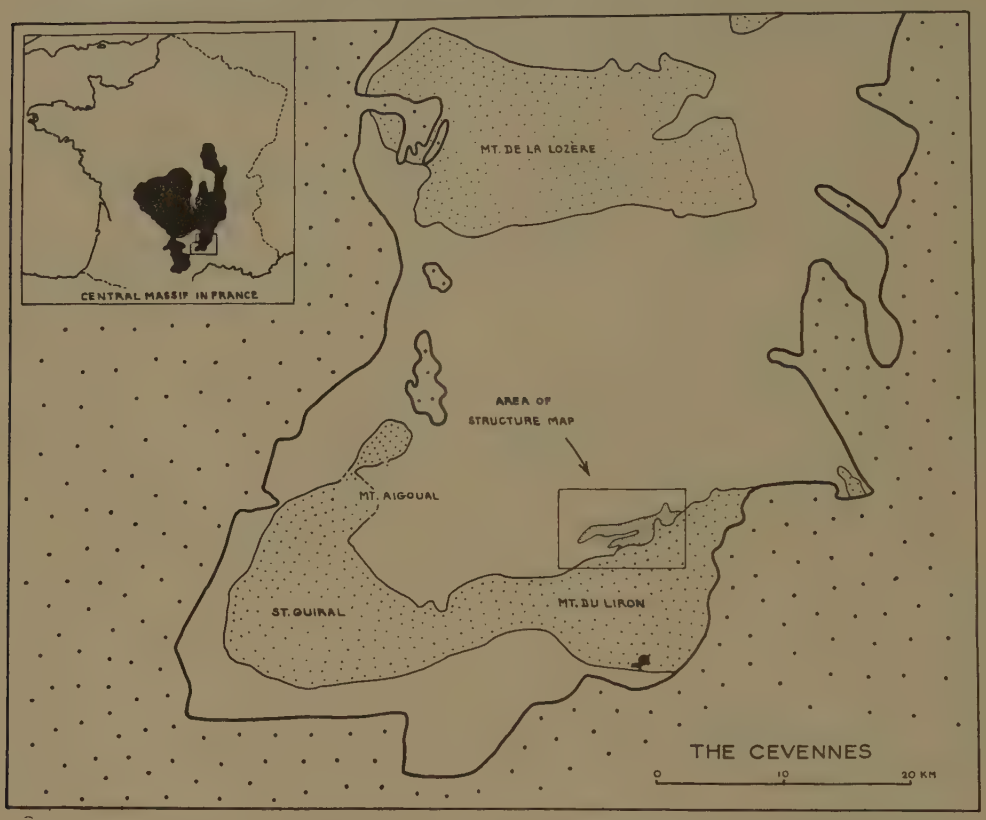

Fig. 1. Map of the southeastern part of the Central Massif showing the geographical position of the mapped area between slates (white) and posttectonic granite massifs (dotted). The surrounding Mesozoic strata are indicated by heavy dots.

were formed. After this the Mont du Liron pluton was intruded, which proved to have tectonic and petrographic characteristics very similar to those of the neighbouring posttectonic Mont Aigoual pluton (Heim, 1949, de Waard, 1949 a, b).

3. *Description of the rocks.*

Country rock.

The country rock is a fine-to-medium grained slate in which both the effects of crystallization and those of deformation are visible. The slates consist of quartz (sometimes interlocking), muscovite, sericite, biotite, chlorite and ore. When seen in a slide the quartz content, though varying, always makes out the bulk of the rock. The larger crystals are mostly streched and show undulatory extinction. Usually the larger crystals are spotted with rows of very small bubbles, passing on from one crystal

to another. Occasionally a cataclastic structure with apparent recrystallization is seen.

The muscovite content is small. In microfolds the bands contain perspicuously less muscovite than the limbs. Occasionally sericite was found in large quantities.

Biotite, mostly present in small flakes, forms a minor constituent of the slates. The larger crystals and a part of the small ones do not follow the schistosity, standing at angles between 45 and 90 degrees. The biotite crystals are often converted into chlorite (see Table I).

We mostly find that the minerals have been mixed arbitrarily in the rock, but occasionally we see bands of quartz alternating with those of muscovite, sericite, biotite and some quartz, the border between two bands showing an excess of sericite.

According to GRUBENMANN (1910) these slates are in part sericite quartzites, in part sericite phyllites ("Glimmerschiefer"). Following the mineralfacies system (ESKOLA, 1939, TURNER, 1948) we arrive at the greenschist facies (the biotite-chlorite and the muscovite-chlorite subfacies). The original given on the Alais sheet is "schistes cristallins". ROQUES (1941) classifies them under his "micaschistes superieur".

In the country rock small folds with an amplitude between 5 and 10 mm are seen here and there. These folds are interbedded between undisturbed layers of the slates, whereas the schistosity of the slates is parallel to the axialplanes (fig. 2). Perhaps we still see here the original bedding. ANGEL

Fig. 2. Microfoldings in the slate.

(1937, p. 190, fig. 34) pictured similar phenomena and interpreted them as shearing-planes folding.

Garbenschiefer.

Garbenschiefer is only found on a few outcrops, with small "Garben", maximum 5 mm long. These slates consist of sericite, muscovite, chlorite and biotite and small laths of ore, probably ilmenite, with very small

TABLE I

	1	2
quartz	24 %	37 %
biotite	75 %	30 %
muscovite		
sericite		30 % *)
chlorite		
accessory minerals	1 %	3 %

1 = country rock.
2 = slate of the contact aureole.
*) part of it sericitized feldspars.

grains of titanite. Around these laths of ore we find chlorite and bleached biotite, not lying with their longer axis in the direction of the cleavage, in contrast with the laths of ore (photo 2).

It seems probable that we are dealing with an altered amphibole, rich in TiO_2, as we found a few aggregates of chlorite and biotite with an outline ressembling a hornblende crystal.Apparently no movement after this alteration of the amphibole took place because of the chlorite and biotite not being oriented.

Slates of the contact aureole.

Near and at the contact with the gneiss the macrostructure of the slates converts to a spotted one. We see a great many small black spots fairly separated from each other. In the immediate vicinity of the contact the spots are larger and more distinct than at some distance away from it. Under the microscope the spots are oblong and consist of sericite. The biotite content is a little less than that of the surrounding parts of the rock. It is rather peculiar that the black spots become light ones under the microscope (photo 1).

Besides the minerals listed under the heading, country rock, we find occasionally a small quantity of *K*-feldspar, albite (5—10 % an), andalusite, cordierite, garnet, apatite or calcite. The feldspars are always strongly sericitized. Andalusite occurs in coarse, columnar aggregates and sometimes as euhedral crystals. It shows a fine pleochroism. Cordierite is present as small polygonal grains, colourless without alteration (see Table I).

In some slides, part of the biotite and the sericite does not lie in the direction of the cleavage.

Occasionally we also find near the contact slates without spots, having in addition to the minerals of the slates feldspars and garnet. The feldspars are always strongly sericitized. Generally, more sericite is found in these feldspar-bearing slates than in the country rock. The garnet has the same appearance as in the spotted slates. These contact slates may be called feldspatized garnet-muscovite-biotite slates.

Gneiss.

The rock is normally light coloured, gray to yellow, sometimes red, gneissose, generally fine grained with or without augen of quartz or feldspar (always smaller than 3 cm). Deformation of the feldspar augen along the cleavage planes is macroscopically visible. The quartz augen are lenticular-shaped. The quartz content is large. It forms together with the orthoclase and some plagioclase (5—10 % an) the bulk of the rock. The quartz is present both in layers and in augen. Sometimes the augen although macroscopically visible disappear under the microscope. The same holds for the schistosity. The quartz augen are formed by a few large crystals with a distinct undulose extinction and small grains on both sides of the augen (in the direction of the schistosity), thus showing a mortar structure. Part of these large crystals are rounded. Occasionally an orthoclase crystal is present in a quartz auge. The quartz in the layers usually consists of small interlocking grains and is locally of a recrystallized mortar structure. The feldspars are more or less sericitized. They do not form layers but have been distributed arbitrarily. Occasionally we see an auge of a streched orthoclase with bands of small quartz grains along the cleavage planes. A few myrmekites and microperthites have been found. The muscovite and sericite occur in bands and are also present in small separate flakes, between the quartz and feldspars. Biotite, in small quantities, is present in small flakes. Andalusite, in larger quantities, is only found in the contact gneisses. The larger andalusite crystals lie in the schistosity. Further constituents are apatite, chlorite and limonite (see Table II).

TABLE II

	1	2	3	4	5	6
quartz	46 %	46 %	50 %	45 %	36 %	43 %
orthoclase *) . . .	44 %	49 %	40 %	50 %	56 %	45.5 %
albite	1 %	—	—	—	—	—
muscovite and sericite	9 %	5 %	2 %	5 %	8 %	11.5 %
biotite	—	—	6 %	—	—	—
andalusite	—	—	2 %	—	—	—

*) The greater part of the *K* feldspars are orthoclase.

Dikes.

These dikes can be distinguished by their schistosity and their boudinage, both are the result of the folding, from the others associated with the Mont du Liron granite. The dikes with a boudinage display rows, more or less straight, of lenses consisting of that particular dike material.

Only those associated with the gneiss will be dealt with.

Granite dikes.

These dikes usually show a boudinage and there is always a marked banding. The mineral content is the same as found in the gneiss, evidently because they are of the same origin. The boundaries of the gneiss against the country rock are sharp and often follow the schistosity of the slates exactly. Usually rather small dikes are found, which lie in the schistosity of the slates. They are found around the gneiss massif (see DE WAARD, 1950).

Quartz porphyry dikes.

In the slates near the gneiss massif we find an outcrop of this dike. Large, rounded, blue quartz phenocrysts have often been built up out of one crystal and some recrystallized very small grains (mortar structure). We further find occasionally acid plagioclase (5 — 10 % an) augen consisting either of one large, rounded crystal — a little sericitized, with on both sides in the direction of the schistosity very small quartz grains —, or of some moderate large plagioclase crystals giving the impression of being torn apart. The groundmass consists of very small quartz grains with sericite and muscovite, "streaming" around the phenocrysts (see Table III).

The large quartz crystals in the slides show rounded borders, bays, inclusions of sericitized feldspar and a very marked undulose extinction. We can distinguish: a) small irregular inclusions, b) regular semi-circular inclusions, c) embayments and d) narrow canals (photo 3 and 4).

The irregular inclusions (a) and the narrow canals (d) are probably of the same origin, and can partly be explained as the skeleton crystals of LAEMMLEIN (1933), and partly by the hypothesis of HEIM (1949), of the drifting together of smaller crystals. With a few exceptions the quartz crystals do not look like having been built up out of other smaller units, they only show undulatory extinction. It seems rather peculiar that units, drifted and welded together according the mechanism proposed by HEIM, should show exactly the same orientation and a strong undulose extinction as well. The explanation given by LAEMMLEIN seems to cover most cases, except for those crystals that have clearly been built up out of some smaller units.

The regular semi-circular inclusions (b) form an other group. Parts of the borders of the crystal with the inclusion show an outline formed by crystalfaces (photo 4). The orientation of the crystal on both sides of the inclusion is the same and a drifting together of smaller crystals is rather improbable, as already pointed out above. The explanation given by HOLMQUIST (1915), a simoultaneous crystallization of quartz and feldspar, will result in an admixure of feldspars during a certain stage of the growth of the crystal. A more natural explanation for this phenomenon is offered by secondary enlargement (see also PETTIJOHN, 1948, p. 39).

The rounded outlines of the crystals found here point to magmatic

corrosion as well as mechanical deformation (mortar structure). MILCH (1905) gives an exposition to show that corrosion is still possible in a rock with quartz phenocrysts and a groundmass rich in quartz, as we have to deal with in this case.

The quartz crystals in this sample show a distinct blue color. This is perhaps due to the dispersion of the light by extremely small bubbles.

Aplite dikes.

The small dikes usually show, besides a boudinage, also ptygmatic folding. We find these dikes both in the slates and in the gneiss. Owing to lack of dark minerals no banding is visible. It is remarkable that the quartz in the aplite exhibits no elongation at all, in contrast with the quartzes of the slates, which show elongated crystals with undulose extinction. Further we see in the slates thin, interrupted sinusoidal aplitic veins, parallel to each other and to the schistosity (fig. 3).

Fig. 3. Interrupted sinusoidal aplitic veins and irregular aplitic inclusions.

Quartz veins.

Only small veins have been observed. They also show ptygmatic folding and sometimes boudinage (see DE WAARD, 1950, photo 2). In the gneiss we find peculiarly formed veins.

Lamprophyre dikes.

In the gneiss we see a lamprophyre with a boudinage and with a distinct schistosity. The boundaries against the gneiss are partly wavy, partly with protuberances into the gneiss (photo 5). The schistosity of this lamprophyre along the boundary does not always coincide with that of the gneiss. It seems that by the "Durchbewegung" some quartz crystals of the gneiss were introduced into the lamprophyre. This boundary suggests large differential movements (photo 6).

The mineral content points to a much altered quartz-kersantite with

a marked schistosity. Very small crystals of biotite, sericite and quartz form almost the whole rock. Furthermore we find feldspars altered to calcite and quartz, rather large flakes of biotite and some chlorite. Occasionally a large quartz crystal, shows a strong undulose extinction, corroded margins, and bays as already described above. One biotite flake has been surrounded by quartz and here a drifting together seems to have taken place.

TABLE III

	1	2
quartz	39 %	53 %
K-feldspar	43 %	24 %
albite	6 %	6 %
muscovite and biotite	12 %	17 %
andalusite	—	±

1 = granite porphyry dike
2 = quartz porphyry dike

4. *Conclusions and summary.*

Around the gneiss massif we find a margin of spotted and feldspatized slates. In comparison with the ordinary slates we thus find feldspars, andalusite, cordierite and a little more biotite. As already pointed out by ROSENBUSCH (1877) it is not necessary to assume an apport of material. This holds in particular for the andalusite, cordierite and biotite. In the cases where much feldspar was found — and this is only very near the contact — it does not seem wrong to suppose an apport —. Because we find these slates only as an aureole around the gneiss we may conclude that the gneiss has supplied the material to form these minerals.

This fact signifies that the gneiss once was a more or less liquid matter with a granitic composition, intruded into the country rock. With the intrusion the associated dikes were formed.

After this intrusion with contact metamorphism the area was affected by the probably sudetic folding with regional epi metamorphism and mechanical deformation. The epi metamorphism and deformation are clearly indicated by the large quantities of sericite, chlorite, the strong undulose extinction and mortar structure of the quartz crystals in the slates. The granite became an orthogneiss, showing augen of quartz and feldspars, often with relicts of mechanical deformation.

The spots of the aureole-slates were sericitized and deformed. The regional metamorphism, however, was not strong enough to obliterate all the effects of the contact metamorphism. Now it is not possible to tell whether feldspars or andalusite, or both of them were responsible for the forming of the original spots.

The dikes associated with the granite (now gneiss) now show schistosity

and boudinage, and it is not possible to establish the sequence of the dike formation, because of this orogeny.

Comparing the mineralogical content of this gneiss with the figures given by ROQUES (1941) we find a rather close resemblance to his anatexites. ROQUES only mentions microcline as the K-feldspar ($2\,V = 80°$). In contrast with it I find that the greater part of the K-feldspars are orthoclase (determination by the Federow methode of the crystallographic orientation of the indicatrix).

From the short description given by DEMAY (1931) of the gneiss found more to the North of the mapped area (the zone of gneiss partly on the Alès sheet, partly on the Largentière sheet), it seems right to conclude that both the gneisses have the same mineralogical content and petrological properties.

Acknowledgments.

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EXPLANATION

Photo 1. Feldspatized slate of the contact aureole, crossed nicols, 20 ×.

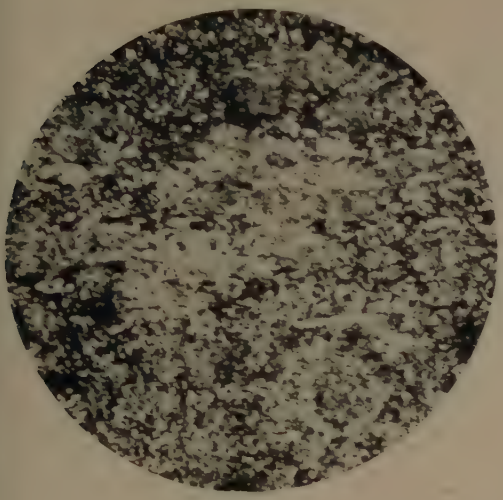
Photo 2. "Garbenschiefer", crossed nicols, 75 ×.

Photo 3. Quartz phenocryst with semi circular inclusion and embayments, crossed nicols, 20 ×.

Photo 4. Quartz phenocryst with inclusions. Notice the boundaries of the large inclusion which are composed of crystalfaces. In part suggesting drifting together of smaller units, crossed nicols, 20 ×.

Photo 5. Protuberance of the lamprophyre into the gneiss, crossed nicols, 20 ×.

Photo 6. Quartz of the gneiss dragged into the lamprophyre, crossed nicols, 20 ×.



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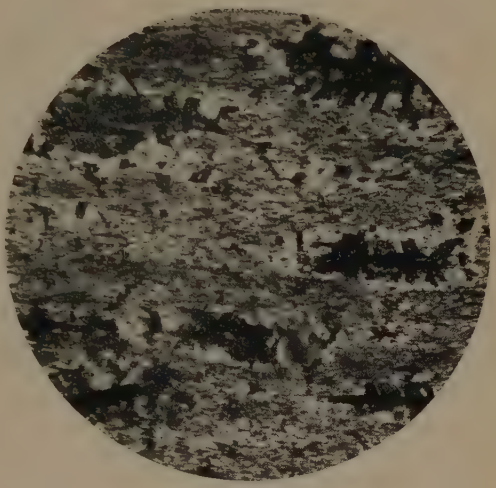

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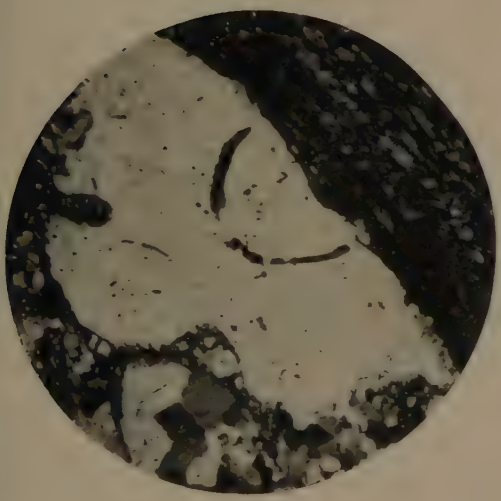

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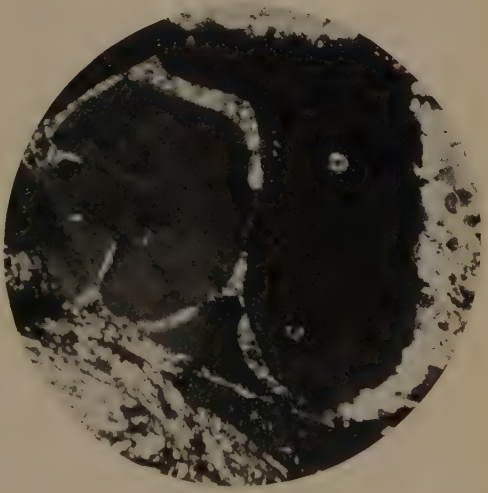

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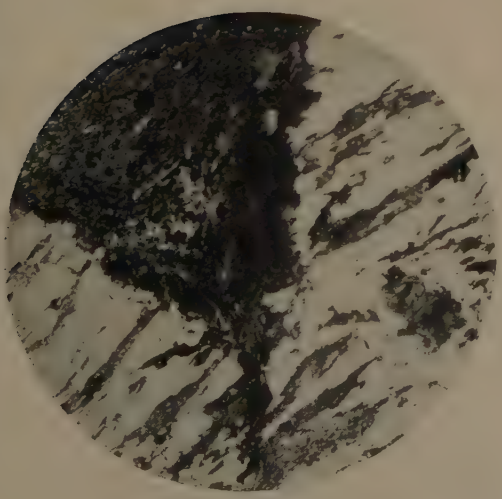

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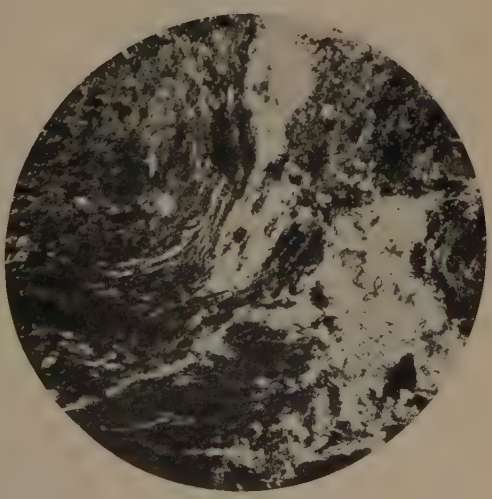

6

THE TEMPERATURE AND HUMIDITY PREFERENCES OF CERTAIN COLEOPTERA

BY

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(Communicated by Prof. CHR. P. RAVEN at the meeting of February 25, 1950)

1. *Introduction.*

Temperature and humidity are well known to be factors of major importance in the lives of terrestrial insects. In this field of research much work has been done on the temperature- and humidity preferences of insect pests, but thus far only a few free living species have been investigated. In the course of previous work by the author on *Forficula auricularia* L., the question arose to what extent the temperature- and humidity preferences might show a marked relation to the natural surroundings of insects.

To this end four species of Coleoptera, living in about the same habitat and being available in sufficient numbers, were chosen for these experiments, the points of investigation being: *a*) Do these insects have similar temperature- and humidity preferences? *b*) Do these preferences show any relations to the temperature and humidity of their habitat?

According to DEAL (1941) the term "preferendum" has been proposed by C. B. WILLIAMS in 1922, while more recently D. L. GUNN suggested "eccritic" temperature. HERTER (1923—'25) used "Thermotactisches optimum", BODENHEIMER c.s. (1928—'31) "Vorzugstemperatur" and NIESCHULZ (1935) "Wahltemperatur". WIGGLESWORTH (1941) called it: "preferred temperature". I will use in this paper the terms: thermo- and hygro-preferendum, (abbreviated: T.P. and H.P.)

2. *The thermopreferendum.*

a) Previous literature.

DEAL (1941) gave an extensive summary of previous publications, so only the more recent ones will be mentioned here: WIGGLESWORTH (1941) tested *Pediculus humanus corporis:* T.P. 25—30° C. According to GUNN and WALSHE (1942), *Ptinus tectus* showed a temperature preference of 3—8° and 22—26° C., the former value probably being that of immobilisation of the insects by low temperature.

HERTER (1943) determined the thermopreferendum of a large number of beetles and bugs. VAN HEERDT (1946) tested *Forficula auricularia:*

26—33° C., Buettiker (1948): *Culex pipiens* and finally Bro Larsen (1949), working on certain Noctuids. It is a rather peculiar fact that the results of experiments with one species may show a notable variation. The thermopreferendum of *Ptinus tectus* stands at 8° C. according to Deal (1941); it is recorded to be 3—8° C. and 22—26° C. by Gunn and Walshe (1942) and lastly 30° C. by Herter (1943). The explanation of this phenomenon may be found in differences of physiological condition of the insects or of the construction of the gradient apparatus used by the various authors. On the other hand, one may find a better agreement between the results, for instance in *Forficula auricularia*, which insect, owing to conditions of war in Western Europe, has been tested independently by three investigators: Deal (1941): 25—30° C.; Herter (1943): 31.79 $\pm$ 0.18° C. and Van Heerdt (1946): 26—33° C.

b) Description of the temperature gradient apparatus.

The apparatus used is made of a bar of aluminium 120 cm long, 5 cm wide and 3 cm in thickness. On this bar a case of zinc has been built, leaving free a 10 cm piece at each end of the bar. This runway is covered by a strip of glass $\frac{1}{2}$ cm in thickness. One free end is placed in a small insulated ice-box, the other is wound with resistance wire and heated by the mains electricity supply.

The thermometers in this type of apparatus are sunk in holes, drilled in the bar and no false floor is used. This is of great advantage in this special case, as these Coleoptera tend to press their bodies against the floor of the gradient apparatus and mainly percieve the temperature of the aluminium bar. Moreover, this construction leaves the floor free of obstacles, an important fact, as these beetles show strong thigmotactic tendencies.

The actual runway being 100 cm long, was divided by painted lines into twenty parts, of 5 cm each: these lines were numbered from 5 to 95. The thermometers were placed at:

5—15—25—35—45—55—65—75—85—95 cm, the average temperatures being: 3—8—12—16—20—24—29—35—42—50° centigrade.

This gradient could be kept constant for a long period, the cold end incidently showing deviations of 2 or 3° C., when the ice in the box was melting down.

It is impossible to have a temperature gradient, without also having an inverse gradient of relative humidity, i.e. the R.H. will be lowest at the warm end of the gradient and highest at the cool end. The only way of maintaining a constant R.H. in a temperature gradient would be either completely drying the air or keeping it saturated (cf. Gunn, 1934; Deal, 1941). It seems then that the observed preferred temperature represents a kind of balance between a pure temperature reaction and a humidity reaction. However, it is the author's opinion that in fresh,

well fed animals the humidity reaction in the temperature gradient apparatus is fairly obscured by the temperature reaction. No relative humidity has been recorded in this series of experiments.

c) Methods.

The experiments were conducted in a room with 3 north windows. The temperature gradient apparatus was placed at a distance of 7 m (about 20 ft.) parallel to the windows and was kept completely dark by covering it with board, cut to fit exactly over the glass plate. This cover was lifted for the purpose of making observations. A dim uniform light was then cast over the gradient, which was an advantage, as a few moments after the cover had been lifted, the insects became restless, so no stirring device was needed.

Each experiment was carried on for one day only, the tested insects being fed and replaced in their terrarium after this period. If the amount of data thus obtained were considered too low, the experiment was continued a few days later with the same individuals.

This is of major importance as many insects, when even so slightly desiccated, tend to lower their preferendum. (Without doubt this is the explanation of the phenomenon stated by DEAL (1941, p. 349) in *Apanteles congestus*).

Every half-hour the positions of the insects were recorded, the first record usually being taken no sooner than one hour after the animals had been introduced into the temperature gradient. Controls showed all of the 4 species of beetles concerned to have definite end preferences, owing to their strong thigmotactic reactions.

Not more than 10 individuals were used in each series of experiments, these species being too large to allow more of them at the time in the apparatus. The number of readings and the temperature were originally plotted against position in centimeters. Each time, when the experiments had been terminated, the apparatus was thoroughly sponged out with cottonwool, soaked in 76 % alcohol and the runway was left open till next morning.

HERTER, BODENHEIMER c.s. and other German workers used to express the T.P. as an average obtained by mathematical operation of the total number of readings along the gradient, e.g. *Liogryllus domesticus*: 35.98 $\pm$ 0.62° C. The statement of the T.P. as a single point, correct to two places of decimals, is of little ecological significance, as it is a range rather than a point (DEAL, 1941, p. 352; BUETTIKER, 1948, p. 64), and knowledge of the range may give us information on the habitat of the insect.

The positions of the beetles in the temperature gradient are expressed in a block graph, every unit in the abscissae corresponding with one division in the temperature gradient. In this way the number of animals staying in each division is shown directly in the graph.

In addition the preferred temperature has been estimated by a mathematical method:

Mean value of the T. P., $M = \frac{\Sigma nT}{N} \pm \frac{S}{\sqrt{N}}$, where $N =$ total number of readings, $n =$ number of readings in each division of the gradient apparatus, $T =$ temperature in each division and $S = \sqrt{\frac{\Sigma n (T-M)^2}{N-1}}$ (standard deviation).

The preferred range, which comprises 67 % of the readings, is limited in this case between $M \pm S°$ C.

It should be understood that the average value of the T.P. has been noted in this case only as a means of comparison with the results of other workers.

3. *The hygropreferendum.*

a) Previous literature.

In the field of humidity preference, there is little work compared with that on preferred temperature, probably owing to hygrometric difficulties. Leclercq (1947) gives an extensive review of the existing publications.

Many papers on humidity preference concern insect pests e.g. Martini and Teubner (1933), De Meillon (1937), Hundertmark (1938), Thomson (1938), Buettiker (1948), working on various kinds of mosquitoes and Gunn and Cosway (1938), Pielou and Gunn (1940), Bentley (1944) on insects in stored products. It has been observed that many insects in normal condition are fairly indifferent to humidity, but show a positive reaction when desiccated (e.g. *Blatta orientalis*, Gunn and Cosway, 1938; *Forficula auricularia*, Van Heerdt, 1946).

The same fact may be noted in species, which are normally hygronegative (e.g. *Tenebrio molitor*, Pielou and Gunn, 1940; *Ptinus tectus*, Bentley, 1944).

Anopheline mosquitoes (according to Buettiker, 1948) preferred in summer 100 % R.H., for hibernation, however, they went to drier places (60—80 % R.H.).

The results of a number of experiments on preferred humidity at different temperatures (Thomson, 1938: *Culex fatigans;* Pielou and Gunn, 1940: *Tenebrio molitor*) suggest that the animals react to the relative humidity rather than to the saturation deficiency. This is an interesting fact, as the water loss caused by evaporation is usually proportional to the S.D.

b) Methods.

Description of the linear humidity gradient apparatus: In the first experiments which were made, the Coleoptera were offered a wide range of humidities. A trough, made of zinc, 60 cm long and 10 cm wide was

provided with a perforated false floor, also made of zinc. There was a distance of about $2\frac{1}{2}$ cm from the false floor to the glass plate covering the trough. This is an important factor, as owing to convection a greater distance counteracts the establishment of a stable gradient. A small door was made in one of the short sides of the apparatus, making possible the introduction of animals without disturbing the humidity gradient.

Under the false floor 6 petri-dishes were placed, containing a series of saturated solutions of various kinds of salts. The gradient, thus obtained, however proved to be not quite satisfactory, so another arrangement was used. The three petri-dishes at one side of the gradient apparatus were filled with $CaCl_2$, those at the opposite side with distilled water, giving a gradient as shown below [1]):

$$CaCl_2 - CaCl_2 - CaCl_2 - H_2O - H_2O - H_2O$$
$$45-45-45-50-62-70-80-88-95-100-100\ \%\ \text{R.H.}$$

This gradient is fairly steep and gives satisfactory results. The R.H. has been recorded by means of a hair hygrometer, which was usually kept at 75 % R.H. This hygrometer was moved up and down the humidity gradient by means of a couple of strings passing through small holes, drilled in the short sides of the apparatus. The hygrometer records a different value (ca. 4 % R.H.) at a certain point, when moved either up or down the gradient. The humidities noted are the average of these deviations. A drawback in this apparatus is found in the "end preference" of the Coleoptera, which tend to gather in the corners.

The "alternative chamber": This apparatus has been described by many workers (Gunn and Kennedy, 1936; Gunn and Cosway, 1938; Pielou and Gunn, 1940; Wigglesworth, 1941) and has been used in the course of these experiments with minor adjustments only, so no description is needed in this paper.

The experiments were conducted in the same room and in the same situation as described for the temperature gradient apparatus, the linear humidity gradient apparatus and the alternative chamber being covered with black paper. Half way through each experiment each apparatus was rotated through 180° in case there were any differences.

These experiments, unlike the temperature preference tests, were carried on for two days. The Coleoptera, if well fed, showed only a weak reaction to humidity. After one day in the gradient, a slight desiccating effect became evident and clear results could be obtained. Every half-hour the positions of the objects were recorded, the first record usually being taken after the animals had been in the gradient for about 24 hours. Controls showed a 50—50 distribution in both sides of the apparatus, end preference being strong, as may be seen in the results. The same number of individuals was used as in the previous experiments.

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[1]) I am much indebted to Dr D. L. Gunn, who gave me useful advice on this subject.

The runway of the linear humidity gradient apparatus was divided into 6 parts of 10 cm each. The numbers of insects were plotted according to their position, in divisions 1—6.

The intensity of reaction can be expressed in two ways: (1) as the ratio of position records on the wetter side to those on the drier side (W/D ratio: GUNN, 1937) and (2) as the excess percentage of records on the wetter side ($\frac{100(W-D)}{W+D}$: GUNN and COSWAY, 1938). (see table 1, 2, 3 and 4). If one half of the trough or chamber is not preferred to the other, the animals should give about 50 % readings in each half. Divergence from this ratio may be tested by the following formula for the normal deviate: $t = \frac{W - W_{th}}{\sigma}$, where W = number of individuals on the wetter side, if this side is preferred; W_{th} = number of individuals, if one half of the apparatus is not preferred to the other (about 50 % readings in each half) = $\frac{1}{2} n$; $\sigma = \sqrt{\frac{1}{4} n}$ (n = total number of readings).

If $t \geqq 3.29$, the probability (P) of this result by random occurrence is $\leqq 1 : 1000$, hence the divergence may be considered significant.

4. *Insects used:*

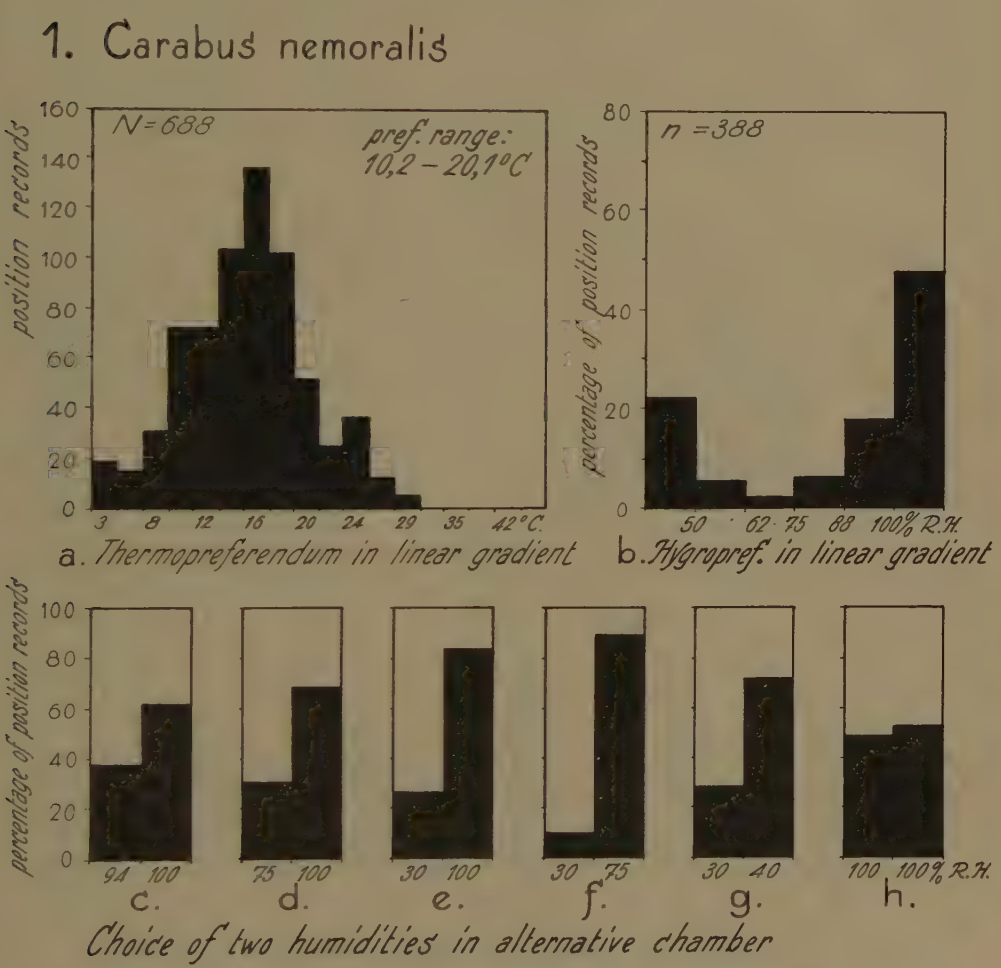
Four species of Coleoptera living in the layer of fallen leaves of a Querceto-betuletum (Baarn, Holland) were used in these experiments: *Carabus nemoralis* Mull., *C. catenulatus* Scop., *Abax ater* Vill. (Carabidae) and *Geotrupes sylvaticus* Panz. (Scarabaeidae). Only newly caught and well-fed individuals were tested in order to eliminate the influence of humidity in the temperature gradient apparatus. During their stay in the laboratory the beetles were kept in a vivarium, where the floor had been covered with a 10 cm layer of damp soil and on top of this, a layer of half-decayed leaves, thus imitating their natural surroundings. The *Carabids* were fed on dead cockroaches or meat, *Geotrupes* on fresh sheep manure. Mortality was low during their stay in the vivarium, the insects apparently being in good health.

The beetles could easily be caught by means of flower-pots dug into the soil with their tops flush with the surface.

5. *Results.*

Carabus nemoralis shows a definite T.P. of 10.2—20.1° C. average 15.2 $\pm$ 0.2° C. This is low compared with the other 3 species tested, except *Abax ater*. (Fig. 1*a*). *C. nemoralis* does not show any avoiding reaction for low temperature, probably because the temperature in the gradient apparatus was not low enough. For high temperature a strong klinokinesis has been recorded for 35° C. In the humidity gradient a hygropreferendum for 100 % R.H. has been observed. (Fig. 1*b*). When tested in the alternative chamber, although the greater number of the insects were gathered in the moister zone, the results were not always significant when a choice was offered of 100 and 94 % R.H. and of

100 and 75 % R.H. (Fig. 1*c*, *d*). A small deviation was observed in the case of a choice of 30 and 75 % R.H. (Fig. 1*f*) where a more intense reaction was initiated than when a choice was offered of 30 and 100 % R.H. (Fig. 1*e*); it may be suspected that a certain difference in physiological condition of the insects was the cause.

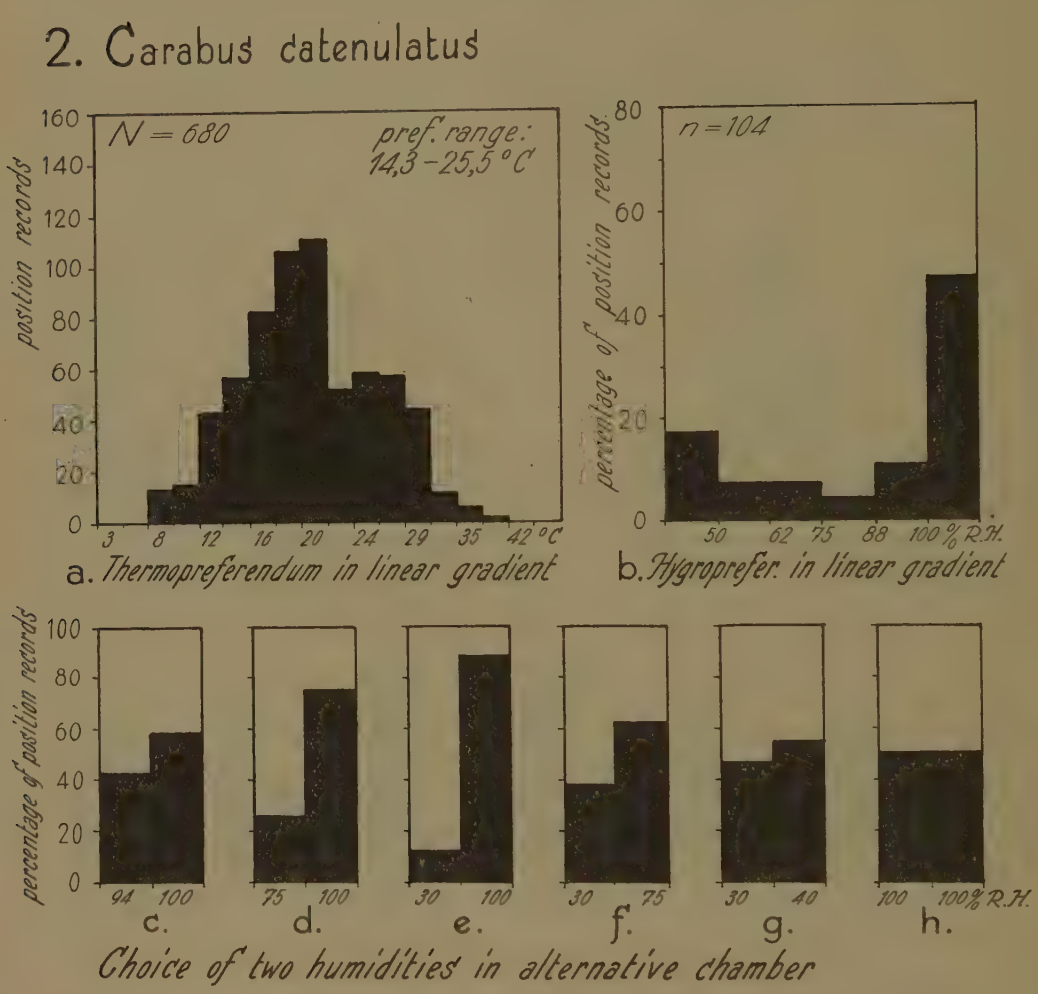


When offered a choice of low humidities, 30/40 % R.H. (Fig. 1*g*) the beetles still give a rather clear (though not quite significant) reaction to the moister side. A control gives a near 50/50 distribution (Fig. 1*h*).

The temperature preference of *Carabus catenulatus* amounts to 14.3—25.5° C. average 19.9 ± 0.2° C. (Fig. 2*a*). Klinokineses are shown for 5—8° C. and 35—42° C. Compared with *C. nemoralis*, this species has a more extensive and somewhat higher T.P., though they are very similar in size and biotope. In the humidity gradient there is a significant hygropositive reaction (Fig. 2*b*). When tested in the alternative chamber the majority tend to congregate at the moist side of the apparatus.

(Fig. 2*c*, *d*, *e*, *f*). At a choice of 30 and 40 % R.H. only a weak reaction may be recorded, but the intensity of the reaction grows with increasing difference of humidity. Controls give an exact 50/50 distribution (Fig. 2*h*).

Abax ater again has a more confined range of temperature preference: 9.7—20.6° C. average 15.1 ± 0.2° C. (Fig. 3*a*), the lowest of the four



species concerned. A few (not all) individuals show an avoiding reaction for 5° C. At the hot side a klinokinesis is observed for as low a temperature as 29° C. A hygropreferendum for 100 % R.H. is shown in the linear humidity gradient. (Fig. 3*b*). The reaction in the alternative chamber is very definite, only the choice of 30 and 40 % R.H. being insignificant. (Fig. 3*c*, *d*, *e*, *f*, *g*). A near 50/50 distribution is obtained in a uniform chamber, (fig. 3*h*).

Finally, *Geotrupes sylvaticus* shows a 11.9—26.9° C. thermopreferendum with an average of 19.3 ± 0.3° C. thus covering a wide range. (Fig. 4*a*).

TABLE 1

Carabus nemoralis

Choice of % R.H.	W/D	excess perc.	W	W_{th}	$\sigma = \sqrt{\frac{1}{4}n}$	t	P
100/94	1.6	23.5	21	17	2.91	1.35	0.1—0.2
100/75	2.2	37.5	33	24	3.46	2.60	ca. 0.01
100/30	5.0	66.6	30	18	3.0	4.00	< 0.001
30/100	5.0	66.6	30	18	3.0	4.00	< 0.001
30/75	9.0	80.0	36	20	3.16	5.06	< 0.001
30/40	2.6	44.4	26	18	3.0	2.66	0.001—0.01

TABLE 2

Carabus catenulatus

Choice of % R.H.	W/D	excess perc.	W	W_{th}	$\sigma = \sqrt{\frac{1}{4}n}$	t	P
100/94	1.3	15.0	23	20	3.16	0.95	0.3—0.4
100/75	3.0	50.0	36	24	3.46	3.46	< 0.001
100/30	7.3	76.0	44	25	3.54	5.36	< 0.001
30/100	7.3	76.0	44	25	3.54	5.36	< 0.001
30/75	1.7	25.0	25	20	3.16	1.58	0.1—0.2
30/40	1.2	8.0	27	25	3.54	0.56	0.5—0.6

TABLE 3

Abax ater

Choice of % R.H.	W/D	excess perc.	W	W_{th}	$\sigma = \sqrt{\frac{1}{4}n}$	t	P
100/94	3.2	52.4	32	21	3.24	3.42	< 0.001
100/75	5.0	66.6	40	24	3.46	4.62	< 0.001
100/30	5.9	71.0	41	24	3.46	4.91	< 0.001
30/100	5.9	71.0	41	24	3.46	4.91	< 0.001
30/75	8.6	78.1	43	24	3.46	5.50	< 0.001
30/40	1.5	20.0	18	15	2.74	1.94	ca 0.05

TABLE 4

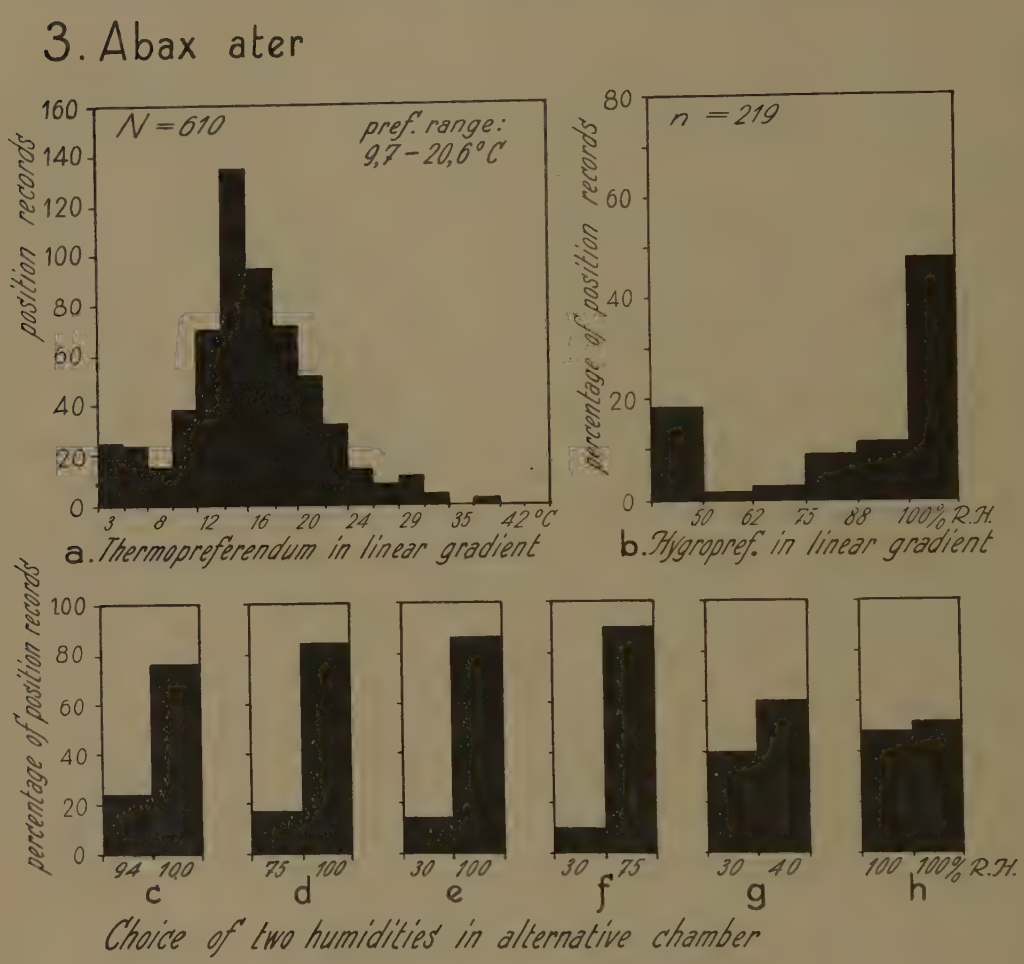
Geotrupus sylvaticus

Choice of % R.H.	W/D	excess perc.	W	W_{th}	$\sigma = \sqrt{\frac{1}{4}n}$	t	P
100/94	2.6	45.0	29	20	3.16	2.84	0.01—0.001
100/75	2.6	45.0	29	20	3.16	2.84	0.01—0.001
100/30	4.6	64.3	46	28	3.74	4.81	< 0.001
30/100	4.6	64.3	46	28	3.74	4.81	< 0.001
30/75	2.2	38.2	38	27.5	3.71	2.83	0.01—0.001
30/40	1.2	8.7	25	23	3.39	0.59	0.5—0.6

W = number of insects on wet side; W_{th} = number of insects in case of even distribution ($= \frac{1}{2}n$).

$$\sigma = \sqrt{\tfrac{1}{4}n} \; ; \; t = \frac{W - W_{th}}{\sigma}.$$

A klinokinesis at the hot side only, could be observed for 45° C., which may be considered fairly high. The humidity reaction in the linear humidity gradient is weak compared with that of the 3 Carabid species, (Fig. 4*b*) the reaction being altogether definite and statistically significant. The area occupied by these dung beetles, however, is less well-defined

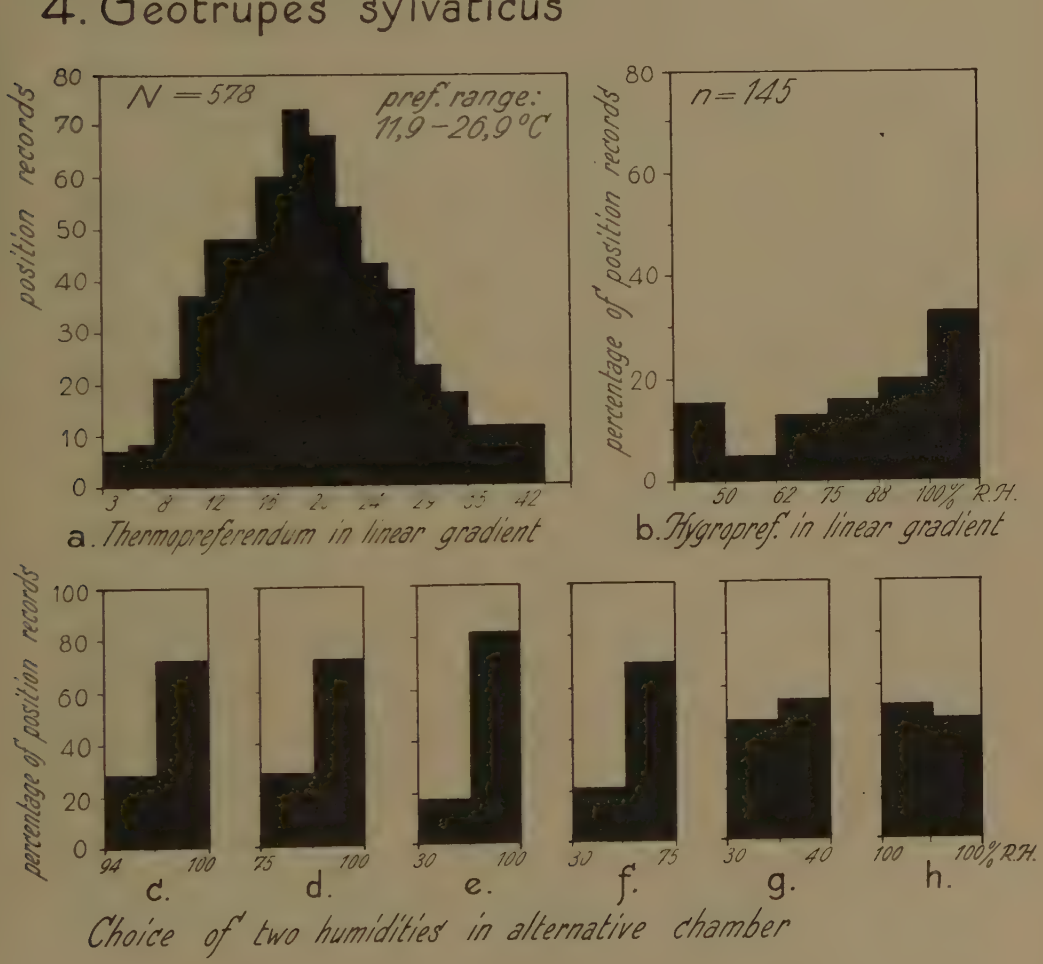


than in the Carabids. In the alternative chamber too, reactions are less definite (Fig. 4*c*, *d*, *f*) only the choice of 30 and 100 % R.H. being really significant. (Fig. 4*e*). Nevertheless, an excess percentage of records on the drier side has never been observed. When a choice is offered of 30 and 40 % R.H. a near 50/50 distribution was recorded, quite similar to the control reaction, (Fig. 4*g*, *h*) so apparently at low R.H. no distinction is made between small differences of humidity.

6. *Discussion.*

It has been proved that the populations of four kinds of Coleoptera,

living in the forest: *Carabus nemoralis*, *C. catenulatus*, *Abax ater* and *Geotrupes sylvaticus* have a preferred temperature which is lower than that of those insects, which live in exposed places (e.g. flies, earwigs) or in stored products (e.g. *Tenebrio*, *Dermestes*). In addition the preferred humidity of the four species has been recorded and certain conclusions have been drawn from a comparative study of these data.



It is evident that the temperature and humidity preferences of these beetles show a marked relation to the corresponding factors in their natural surroundings. Moreover, the thermopreferendum of these 4 species is very low if compared with, for example, *Musca domestica:* 33.5° C.; *Stomoxys calcitrans:* 28° C.; *Musca vicina:* 33° C. (according to NIESCHULZ, 1933—'35); *Culex pipiens* 31° C. (BUETTIKER, 1948), insects living in drier places e.g. *Forficula auricularia:* 26—33° C., (VAN HEERDT, 1946) or those, living in stored products e.g. *Tribolium confusum:* 29° C.

Dermestes vulpinus 30° C. (Deal, 1941); *Tenebrio molitor:* 32° C.; *Anthrenus verbasci:* 32° C.; *Ergates faber:* 35° C., (according to Herter, 1943).

Their humidity preferences, on the other hand, tend to be higher than in the groups of Insects previously mentioned: *Anopheles maculipennis* var. *messeae:* 40 % R.H. (Buettiker, 1948); *Blatta orientalis:* when offered a choice of 20 and 90 % R.H. prefers 20 % R.H. (Gunn and Cosway, 1938); *Tenebrio molitor* choice of 94 and 100 % R.V. prefers 94 % R.H.; *Carabus auratus:* choice of 55 and 86 % R.H. prefers 55 % R.H. (Leclercq, 1947).

Carabus nemoralis, C. catenulatus and *Abax ater* are insects which live throughout the year in the damp layer of fallen leaves of a Querceto-betuletum. The temperature of their biotope never rises higher than about 25° C. even in hot summertime. The relative humidity, which during the day tends to fall considerably, however, rises again at night and soon reaches 100 % R.H. when the temperature drops after sunset. Owing to its hygroscopic abilities, the desiccated upper layer of dead leaves absorbs the humidity of the air and takes up still more water whenever the temperature drops below dewpoint. This water is lost again during the following day, but, as will be easily understood, provides a damp medium, where extreme conditions of heat and drought are never apt to occur. As these beetles are animals of the night, they never show themselves at daytime and, owing to their low temperature and high humidity preferences, they will fit very well into their natural haunts.

Geotrupes sylvaticus, a dung beetle, has a far wider range of temperature and humidity preference. In accordance with these facts, the habits of this insect are quite different from the Carabids previously mentioned. They are often seen at daytime, busily walking about in search of food, which mainly consists of animal droppings. They are not confined to the damp interior of the woods and are more euryoecous than the Carabids.

As a matter of fact a comparative study of the temperature and humidity preferences of the Carabid beetles and of *Geotrupes sylvaticus* reveals a clear difference, the preferences of the former being more restricted than those of the latter.

Comparing the separate results, it can be stated, that *C. nemoralis* shows a lower T.P. than *C. catenulatus*, although both species are about equal in size, behaviour and habitat. Their humidity preferences are much the same, so it is difficult to give a plausible explanation for this phenomenon.

Either of the 4 species, when tested in the linear humidity gradient apparatus (extending from 45—100 % R.H.) shows a humidity preference of 100 % R.H. This preference, generally corresponds with their behaviour in their natural surroundings, the reaction of *Geotrupes sylvaticus* being

weaker than of the Carabids. A definite end preference is observed, owing to the strong thigmotactic reactions of these insects.

In the alternative chamber a marked hygropositive reaction in a humidity gradient extending from 30—100 % R.H. and from 30—75 % R.H. was observed in all cases. A similar reaction in a gradient from 100—94 % R.H. and 100—75 % R.H. is, although always positive, much weaker in *C. nemoralis*, *C. catenulatus*, and *Geotrupes sylvaticus*; *Abax ater* being the only case, in which the reaction is really significant. A choice of 30—40 % R.H., however, always proves insignificant, even in *Abax*. In *Carabus catenulatus* and *Geotrupes sylvaticus* a choice of 30 % R.H. in the presence of 40 % R.H. gives a near 50/50 distribution, only slightly different from that shown by controls in a uniform chamber. Still, stress must be laid on the fact that none of the experiments (about 200 in all) ever showed an excess percentage on the drier side.

Thomson (1938) in *Culex*, Pielou and Gunn (1940) in *Tenebrio* and Wigglesworth (1941) in *Pediculus* observed that their objects show in general a greater sensivity within the higher range of humidity. For example, the louse shows a striking preference of 92 % R.H. in the presence of 95 % R.H. while the mosquito in this range even reacts to differences of 1 % R.H.

In our case too, the reaction in the higher range of humidity is more pronounced than in the lower range (e.g. in *Abax ater*, a choice of 94 and 100 % R.H. gives significant results, but of 30/40 % R.H. proves insignificant). Moreover, it may be pointed out, that the greater the difference of humidities offered, the stronger the reaction, (e.g. in *Carabus catenulatus*, when offered a choice of 100/94 % R.H., 100/75 % R.H. and 100/30 % R.H., the *W/D* ratio is steadily increasing: 1,3, 3.0 and 7.3).

As a whole, the reactions of *Abax ater* prove to be the most positive of the 4 species tested, the results of *Abax*, being, however, very similar to those of *Carabus nemoralis*.

The mechanisms of the reactions, to temperature as well as humidity, clearly show the same kind, that is: Klinokinesis. On the entry of an adverse region, the beetles no longer walk straight, but follow a hesitant course. In some cases a directed response has been observed (abruptly turning to the favorable region). An amputation of sense organs has not been made, so no more can be said concerning this subject.

These results show in a very positive way the existance of a relation between thermo- and hygropreferendum and the natural surroundings in Insects. These Coleoptera, living in the cool, damp layer of fallen leaves in the woods, have preferences quite contrary to those of flying, free living species or stored products pests.

Summary.

1. The populations of *Carabus nemoralis*, *C. catenulatus*, *Abax ater*

and *Geotrupes sylvaticus*, 4 species of Coleoptera, living in the Querceto-betuletum (Baarn, Netherlands) have a low temperature- and a high humidity preference, compared with other kinds of insects which live in more exposed biotopes or in stored products.

2. The preferred ranges of the 3 Carabid species are more restricted than those of Geotrupes, which agrees with the euryoecous way of life of the latter.

3. Although the humidity reaction is not always significant, the number of insects observed at the moister side of the alternative chamber, is always more than the number at the drier side.

4. The reactions to differences in humidity are more definite, when a choice is offered of 2 high humidities than of 2 low ones.

Acknowledgements:

I am much indebted to Prof. Dr Chr. P. Raven and Mr Peter S. Digby for their corrections of the M.S. and to Ir J. J. Bezem for his advice on the mathematical calculations.

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DE F_5-ZAADGENERATIE VAN 1936 NA KRUISINGEN VAN TWEE ZUIVERE LIJNEN VAN PHASEOLUS VULGARIS. II

BY

G. P. FRETS

(Communicated by Prof. J. BOEKE at the meeting of January 28, 1950)

Cl. 2b. 35 gevallen. De uitgangsbonen van F_4-1935 hebben de formule $L_1\, l_2$ B th, cl 2b (tab. 7). In 15 van deze 35 gevallen is de formule van de gemiddelden der bonenopbrengsten van F_5-1936, ook L B th. In één van deze gevallen (pl. 674) is de form. van de gemiddelden $L_1\, L_2$ B th, cl 2a.

Pl. 674. De uitg. boon is van pl. 211 (fig. 7, K 4 en tab. 5d en 5e). Zie ook pl. 675, blz. 12. Ze is de enige boon van de peul van de uitgangsboon, die een grote lengte heeft (dan volgen l = 13.9—13.4 mm). De grootste lengte van de 23 bonen met een zeer grote lengte van pl. 211 is, l = 17.7 mm. Zoals de uitgangsboon, zijn er meerdere bonen van de I-lijn van 1935. Van de bonenopbrengst van pl. 674 is de gemidd. lengte groter dan bij bonenopbrengsten van de I-lijn van 1936 voorkomt (l_m = 15.5 mm). Er is hier transgressieve variabiliteit. De gemiddelde indices zijn als van bonenopbrengsten van de I-lijn van 1936. Dit geldt ook voor de classificatie. Bij de bonenopbrengst staat aangetekend „heel goed". Onder de bonen van pl. 674 is er geen, zoals ze ook niet bij de I-lijn van 1936 voorkomen. De uitgangsboon van pl. 211 voor pl. 674 is volgens het phaenotype van pl. 674 homozygoot voor de formule $L_1\, L_2$ B th van de bonen van de I-lijn, doch bevat misschien meer L L-verbindingen.

Pl. 678. Er staat aangetekend „niets van terecht gekomen". Toch zijn er nog 8 bonen geoogst en gemeten. Voor zover hier een oordeel mogelijk is, is het genotype van de uitg. boon als van bonen van de I-lijn.

Ook bij *pl. 562* (fig. 6, K 10) hebben we met een mooi geval van een bonenopbrengst te doen, die overeenkomt met bonenopbrengsten van de I-lijn.

Pl. 478. De uitg. boon is van pl. 124 (fig. 3, K 15 en tab. 5d en 6d). Zoals de uitgangsboon, zijn er ook bonen van de I-lijn van 1935. Ook de gemiddelden van de bonenopbrengst van pl. 478 vinden we zo bij bonenopbrengsten van de I-lijn van 1936. De bonenopbrengst van pl. 478 heeft 2 bonen met de form. L B Th, cl 1; de dikte, th = 6.6 mm, is dus iets groter dan de grenswaarde (th = 6.5 mm). Eén van deze bonen is in 1937 uitgezaaid en leverde pl. 323, F_6-1937. Zoals de uitg. boon van pl. 478 voor pl. 323 is, zijn er ook bonen van de I-lijn van 1936. De formule van de gemiddelden van pl. 323, is L B th, cl 2. Bonenopbrengsten met dgl gemiddelden zijn er ook bij de I-lijn van 1937. De gemidd. B Th-index van pl. 323, die B Th_m = 71 is, komt onder de gemidd. B Th-indices van de bonenopbrengsten van de I-lijn van 1937 4 maal voor. In de bonenopbrengst van pl. 323 is een boon met de B Th-index B Th = 85. Als exceptie vinden we ook een dgl boon

in een bonenopbrengst van de I-lijn van 1937. Door deze boon voort te kweken kan blijken, of ze een niet-erfelijke variant is van de I-lijn dan wel een erfelijke variatie, een uitsplitsing, aanwijzende, dat de uitgangsboon voor pl. 323 niet geheel homozygoot is (vgl. ook de ascendentie van pl. 478, tab. 5d—5a en 6d—6a).

Pl. 546. De uitg. boon is van pl. 157 (fig. 6, K 10). De bonenopbrengst van pl. 546 heeft bijna alleen bonen in cl 2. De gemidd. breedte is wat groter. Er zijn hoge L B- en lage B Th-indices, (L B = 64—72, B Th = 61—69, dan 67). De bonenopbrengst is gelijkmatig. De uitg. boon van pl. 157 heeft een grote mate van homozygotie voor bonen van de I-lijn. Ze heeft veel B-factoren, een kleiner aantal L-factoren. Dit geldt ook voor de uitg. boon van pl. 124 voor pl. 478 (het vorige geval) en voor *pl. 669* met de uitg. boon van pl. 210 (fig. 7, K 4).

Pl. 765. De uitg. boon is van pl. 284 (fig. 7, K 4). Ook een bonenopbrengst met het phaenotype van de I-lijn. Er zijn hier veel bonen met een hoge B Th-index (B Th = 70 en iets hoger). Ook van *pl. 694* met uitg. boon van pl. 214 (fig. 7, K 4) komen bonenopbrengst en uitg. boon overeen met de I-lijn.

Pl. 1081. De uitg. boon is van pl. 174 (fig. 6, K 10). Alle 4 bonen van de peul van de uitg. boon voor pl. 1081 van pl. 174 zijn uitgezaaid (zie blz. 219). Het zijn, wat de indices betreft, overeenkomstige bonen. De bonenopbrengst van pl. 1081 is „heel goed". Ze heeft het phaenotype van de I-lijn.

Pl. 786. De uitg. boon is van pl. 300 (fig. 1, K 33); zie ook pl. 784 en 785 (blz. 215). Er is veel overeenkomst met bonenopbrengsten van de I-lijn van 1936. Er zijn veel hoge L B-indices en waarschijnlijk ook te veel hoge B Th-indices.

Pl. 775. De uitg. boon is van pl. 295 (fig. 1, K 33). Zie ook pl. 776 (blz. 214). De gemidd. dikte is te groot. Ook de classificatie verschilt iets van die van bonenopbrengsten van de I-lijn (4 bonen in cl 6). Onder de individuele bonen zijn er enkele, zoals er onder de vgl-bonen van de I-lijn van 1936 niet voorkomen (b.v. een boon met l = 14.5 en th = 7.5 mm). De bonenopbrengst wijkt iets af van bonenopbrengsten van de I-lijn van 1936. Dit geldt ook voor *pl. 464*, met de uitg. boon van pl. 121 (fig. 3, K 15). Pl. 121, F_4-1935, bevat enige van de grootste bonen van F_4-1935 (b.v. l = 19.0 en = 18.3 mm; gew. = 112 en 113 cG). De 5 bonen van de peul van de uitg. boon zijn niet zeer groot (l = 13.6—14.5 mm); de eerste boon is de uitg. boon voor pl. 463 (blz. 364).

Pl. 544. De uitg. boon is van pl. 157 (fig. 6, K 10). De gemiddelden van de bonenopbrengst van pl. 544 komen zo ook bij de bonenopbrengsten van de I-lijn van 1936 voor; de gemidd. indices zijn aan de hoge kant. Onder de individuele bonen zijn er enkele, die, in deze combinatie van de indices, bij bonen van de I-lijn van 1936 niet voorkomen. Er is bij de bonenopbrengst van pl. 544 enig verschil met bonenopbrengsten van de I-lijn van 1936.

Pl. 454. De uitg. boon is van pl. 120 (fig. 3, K 15 en tab. 5d en 6d). Ook hier is een grote overeenkomst met bonenopbrengsten van de I-lijn. De gemidd. L B-index is hoog.

Pl. 470. De uitg. boon is van pl. 122 (fig. 3, K 15). De breedte is groot. Van de gemiddelden van de bonenopbrengst van pl. 470 zijn de dikte en de breedte groter dan bij vergelijk-bonenopbrengsten van de I-lijn van 1936. Volgens de classificatie zijn er te veel bonen in cl 1. De bonen zijn „heel goed", enige bonen zijn

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Fig. 9. Characterograms of pl. 765, F_5-1936 and its ascendants. a. characterogram of the means of the beanyield of K_4, F_2-1933. b. charact. of the initial bean of K_4, F_2-1933 for pl. 195, F_3-1934. c. charact. of the means of pl. 195, F_3-1934. d. charact. of the initial beau of pl. 195, F_3-1934 for pl. 284, F_4-1935. e. charact. of the means of pl. 284, F_4-1935. f. charact. of the initial bean of pl. 284, F_4-1935 for pl. 765, F_5-1936. g. the charact. of the means of the beanyield of pl. 765, F_5-1936. s = Standardcharacterogram. These characterograms resemble the characterogram of the I-line.

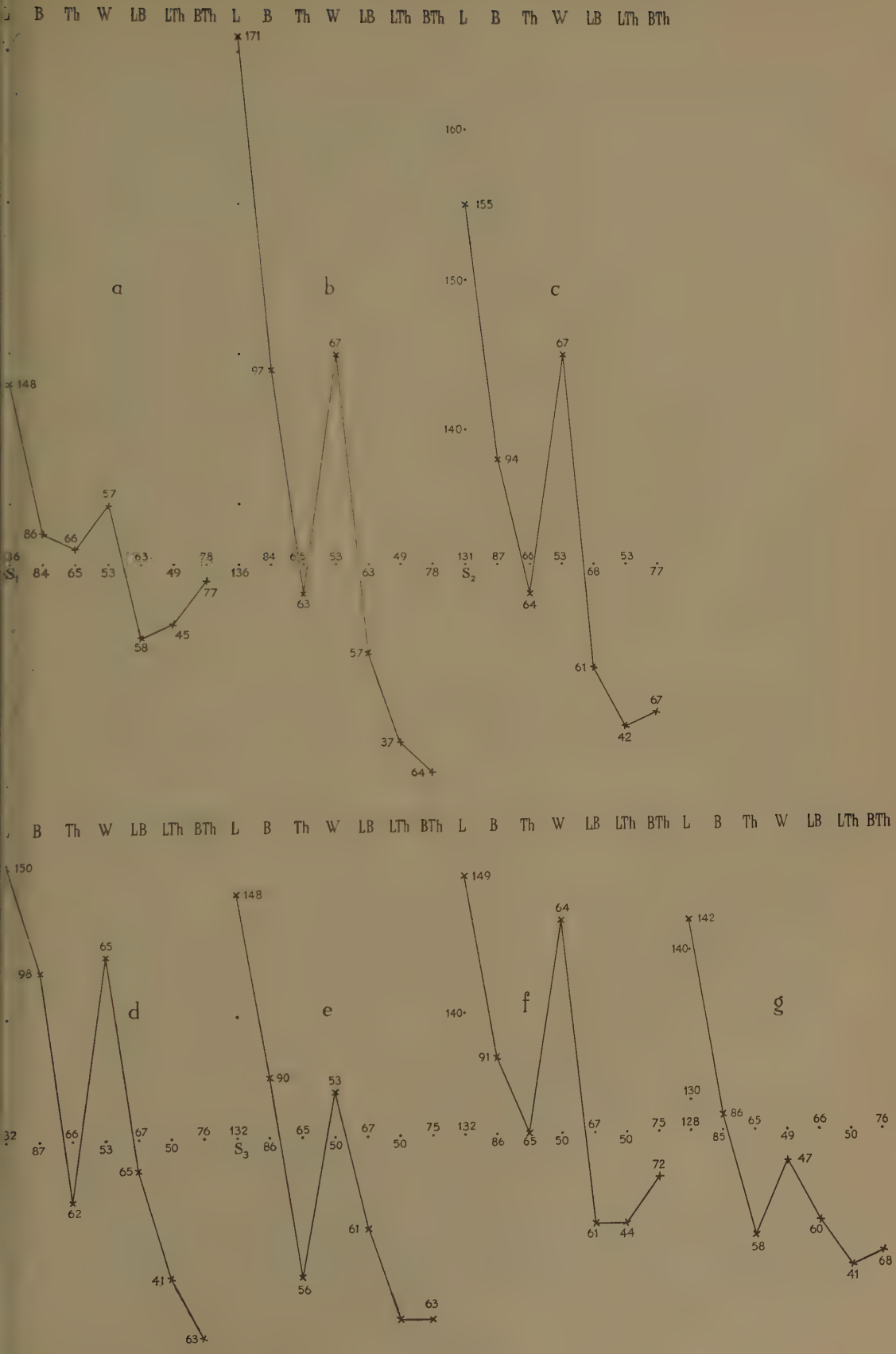
L B Th W LB LTh BTh L B Th W LB LTh BTh L B Th W LB LTh BTh
a
b
c
d
e
f
g
S1
S2
S3

Fig. 9.

„abnormaal". Niet alle bonen komen zo ook voor bij de bonen van de I-lijn van 1936. In 1937 is één boon van pl. 470 voortgekweekt; ze leverde pl. 318, F_6-1937. Onder de bonen van de I-lijn van 1936 zijn er, die overeenkomen met de uitgangsboon van pl. 470 voor pl. 318. Onder de bonenopbrengsten van de I-lijn van 1937 zijn er geen, wier gemiddelden overeenkomen met die van pl. 318. De samenstelling van de bonenopbrengst is gelijkmatig. Het phaenotype van de bonenopbrengst van pl. 318 beantwoordt aan een genotype van de uitgangsboon met veel B-factoren in homozygote vorm en minder L-factoren dan dat van de I-lijn. Ze sluit aan bij pl. 470 en haar uitgangsboon.

Om de bonenopbrengsten van F_5-1936, die, voor de door ons onderzochte eigenschappen, geheel overeenkomen met bonenopbrengsten van de I-lijn van 1936, te kunnen beoordelen, hebben we deze bonenopbrengsten in een tabel bijeengebracht, volgens de grootte van de gemiddelde dikte gerangschikt. Het zijn 28 bonenopbrengsten. In een andere tabel zijn, op overeenkomstige wijze gerangschikt, de 64 bonenopbrengsten van de I-lijn van 1936 opgenomen. Uit deze tabellen is tab. 8 samengesteld. We zien er uit, dat het aantal bonenopbrengsten met een kleine gemiddelde dikte bij de bonenopbrengsten van F_5-1936 kleiner is dan bij de vergelijk-bonenopbrengsten van de I-lijn van 1936. Zo zijn er, — omgerekend tot 64 bonenopbrengsten —, 2.3 bonenopbrengsten van F_5-1936 tegen 8 van de I-lijn, waarvan de gemidd. dikte = 5.5 mm en kleiner; er zijn er 11.4 tegen 28, waarvan de gemidd. dikte = 5.8 mm en kleiner. Voor 6.1 mm en kleiner zijn deze aantallen 38.8 en 53, voor 6.2 mm 45.7 en 57 (tab. 8). Onder de bonenopbrengsten van F_5-1936, die, voor de onderzochte eigenschappen, geheel overeenkomen met bonenopbrengsten van de I-lijn zijn er te veel met de grote gemiddelde dikten. Niet al deze bonenopbrengsten zijn dus bonenopbrengsten als van de I-lijn. Een dergelijke bewerking voerden we voor de gemiddelde L Th-indices uit. Hier is het verschil tussen het kleine aantal bonenopbrengsten met een kleine gemiddelde L Th-index van de bonenopbrengsten van F_5-1936, die overeenkomen met bonenopbrengsten van de I-lijn en het veel grotere aantal van de bonenopbrengsten van de I-lijn van 1936 nog duidelijker (tab. 8).

Volgens hun positie in tab. 8 zijn de bonenopbrengsten van pl. 1066, 674, 675, 765 en 562 het best in overeenstemming met bonenopbrengsten van de I-lijn van 1936. Om de ras-zuiverheid van deze bonenopbrengsten te kunnen vaststellen, zou van ieder van deze planten een groot aantal bonen moeten zijn voortgekweekt. Van pl. 765 en haar ascendentie zijn de characterogrammen getekend (fig. 9). We zien hier de erfelijkheid van bonen met de form. L B th, cl 2.

Bij de 36 gevallen, waar de uitgangsboon de form. L B th, cl 2 heeft, zijn *5 gevallen*, waar de formule van de gemiddelden L B Th, cl 1 is. In één van deze gevallen is deze formule L_1 L_2 B Th, cl 1a.

Pl. 463. De uitg. boon is van pl. 121 (fig. 3, K 15 en tab. 5d en 6d). Ze komt zo ook onder vgl-bonen van de I-lijn van 1935 voor. De gemidd. afmetingen van de weliswaar kleine bonenopbrengst van pl. 463 zijn

zeer groot. Ze zijn groter dan ze bij bonenopbrengsten van de I-lijn van 1936 voorkomen. De uitg. boon van pl. 121 voor pl. 463 (en 464, blz. 362) is niet zeer groot. De bonenopbrengst van pl. 121 bevat zeer grote bonen (o.a. de uitg. bonen voor de pl. 457 en 459, blz. 214 en 215), doch ook kleine. Er zijn in de bonenopbrengst van 51 bonen van pl. 121, 20 bonen, waarvan de lengte, l = 19.0—16.8 mm en ook 20, waarvan l = 12.9—14.5 mm is. Het is waarschijnlijk, dat de kleine bonen vooral niet-erfelijke variaties zijn. Onder de individuele bonen van pl. 463 zijn er enkele (b.v. th = 7.9 mm), zoals ze onder de bonen van de I-lijn van 1936 niet voorkomen.

Van pl. 463 is in 1937 een boon voortgekweekt, ze leverde pl. 317, F_6-1937. Bonen als de uitg. boon van pl. 463 voor pl. 317 komen ook onder de bonen van de I-lijn van 1936 voor. De overige 4 bonen van de peul van de uitg. boon komen met deze overeen. De grote gemiddelde afmetingen van de kleine bonenopbrengst van 19 bonen van pl. 317 komen zo tezamen bij de gemiddelden van de bonenopbrengsten van de I-lijn van 1937 niet voor: de gemidd. dikte is zeer groot. Onder de individuele bonen zijn er, — vergeleken met bonenopbrengsten van de I-lijn —, te veel met hoge indices. Het phaenotype van de bonenopbrengst van pl. 317 is L_1 L_2 B Th, cl 1a. De grote afmetingen van de uitg. boon van pl. 463, F_5-1936 zijn hier op die van pl. 317, F_6-1937 overgegaan.

Pl. 526. De uitgangsboon is van pl. 143 (fig. 2, K 13); ze heeft een grote breedte. De bonenopbrengst van pl. 526 is samengesteld, met zeer veel bonen in cl 1b; de uitg. boon is niet homozygoot voor de form. L B th, cl 2. Van een peul met 5 bonen van pl. 526 zijn alle bonen in 1937 voortgekweekt. Ze leverden pl. 366—370, F_6-1937. De uitgangsbonen hebben alle de form. L B Th, cl 1b. De bonenopbrengsten komen overeen, behoren alle tot het gebied van cl 1b, form. L B Th en zijn nog al samengesteld. De heterozygotie van de uitg. boon van pl. 143 voor pl. 526 vinden we nog terug in de bonenopbrengsten van de pl. 366—370, F_6-1937.

Pl. 475. De uitg. boon is van pl. 123 (fig. 3, K 15). Zie ook pl. 472, 473 en 474 alle van cl 1a (blz. 215). In de bonenopbrengst van pl. 475 zijn veel bonen met zeer grote afmetingen. Van pl. 475 is in 1937 een boon met de form. L B Th, cl 1b, en een grote breedte, uitgezaaid; ze leverde pl. 322, F_6-1937. De uitg. boon komt zo ook onder bonen van de I-lijn van 1936 voor. De gemiddelden van de bonenopbrengst van pl. 322 vinden we zo ook bij bonenopbrengsten van de I-lijn van 1937. Volgens de classificatie zijn er zeer veel bonen in cl 2. Onder de individuele bonen zijn er zeer veel met een hoge L B-index (L B = 69), veel meer dan er bij de bonen van de bonenopbrengsten van de I-lijn van 1937 zijn. De uitg. boon van pl. 475 voor pl. 322 is in hoge mate homzozygoot voor de formule van bonen van de I-lijn, doch bevat óf meer B B-verbindingen (transgressieve-variabiliteit) óf minder L L-verbindingen dan bonen van de I-lijn.

Pl. 1054. De uitg. boon is van pl. 174 (fig. 6, K 10). Zie ook blz. 219). De bonenopbrengst is samengesteld, met veel bonen in cl 1, ook meerdere in cl 2, 3 en 4. Er zijn 2 bonen met een zeer grote lengte (l = 15.7 mm).

Pl. 590. De uitg. boon is van pl. 179 (fig. 6, K 10). Een zo grote gemidd. dikte als van de bonenopbrengst van pl. 590 komt bij de bonenopbrengsten van de I-lijn van 1936 niet voor. Volgens de classificatie zijn er zeer veel bonen in cl 1 en 2.

De 13 bonen van cl 1, form. L B Th laten zich van de 12 bonen van cl 2, form. L B th, ook door hun indices, (cl 1 heeft hogere L Th- en B Th-indices) goed onderscheiden. De 13 bonen van cl 1, 11 bonen van cl 2 en de ene boon van cl 4 (laatste boon in de rij van de peul) van pl. 590 zijn in 1937 uitgezaaid. Ze leverden de pl. 463—488, F_6-1937. Van de uitvoerige aantekeningen van deze 25 bonenopbrengsten van uitgangsbonen, die de hele gemeten bonenopbrengst van pl. 590 uitmaakt, nemen we het volgende over. Van de 13 bonenopbrengsten, waarvan de uitgangsboon de form. L B Th, cl 1 heeft, is van 5 bonenopbrengsten de form. van de gemiddelden ook L B Th. Ze vertonen een grote overeenkomst. Van de meeste komt de uitg. boon zo ook onder de bonen van de I-lijn van 1936 voor. In één geval zijn er volgens de classificatie, bonen in de meeste classen, in de overige 4 gevallen zijn er zeer veel bonen in cl 1 en enige in cl 2. In de overige 8 van de 13 gevallen, waar de form. van de uitgangsboon L B Th, cl 1 is, is de form. van de gemiddelden van de bonenopbrengsten L B th, cl 2. Ook deze bonenopbrengsten tonen onderling overeenstemming en verschillen niet veel van de 5 te voren besprokene, waarvan de form. van de gemiddelden L B Th, cl 1 is. In een tabel zijn deze 13 bonenopbrengsten, F_6-1937 van uitg. bonen met de form. L B Th, opgenomen in de volgorde van de gemiddelde dikte: we zien de geleidelijke daling van de gemiddelde indices en in de classificaties de geleidelijke daling van het aantal bonen in cl 1 en idem stijging in cl 2. De uitgangsbonen tonen enige overeenstemming met de gemiddelden der bonenopbrengsten; er is enige, min of meer regelmatige, daling van de L Th- en B Th-indices. Hierin komen de niet-erfelijke variabiliteit en ook de erfelijkheid door polymere factoren tot uitdrukking. In 11 gevallen hebben de uitgangsbonen, die van de bonenopbrengst van pl. 590 zijn voortgekweekt en gemeten, de form. L B th, cl 2. Van 4 der bonenopbrengsten is de form. der gemiddelden ook L B th, cl 2, van de 7 overige is ze L B Th, cl 1. Ze tonen geen volledige overeenstemming met bonenopbrengsten van de I-lijn van 1937. In 2 gevallen is de gemidd. dikte te groot. In een ander geval wijst de classificatie op een hoge mate van homozygotie van de uitg. boon voor de form. L B th, cl 2. Twee gevallen zijn goede voorbeelden van bonenopbrengsten met het phaenotype L B Th. In het ene geval zijn er veel bonen in cl 1a en 1b, ook in 2a en 2b. In 2 gevallen is de form. van de uitgangsboon van pl. 590, L_1 L_2 B th, cl 2a en die van de gemiddelden der bonenopbrengsten resp. L_1 l_2 B th, cl 1b en L_1 l_2 B th, cl 2b. In 2 gevallen is de formule van de gemiddelden L b th, cl 4. In het ene geval ten slotte, waar de form. van de uitg. boon L b th, cl 4 is, is die van de gemiddelden van de bonenopbrengst L B th, cl 2.

De 25 bonenopbrengsten van pl. 463—488, F_6-1937, waarvan de uitgangsbonen de hele gemeten en gewogen bonenopbrengst vormen van pl. 590, F_5-1936, geven ons een blik op de erfelijke variabiliteit, die deze bonenopbrengst nog bevat. Ze wijzen op kleine geno-typische verschillen van de uitgangsbonen, die overeenkomen met de gegevens over de bonenopbrengst van pl. 590 en de opvatting over de uitgangsboon van pl. 179, F_4-1935 voor pl. 590.

Er zijn *6 gevallen*, waar de uitgangsboon van F_4-1935 voor F_5-1936 de form. L B th, cl 2b heeft en de formule van de gemiddelden van de bonenopbrengsten van F_5-1936, L b th, cl 4 is.

Pl. 671. De uitg. boon is van pl. 210, F_4-1935 (fig. 7, K 4). Volgens de classificatie van de bonenopbrengst zijn er alleen bonen in cl 2, cl 4 en cl 8. De aantekening is „slecht, niet uitgegroeid". Van de *pl. 1076*, *1082*, *1073*, *1075* en *1053* zijn de uitg. bonen alle van pl. 174, F_4-1935 (fig. 6, K 10). Pl. 1075 (tab. 5d en 6d) heeft alleen bonen in cl 4 (en één boon in cl 8). Al deze bonenopbrengsten hebben

veel bonen in cl 4 en in cl 2, enige hebben er ook in cl 1. Ze sluiten nauw bij elkaar aan, ook bij pl. 1054 en pl. 1063 en 1064 (blz. 367).

In *één geval* is de formule van de uitgangsboon L B th, cl 2b en die van de gemiddelden van de bonenopbrengst 1 B th, cl 6.

Pl. 553. De uitg. boon is van pl. 171 (fig. 6, K 10). De bonenopbrengst van pl. 171 bevat veel bonen in cl 6 en in cl 5. Van alle bonen van pl. 171, ook van de uitgangsboon voor pl. 553, is de L B-index hoog, wat kenmerkend is voor bonen van cl 6, form. 1 B th. De uitg. boon voor pl. 171, F_4-1935 is van pl. 178, F_3-1934. Hierbij staat vermeld „bonentype II". Volgens de classificatie van pl. 553 zijn er bonen in cl 6, ook in cl 2 en 1, resp. met een kleine en een grote dikte. De L B-indices zijn hoog. Er is wel uiting van splitsing in deze bonenopbrengst, doch ze behoort overwegend tot het gebied van cl 6 en cl 5, de bonen hebben een grote breedte. Er staat aangetekend „goed met brede peulen".

Er zijn *9 gevallen*, waar de formule van de uitgangsboon L B th, cl 2b is en die van de gemiddelden van de bonenopbrengsten 1b th, cl 8 is. Het zijn: *pl. 432*, fig. 5 K 16; *pl. 663*, *693* en *734*, fig. 7 K 4; *pl. 1063* en *1064*, fig. 6 K 10; *pl. 788*, fig. 1 K 33 en *pl. 800* en *811*, fig. 8 K 64.

Blijkens de classificatie bevatten de meeste van deze opbrengsten ook bonen in de cl 1, 2 en 4. Van deze bonen is de lengte groter dan l = 13.0 mm, groter dan de grootste variaties van de lengte van bonen van de II-lijn; de formule van de lengte bevat dus ook L L- of L l-verbindingen. Er zijn bonenopbrengsten, die door hun indices in hoge mate, doch niet geheel overeenkomen met bonenopbrengsten van de I-lijn, andere met die van de II-lijn, enkele met die van de form. l b th, cl 8c. Zie ook bij groep cl 8, (eerste pag. van deel III van dit artikel).

Cl 3. Form. L b Th (tab. 7) *3 gevallen.* De formule van de uitgangsboon is L b Th. In één geval, *pl. 568*, is de formule van de gemiddelden L B Th, cl 1, (fig. 7, K 4). Alle bonen op één na (deze is in cl 4) zijn in cl 1a en 1b; er is dus geen enkele boon in cl 3. Er is hier schijnbaar een groot verschil met de uitg. boon. Deze is van pl. 250. Pl. 250 heeft zeer veel bonen in cl 3 ook in cl 1. Het zijn kenmerkende bonen. De L B-indices zijn slechts iets hoger dan die van bonen met de form. L B th, cl 2, omdat de lengte der bonen relatief klein is. Alle bonen van pl. 250 hebben een zeer hoge B Th-index (B Th = 83—88), omdat de dikte groot en de breedte klein is. Ook de L Th-indices zijn hoog (L Th = 53—54). We nemen aan, dat in de uitgangsboon de erfelijke constitutie L b Th is uitgedrukt. Alleen verder voortkweken van pl. 568, die ons het genotype van de bonen der bonenopbrengst van pl. 568 zou hebben geleerd, zou de tegenstrijdigheid tussen bonenopbrengst van pl. 568 en haar uitg. boon van pl. 250 hebben kunnen oplossen.

Van het 2de geval, *pl. 557* (fig. 6, K 10) is de formule der gemiddelden L b th, cl 4. We zijn hier in het gebied van cl 2 en de grensgebieden cl 4 en cl 3. Toch hebben we met kleine erfelijke verschillen te doen. Van *pl. 595* (fig. 6, K 10) is de bonenopbrengst samengesteld en „de peulen zijn klein en nog al groen". Er zijn zeer veel bonen in cl 8b en 8c met hoge en minder hoge B Th-indices; de eerste behoren door hun indices tot cl 3 of cl 7.

Een goed voorbeeld van de groep cl 3, bevat ons materiaal niet.

Cl 4. Form. L b th. 9 gevallen (tab. 7). In 5 van deze gevallen, is de formule van de gemiddelden van de bonenopbrengst ook L b th, cl 4.

Van de *pl. 1077*, *1084* en *1056* zijn de uitgangsbonen van pl. 174 (fig. 6, K 10). Volgens de classificatie zijn we hier in het gebied van cl 2 en cl 4 (zie ook pl. 1075).

Pl. 727 en 929. De uitgangsbonen zijn van pl. 228 (fig. 7, K 4 en tab. 5d en 6d). De uitg. boon van pl. 228 voor pl. 729 is van een peul met 5 bonen, die alle tot cl 4, form. L b th, behoren. De 8 bonen in cl 1 van pl. 228 hebben geen grote dikte. Volgens de classificatie brantwoordt de bonenopbrengst van *pl. 727* niet geheel aan het genotype L b th. Onder de bonen van cl 4 van *pl. 729* zijn er met een vrij grote lengte (l = 15.1, 15.0, 14.5 mm, e.v.). Van 4 bonen van cl 8 hebben er 2 een niet zeer kleine lengte (l = 13.0 en = 12.7 mm); de overige 2 zijn de laatste in de rij van de peul en hebben overeenkomstige indices als de bonen van cl 4 van de bonenopbrengst. We hebben bij pl. 729 met een bonenopbrengst te doen, waarvan de uitg. boon in hoge mate homozygoot is voor de form. L b th, cl 4.

In *3* van de 9 gevallen, waarvan de uitg. boon de form. L b th, cl 4 heeft, is die van de gemiddelden L B th, cl 2. *Pl. 713* (fig. 7, K 4) is een vrij samengestelde bonenopbrengst. *Pl. 448* (fig. 3, K 15) stemt zeer overeen met bonenopbrengsten van de I-lijn. Er zijn daarbij bonen in cl 1a en 2a met een zeer grote lengte (l = 15.8—17.1 mm). We nemen aan, dat de uitg. boon van pl. 119 een niet-erfelijke minus-variant is. Van *pl. 1057* met uitg. boon van pl. 174 (fig. 6, K 10) komt de bonenopbrengst overeen met bonenopbrengsten van de I-lijn (zie blz. 219). *Pl. 1083* met zeer kleine gemiddelden, form. l b th, cl 8, heeft ook de uitg. boon van pl. 174 (fig. 6).

Cl. 5, Form. l B Th (tab. 7). *15 gevallen.* De formule van de uitg. boon is l B Th, heeft een kleine lengte. In 2 van deze gevallen, is de formule van de gemiddelden van de bonenopbrengsten ook l B Th.

Pl. 1087 en 1088. De uitg. boon is van pl. 185 (fig. 6, K 10 en tab. 5d en 6d). De uitg. bonen zijn de 1ste en 2de boon van dezelfde peul van 6 bonen. Ook de 3e boon van de peul, — d.i. de uitg. boon voor pl. 1089 —, behoort tot cl 5. De 4de, 5de en 6de boon, waarvan de 5de de uitg. boon van pl. 1091 is, behoren tot cl 7. Ook de uitg. boon voor pl. 619, eveneens van pl. 185, behoort tot cl 5. Volgens de classificatie zijn er in de bonenopbrengst van pl. 185 veel bonen in cl 5 en cl 7. Bijna alle bonen hebben een grote dikte. Er zijn van de bonenopbrengsten van pl. 1087 en 1088 (deze laatste bonenopbrengst is zeer klein, n = 6) zeer veel bonen in cl 5 (tab. 6d). Van vele ligt de lengte niet ver onder de grenswaarde, (van 14 bonen is l = 13.0—12.3 mm).

De L B-index is hoog (L B = B/L × 100), hoger dan van bonen van de II-lijn; de B th-index (B Th = Th/B × 100) is lager dan van bonen van de II-lijn. De bonen in cl 1, cl 7, cl 3 en cl 8b van pl. 1087 zijn milieu-variaties van bonen met kleine genotypische verschillen ten opzichte van bonen van cl 5. Van *pl. 1089* is de form. der gemiddelden l B th,

cl 6. De bonenopbrengst is verwant aan die van pl. 1087 en 1088; ze is een goed voorbeeld van een bonenopbrengst van cl 6.

Van *één* van de 15 gevallen, waar de uitg. boon voor de bonenopbrengst de form. l B Th, cl 5 heeft, is de form. van de gemiddelden der bonenopbrengst L B th, cl 2. *Pl. 830*; de uitg. boon is van pl. 350 (fig. 8, K 64), ze ligt dichtbij cl 1. Het geval behoort tot het gebied van cl 1 en 2.

In *drie* van de 15 gevallen, is de form. van de gemiddelden l b Th, cl 7. *Pl. 699*. De uitg. boon is van pl. 217 (fig. 7, K 4). De 2 overige bonen van de peul van de uitgangsboon behoren tot cl 7. De bonenopbrengst van pl. 217 behoort tot het gebied van cl 7 en cl 5. Van de gemiddelden van de bonenopbrengst van pl. 699 is de gemidd. lengte ($l_m = 12.8$ mm) groter dan de grootste gemidd. lengte ($l_m = 12.6$ mm) van vergelijk-bonenopbrengsten van de II-lijn van 1936. Er zijn te veel bonen met een grote lengte. De uitg. boon van pl. 217 voor pl. 699 is niet geheel homozygoot voor het genotype van cl 7 en cl 5.

Pl. 499. De uitg. boon is van pl. 129 (fig. 3, K 15, tab. 5d en 6d). Ze is uit een peul met 5 bonen, die alle tot cl 5 behoren. Volgens de classificatie zijn er in de bonenopbrengst van pl. 129 (tab. 6c) zeer veel bonen in cl 7, vele in cl 5 en enkele in cl 3 en cl 1. Van deze laatste is de grootste lengte l = 14.6 mm. Het phaenotype van de bonenopbrengst van pl. 129 is dus niet geheel dat van de form. l B Th en l b Th. Van de gemiddelden van pl. 499 is de gemidd. lengte te groot voor bonenopbrengsten van de II-lijn van 1936. Ook de samenstelling verschilt er, blijkens de classificatie, (4 bonen in cl 2, tab. 6d), zeer van. De uitg. boon van pl. 129 voor pl. 499 heeft een iets ander genotype dan dat van cl 5 en cl 7.

Van pl. 499 zijn in 1937 2 bonen uitgezaaid. Ze leverden de pl. 347 en 348, F_8-1937 (tab. 5e en 6e). De uitg. boon voor pl. 347 heeft de form. l B Th, cl 5; evenzo de 4de en 5de, laatste, boon van de peul. De 1ste en 2de boon hebben de form. L B Th, cl 1b. De form. van de gemiddelden van pl. 347 is l b th, cl 8b. Van de 17 bonen in cl 8, form. l b th, hebben er 6 indices, die overeenkomen met die van de bonen in cl 7, form. l b Th; 8 bonen hebben indices als van cl 5. De grootste lengte der bonen is l = 13.4 mm (de ene boon van cl 4). Het phaenotype van pl. 347 is als van bonenopbrengsten van cl 5 en cl 7 en is in overeenstemming met de form. van de uitg. boon. De uitg. boon voor pl. 348 heeft de form. L B Th, cl 1b. Alle 5 bonen van de peul van de uitg. boon hebben deze formule (l = 14.6 — 13.4 mm). De formule voor de gemiddelden van pl. 348 is l b th, cl 8. Deze bonenopbrengst heeft, blijkens de classificatie, in mindere mate het phaenotype van bonenopbrengsten van cl 5 en cl 7; dit is in overeenstemming met de uitgangsboon.

Pl. 497. De uitg. boon is van pl. 128 (fig. 3, K 15 en tab. 5d en 6d). Vier bonen van de peul van de uitgangsboon behoren tot cl 5, de 5de, laatste, behoort tot cl 7. Volgens de classificatie van pl. 128 zijn er zeer veel bonen in cl 7 en 5. Van de gemiddelden van pl. 497 is de gemiddelde dikte iets kleiner dan van bonenopbrengsten van de II-lijn van 1936. Volgens de classificatie zijn er veel bonen in cl 5, 7 en 8. De grootste lengte van de bonen is l = 12.8 (dan = 12.4 mm). Er zijn veel bonen met een hoge L B-index, omdat de breedte groot is. Er zijn bonen met een hoge B Th-index en ook enkele met een lage B Th-index, omdat de dikte klein en de breedte groot is (B Th = 81 = 86, als cl 7 en 5;

B Th = 73—79, als cl 6). De bonenopbrengst heeft in grote mate het phaenotype van cl 7 en 5. Er zijn 8 van de 15 bonenopbrengsten met de form. l B Th, cl 5, van de uitg. boon, waar de formule van de gemiddelden der bonenopbrengsten l b th, cl 8 is.

Pl. 665; de uitg. boon is van pl. 205 (fig. 7, K 4). Ze heeft een grote dikte (th = 6.8 mm) en is de enige boon in cl 5 van pl. 205, die overigens zeer veel bonen in cl 2 heeft. De bonenopbrengst van pl. 665 is in overeenstemming met de formule van de uitgangsboon en behoort, volgens haar classificatie, tot het gebied van cl 7 en cl 5. *Pl. 480*. De uitg. boon is van pl. 125 (fig. 3, K 15 en tab. 5d en 6d). Pl. 125 heeft zeer veel bonen in cl 7 en vele in de overige classen van bonen met een grote dikte. Pl. 480 heeft volgens de classificatie veel bonen in cl 7 en 5. De 19 bonen in cl 8 behoren volgens de indices tot cl 7 en cl 5. De bonenopbrengst is in overeenstemming met de formule van de uitgangsboon. Van pl. 480 zijn in 1937 4 bonen voortgekweekt; ze leverden pl. 324—327, F_6-1937. Pl. 325, met een zeer kleine uitg. boon, heeft volgens de classificatie 21 bonen in cl 7. De gemidd. lengte van de bonenopbrengst is l_m = 12.2 mm; van die van bonenopbrengsten van de II-lijn van 1936, is de grootste gemidd. lengte l_m = 11.6 mm (pl. 140 en 106). Alle 4 bonenopbrengsten van pl. 324—327 hebben meer of minder duidelijk het phaenotype van cl 7 en cl 5; ze sluiten aan bij hun uitgangsbonen: er is erfelijkheid.

Pl. 637. De uitg. boon is van pl. 198 (fig. 6, K 10). De bonenopbrengst van pl. 198 behoort duidelijk tot het gebied van cl 7, form. l b Th. De gemiddelden van de bonenopbrengst van pl. 637 zijn zeer klein; de gemidd. dikte is kleiner dan van vgl.-bonenopbrengsten van de II-lijn van 1936. Ze behoort tot het gebied van bonenopbrengsten van de II-lijn en is goed in overeenstemming met de formule van de uitgangsboon en de bonenopbrengst van pl. 198. *Pl. 602*. De uitg. boon is van pl. 184 (fig. 6, K 10). De bonenopbrengst van pl. 602 en die van pl. 184 van de uitg. boon hebben overeenkomst met bonenopbrengsten van de II-lijn. Evenzo tot het gebied van cl 5 en cl 7 behoort *pl. 647* met uitg. boon van pl. 200 (fig. 6, K 10). Een iets meer samengestelde bonenopbrengst, wel nog binnen het gebied van cl 5 en cl 7, heeft *pl. 619*, met uitg. boon van pl. 195 (fig. 6, K 10). Ook de *pl. 654* en *656* met uitg. boon van pl. 201 (fig. 6, K 10) zijn goede voorbeelden van bonenopbrengsten van het gebied van cl 5 en cl 7.

Van de 15 gevallen, waar de formule van de uitgangsboon van de bonenopbrengst l B Th, cl 5 is, behoren de meeste tot het gebied van cl 5 en cl 7. Bonenopbrengsten, die geheel overeenkomen met bonenopbrengsten van de II-lijn van 1936 treffen we er niet bij aan. Onder de 8 gevallen, waar de formule der gemiddelden van de bonenopbrengsten l b th, cl 8 is, is er geen met de form. l b th I, cl 8a.

Cl. 6 (tab. 7). In *2 gevallen* heeft de uitgangsboon de form. l B th, cl 6. De formule van de gemiddelden van de bonenopbrengst is in beide gevallen l b th, cl 8. Zie ook blz. 368, pl. 1089. *Pl. 806*. De uitg. boon is van pl. 333 (fig. 8, K 64 en tab. 5d en 6d). Ook de eerste boon van de peul van de uitg. boon behoort tot cl 6 en de dikte heeft ook hier de grenswaarde (th = 6.5 mm). De 2de boon behoort tot cl 5, de 4de, laatste, tot cl 8 met indices als de andere bonen van de peul. Volgens de classificatie van pl. 333 zijn er veel bonen in cl 5, 6, 7, 1 en 8, ook enkele in cl 2. (Zie ook pl. 331 Genetica 1937). In de bonenopbrengst van pl. 806 zijn veel bonen in cl 6; verreweg de meeste bonen zijn in cl 8.

De grootste lengte van de bonen is l = 13.5 mm. Van de 26 bonen van cl 8 hebben de meeste indices, die overeenkomen met bonen van cl 6. De bonen van cl 6 hebben een hoge L B-index (ind. L B = B/l × 100). We vinden voor die van pl. 806, (niet tot cl 8 behorende), L B = 69—75. Van de 26 bonen van cl 8 hebben er 6 een L B-index van 69 en hoger, van de overige 23 bonen in cl 8 is de L B-index 63—68. De bonen van cl 6 hebben een L Th-index, die intermediair is tussen die van de bonen van de I- en de II-lijn (Ind. L Th = th l × 100; van de I-lijn is L Th = th L × 100 en van de II-lijn = Th l × 100). De L Th-index van de bonen van pl. 806 is L Th = 48, dan 50—53, dan 55. Die van de 28 bonen van cl 8 is L Th = 45, dan 48—55, dan 57. De B Th-index van de bonen van cl 6 is even hoog als die van bonen van de I-lijn (B Th = th B × 100). We vinden voor de 7 bonen van pl. 806, die niet tot cl 8 behoren, B Th = 67—75. De B Th-index van de 29 bonen van cl 8 is, B Th = 69—82, daarvan van 19 bonen hoger dan 75. De B Th-indices zijn dus te hoog (verwantschap ook met cl 5). De uitg. boon voor pl. 333 is van pl. 359, F_3-1934. De form. van deze uitg. boon is l b th, cl 8 en de indices zijn als die van cl 6. De bonenopbrengst van pl. 359 bevat veel bonen in cl 6, 7 en 8 en enkele in de meeste andere classen. De grootste lengte van de bonen is l = 13.9 mm. De uitgangsboon voor pl. 359, F_3-1934 is van K 64. De bonen van K 64 zijn alleen gephotografeerd. De bonenopbrengst van pl. 806 is een goed voorbeeld, ook in haar ascendentie, van een geval, waar de formule van de uitg. boon (van pl. 333), l B th cl 6, is en de bonenopbrengst (pl. 806) het phaenotype l B th heeft.

Pl. 664 (fig. 7, K 4). Een kleine bonenopbrengst van 9 bonen, die slecht is.

Cl 7. Er zijn *18 gevallen*, waar de uitgangsboon de formule lb Th, cl 7 heeft (tab. 7). In één van deze gevallen (pl. 652) is ook de formule van de gemiddelden lb Th. cl 7. *Pl. 652.* De uitgangsboon is van pl. 200 (fig. 6, K 10). Ze is van een peul met 5 bonen, die alle de form. l b Th, cl 7 hebben. De bonenopbrengst van pl. 200 heeft zeer veel bonen in cl 5 en in cl 7. Zoals de uitgangsboon van pl. 200 voor pl. 652, komen er ook voor onder de vgl-bonen van de II-lijn van 1935. Van de gemiddelden van de bonenopbrengst van pl. 652, is de gemiddelde lengte groter dan ze bij vgl-bonenopbrengsten van de II-lijn van 1936 voorkomt; de gemiddelde L B- en L Th-indices zijn te laag. De grootste lengte van de bonen van pl. 652 is l = 13.5 mm, dat is groter dan van de bonen van de II-lijn van 1936. Van de 9 bonen van cl 7 is de dikte niet zeer groot (th = 6.7, dan = 6.6 mm). Van de 5 bonen van cl 5 is ze iets groter (th = 6.6—6.9 mm). We hebben hier niet te maken met een bonenopbrengst zoals van de II-lijn. De bonenopbrengst is gelijkmatig. De uitgangsboon is in hoge mate homozygoot voor een genotype, dat iets verschilt van dat van bonen van de II-lijn; het bezit niet alle Th Th-verbindingen van de bonen van de II-lijn en niet uitsluitend l l-verbindingen.

In *één* van de 18 gevallen van cl 7 is de form. van de gemiddelden van de bonenopbrengst L B Th, cl 1. *Pl. 487*; de uitg. boon is van pl. 126 (fig. 3, K 15). De bonenopbrengst bevat alleen bonen in cl 1. De uitg. boon heeft een grote dikte. We nemen aan, dat ze een modificatie is van bonen van cl 1, verwant aan die van cl 7.

In *drie* van de 18 gevallen van cl 7 is de form. van de gemiddelden van de bonenopbrengsten l B Th, cl 5. *Pl. 528*. De uitgangsboon is van pl. 144 (fig. 2, K 13). De uitg. boon komt zo ook onder de bonen van de II-lijn van 1935 voor. De gemiddelde lengte van de bonenopbrengst heeft de grenswaarde (l_m = 13.0 mm) voor bonen van cl 1, form. L B Th. Volgens de classificatie zijn er zeer veel bonen in cl 1, vele in cl 5 en 8. Ze komen zeer overeen wat de indices betreft, die we zo ook in cl 5 en cl 1 aantreffen. Dit geval behoort ook tot het tussengebied van cl 1 en cl 7.

Pl. 645. De uitg. boon is van pl. 200 (K 10, fig. 6, tab. 5d en 6d, fig. 10). Vier van de 5 bonen van de peul van de uitg. boon hebben de form. l b Th, cl 7, de 5de, laatste, boon behoort tot cl 8b. Volgens de classificatie van de bonenopbrengst van pl. 645 zijn er veel bonen in cl 5, cl 7 en cl 1. Van één boon in cl 2 heeft de dikte de grenswaarde (th = 6.5 mm). Van de boon met de grootste lengte is l = 13.9 (dan = 13.7, 13.5, 13.4 mm). De bonen komen zeer overeen, ook wat de indices betreft. De grootste verschillen tonen de B Th-indices. De bonenopbrengst behoort tot het tussengebied van cl 7,5 en 1, met kleine genotypische verschillen der bonen. De uitgangsboon van pl. 200 beantwoordt dus niet geheel aan het genotype l b Th, cl 7; wel is ze in hoge mate homozygoot. Van pl. 645 zijn in 1937 enige bonen uitgezaaid. Ze leverden de planten *830—847, F_6-1937* (tab. 5e en 6e). De pl. 838, 832, 830, 835, 836 en 837 stemmen zeer overeen. Er zijn veel bonen in cl 7, cl 5, cl 8b en enkele in cl 1. De lengte van de uitg. bonen is niet zeer klein. Van pl. 831 is de form. van de uitg. boon l B Th, cl 5 (l = 12.4 mm). Er zijn, volgens de classificatie, in de bonenopbrengst bonen in cl 5 en cl 7, doch zeer vele in cl 8. Er staat aangetekend „gave kleine boontjes". Er zijn 15 van de 25 bonen in cl 8; daarbij zijn er 3 zeer kleine boontjes. Van pl. 834 heeft de uitgangsboon de form. l b Th, cl 7 (tab. 5e en 6e, fig. 10). De form. van de gemiddelden van pl. 834 is l b th, cl 8, doch de gemiddelde dikte heeft de grenswaarde (th_m = 6.4 mm), zodat de bonenopbrengst bijna tot cl 7, form. l b Th kan gerekend worden. Er zijn, volgens de classificatie, alleen bonen in cl 7 en cl 8b en 8c. Ze verschilt van bonenopbrengsten van de II-lijn van 1937: de gemiddelde lengte is ten opzichte van vgl-bonenopbrengsten van 1937 te groot; de gem. L B- en L Th-indices zijn te laag. Onder de individuele bonen zijn er geen, zoals we ze als uitzonderingen ook niet onder bonen van de II-lijn van 1937 aantreffen. Pl. 834 en haar uitgangsboon is een goed voorbeeld van cl 7, form. l b Th, doch komt niet geheel met bonenopbrengsten van de II-lijn overeen.

Pl. 661. De uitg. boon is van pl. 204 (fig. 6, K 10); ze komt zo ook onder de bonen van de II-lijn van 1935 voor. De gemidd. dikte is kleiner, de gemidd. lengte groter dan van bonenopbrengsten van 1936. De

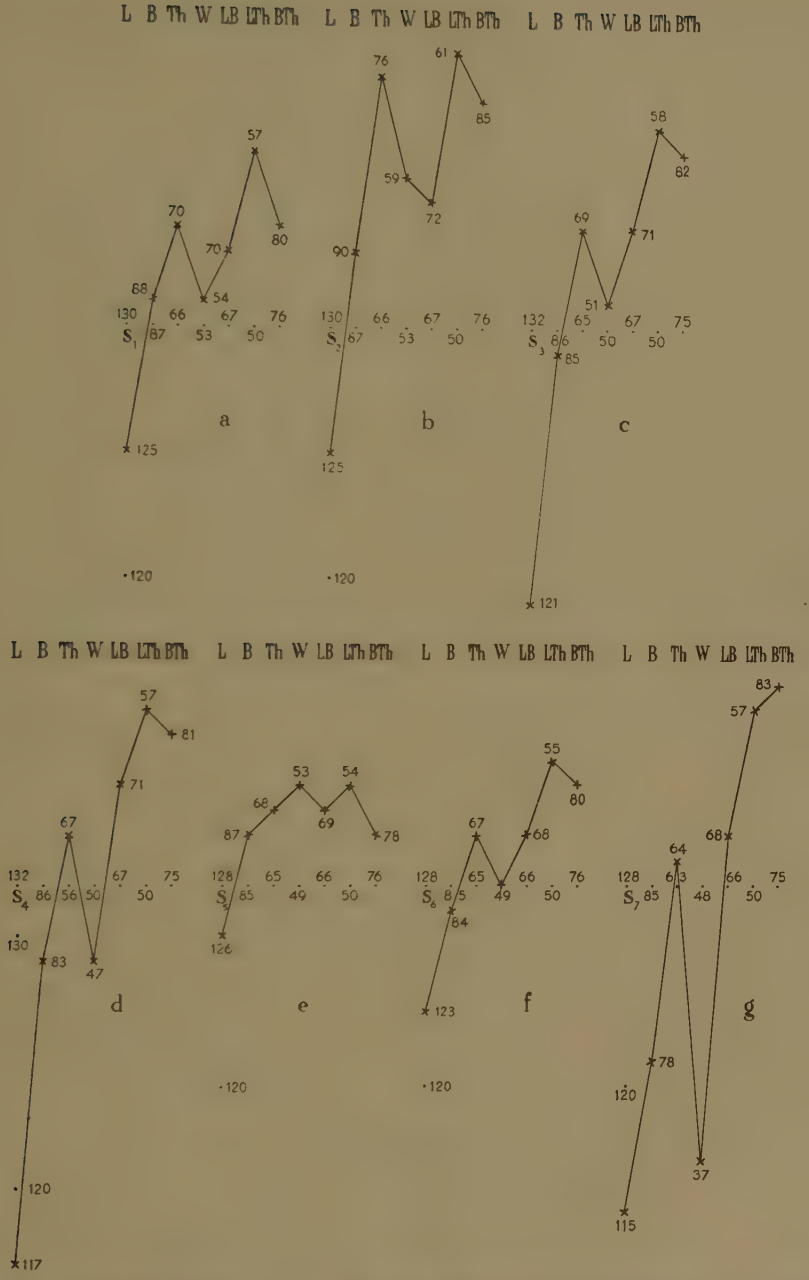

Fig. 10. Characterograms of pl. 645, F_5-1936 and its ascendants in 1935 and 1934 and its descendants in 1937. *a*. characterogram of the mean dimensions, etc. of the beanyield of pl. 180, F_3-1934. *b*. characterogram of the initial bean of pl. 180, F_3-1934 for pl. 200, F_4-1935. *c*. characterogram of the mean dimensions etc. of pl. 200, F_4-1935. *d*. characterogram of the initialbean of pl. 200, F_4-1935 for pl. 645, F_5-1936. *e*. charact. of the means of the bean yield of pl. 645, F_5-1936. *f*. charact. of the initial bean of pl. 645, F_5-1936 for pl. 834, F_6-1937. *g*. charact. of the mean dimensions etc. of the beanyield of pl. 834, F_6-1937. s =standard-characterogram. These characterograms resemble the characterogram of the II-lire (cf fig. 9).

bonenopbrengst van pl. 661 heeft niet geheel het phaenotype van bonenopbrengsten van de II-lijn.

Er zijn *13 gevallen* van de groep cl 7, waar de formule van de gemiddelden der bonenopbrengsten l b th, cl 8 is.

Pl. 514. De uitg. boon is van pl. 136 (fig. 2, K 13). De bonenopbrengst van pl. 514 heeft een kleine gemidd. dikte en een vrij lage gemidd. B Th-index; alle bonen behoren in cl 8. De bonenopbrengst is „vrij matig", te „onrijp". Het kan zijn, dat sommige bonen niet volgroeid zijn; vooral de dikte is zeer klein en deze groeit het laatst. Alle bonen hebben een hoge L B-index als van bonen met de form. l B th, cl 6. De bonenopbrengst behoort min of meer tot het gebied van cl 6.

Er zijn in 1937 8 bonen van pl. 514 voortgekweekt. Ze leverden pl. 352—359, F_6-1937. Van al deze planten is de form. van de uitg. boon l b th, cl 8. De dikte is steeds klein, de B Th-index niet hoog. Van pl. 352 en 357 is de form. van de gemiddelden L B Th, cl 1, met veel bonen in cl 1, cl 5 en cl 7. Van pl. 353, 355, 358 en 359 is de form. der gemiddelden l B th, cl 5 met veel bonen in cl 5 en cl 1, ook cl 6. Van pl. 356 ten slotte, is de form. der gemiddelden l b Th, cl 7. De uitg. boon heeft de kleinste lengte van alle 8 uitg. bonen. In de bonenopbrengst van pl. 356 zijn zeer veel bonen in cl 5, 7 en 8. Het is een zeer gelijkmatige bonenopbrengst. De bonenopbrengsten van pl. 353—359 komen zeer overeen. Het genotype van de uitg. boon is l B Th, cl 5, met genotypische verschillen ontleend aan cl 1, cl 7 en cl 6. Het phaenotype van pl. 514 is waarschijnlijk beïnvloed door de onrijpheid van de bonen, waardoor veèl bonen een zo kleine dikte hebben.

Pl. 492; de uitg. boon is van pl. 127 (fig. 3, K 15, tab. 5d en 6d). Alle 5 bonen van de uitg. boon hebben de form. l b Th ,cl 7. Volgens de classificatie behoren de bonen van pl. 127 alle in cl 7,5 en 8b. Het phaenotype is overwegend als van bonenopbrengsten van de II-lijn van 1935. Zoals de uitg. boon van pl. 127 voor pl. 492 is, zijn er ook bonen van de II-lijn van 1935. Bonenopbrengsten met dergelijke gemiddelden als van pl. 492, komen ook bij de II-lijn van 1936 voor. Volgens de classificatie heeft pl. 492 alleen bonen in cl 7 en 8b. We vinden dgl. classificaties ook bij de II-lijn van 1936.

Van pl. 492 zijn in 1937 13 bonen uitgezaaid, die de pl. 333—346, F_6-1937 leverden. Deze 13 bonenopbrengsten zijn zeer overeenkomstig; ze hebben uitsluitend bonen in cl 7 en 8b (tab. 5e en 6e). Het is opmerkelijk, dat de gemiddelde afmetingen klein zijn. Er is hier transgressieve variabiliteit van minus-variaties.

Van *pl. 565* is de bonenopbrengst „slecht". Ze behoort waarschijnlijk ook tot het gebied van cl 5 en cl 7.

Pl. 425. De uitg. boon is van pl. 112 (fig. 5, K 15, tab. 5d en 6d). Alle 4 bonen van de uitg. boon behoren tot cl 7. In de bonenopbrengsten van pl. 112 zijn zeer veel bonen in cl 7, vele in cl 5 en cl 6. De meeste (35) bonen behoren tot cl 8 en volgens de indices heeft het genotype overeenkomst met dat van cl 7, l b Th. De bonenopbrengst van pl. 425 is „goed". Het phaenotype wijkt iets meer af van dat van bonenopbrengsten van de II-lijn dan dat van pl. 112. Het komt meer overeen met het phaenotype van l b th, cl 8c (eerste pag. van deel III van dit artikel en tab. 7). In 1937 is één boon van pl. 425 voortgekweekt. Ze leverde *pl. 274*, F_6-1937 (tab. 5e en 6e). De uitg. boon van pl. 425 voor pl. 274 heeft de form. l b th, cl 8c, evenals de 2de, de 4de en 5de, laatste, boon van de peul van de uitg. boon; de eerste behoort tot cl 4 (l = 13.2 mm). De form. van de

gemiddelden van de bonenopbrengst van pl. 274 is l b th, cl. 8c. De gemidd. L B-index is iets te laag. Ook volgens de classificatie zijn er zeer veel bonen in cl 8c (tab. 6e). We hebben hier met een goed voorbeeld van een bonenopbrengst van het phaenotype cl 8c te doen, waarvan ook het genotype van de uitg. boon l b th, cl 8c, is. Ze is hier ontstaan als uitsplitsing, d.i. als klein genotypisch verschil ten opzichte van bonen van cl 6, form. l B th, cl 7, form. l b Th en cl. 5, form. l B Th. Ook *pl. 527* is verwant aan cl 5 en cl 7 en een goed voorbeeld van een bonenopbrengst met het phaenotype l b th, cl 8c.

In de nog overige 7 gevallen van cl 7 en waarvan de formule der gemiddelden l b th cl 8 is, is er een grote overeenstemming van de kleine lengte en breedte van de uitgangsbonen en de kleine gemidd. lengte en breedte van de bonenopbrengsten. We vermelden hier slechts *pl. 549* (fig. 6, K 10), *pl. 493*, zie ook pl. 492, blz. 374 (fig. 3, K 15) en *pl. 635*, zie ook pl. 637, blz. 370 (fig. 6, K 10). Van *pl. 1091* is de uitg. boon van pl. 185 (fig. 6, K 10); zie ook blz. 368). In de bonenopbrengst van pl. 1091 (aant. „platte, gesnoerde peulen, gerimpelde bonen") zijn zeer veel bonen in cl 8, waarvan vele met een lage B Th-index als bonen van cl 6. Van pl. 1091 zijn in 1937 enige bonen uitgezaaid. Ze leverden de pl. 1047—1050, F_6-1937. Deze bonenopbrengsten stemmen goed overeen met de uitg. bonen. Ook hier zijn veel bonen in cl 8 met een lage B Th-index, hierin dus overeenkomende met bonen van cl 6. *Pl. 655* (fig. 6, K 10) heeft grote overeenkomst met bonenopbrengsten van de II-lijn van 1936. De formule van de uitgangsboon is niet geheel identiek met die van bonen van de II-lijn. Ook van *pl. 622* (fig. 6, K 10) verschilt het genotype van de uitgangsboon, op grond van de samenstelling van de bonenopbrengst, slechts iets van dat van bonen van de II-lijn. Hetzelfde geldt ten slotte van *pl. 634* en *635* met de uitg. bonen van pl. 196 (fig. 6, K 10). De bonen van pl. 635 zijn klein; het gemidd. gewicht is zeer klein (w_m = 36 cG). Er zijn ook dgl. kleine gemidd. gewichten bij bonenopbrengsten van de II-lijn van 1936. De bonenopbrengst behoort tot het gebied van cl 7, 6 en 5.

Zo vinden we dus, dat, in het grote aantal gevallen, waar de uitgangsboon de form. l b Th, cl 7 heeft en de form. van de gemiddelden der bonenopbrengsten l b th, cl 8 is, de samenstelling van deze bonenopbrengsten in veel gevallen wijst op een grote mate van homozygotie van de uitgangsboon met het genotype l b Th, cl 7 en enkele L L-, B B- en th th-verbindingen, verwantschap uitdrukkend met de bonen van cl 5, 6 en 1.

THEORY OF THE ELECTRIC DOUBLE LAYER OF STABILIZED EMULSIONS

BY

E. J. W. VERWEY

(Communicated at the meeting of February 25, 1950)

Introduction.

The electrical double layer at the interface of two inmiscible liquids is of the double diffuse type. It is a well known fact that this double layer is generally insufficient to stabilize an emulsion, i.e. to prevent coalescence of the droplets of one phase dispersed in the other. Special measures have to be taken to obtain such a stability; one of them is the addition of a suitable capillary active electrolyte, i.e. an electrolyte for which one ionic species accumulates at the interface. The system of charges as it is formed at the interface when the electrical double layer is modified by the presence of such an "emulsifier" has never been investigated in detail. This problem is of importance for a better understanding of the colloid chemical properties of an emulsion stabilized in this way. In the present paper we will therefore study the distribution of the charges and the electrical potential for such a system.

1. *Theoretical.*

The original double layer, present at the interface of two liquids in the absence of capillary active ions, is caused by the circumstance that the positive and negative ions present in the system will generally tend towards a mutually unequal distribution between both phases. In a previous paper [1]) we have derived equations for the distribution of the charges and the electrical potential function of such a double diffuse double layer, on the basis of a GOUY-CHAPMAN type theory. The total electrical potential drop is divided into two parts, one on either side of the interface. It is found that most of the electrical potential drop occurs in the phase with the lowest value of the product $n\varepsilon$, i.e. generally in the phase with the lowest dielectric constant (ε) which nearly always will be the phase with the lowest ionic concentration (n) as well. For sake of convenience this phase will be indicated as the "oil" phase; the other one as the "water" phase.

The equations were derived for a flat double layer. Hence they lose their quantitative validity, for instance, when the dimensions of one phase

[1]) VERWEY, E. J. W. and K. F. NIESSEN, Phil. Mag., 28, 435, (1939).

(e.g. oil droplets in an oil-in-water emulsion) are of the order of magnitude of DEBYE's characteristic length $1/\varkappa$ in this phase $\left(\varkappa^2 = \frac{8\pi n v^2 e^2}{\varepsilon kT}\right)$ or smaller (v = valency of the ions of the electrolyte, supposed to be of the symmetrical type; n = concentration of positive or negative ions per cm^3; e, k and T have their usual meaning). For a droplet size of 10^{-3} cm ($10\,\mu$) this will be the case when the ionic concentration becomes less than 10^{-10} eq./l. In that case the diffuse charge in the oil droplets and accordingly the whole double layer cannot develop completely owing to a lack of space [2]).

In the following considerations it is assumed that either the dimensions of the oil droplets or the ionic concentrations in the "oil" are sufficiently large that this complication does not occur.

In the present paper we want to consider the electrical double layer for a two-liquid system, to which an amount of capillary active electrolyte has been added with the result that either the positive or negative ions of this electrolyte have been accumulated in the interface. For the system of charges, as present at the interface, we shall use a model similar to that used in the previous paper. The charge layer due to the adsorbed ions will be treated as a surface charge, coinciding with the mathematical plane of the interface separating the continuous phases 1 and 2. For the charge layers in these phases we use again GOUY's theory of the diffuse layer. Hence the dimensions of the ions, including those of the capillary active ions, are neglected. As the theory is only applied to systems containing very small amounts of electrolytes the approximations involved in this model seem to be fairly justified.

The number of capillary active ions which can be adsorbed at the interface is obviously more or less limited, either for sterical reasons, or because the electrical potential at the interface reaches such high values that a further adsorption is inhibited. In this paper we will not be dealing with this distribution equilibrium of the capillary active ions themselves. Considering only the electrical problem, we shall treat the surface charge as a given quantity, effected, as a first approximation, by the complete adsorption of such ions added to the system. This assumption of a complete adsorption seems to be a reasonable one for those cases where one phase has a large dielectric constant or a strongly polar character (e.g. water) and the second phase has a low dielectric constant or a non-polar character (e.g. a hydrocarbon). The capillary active ions are generally ions containing a relatively large non-polar group (e.g. a hydrocarbon chain) and a polar, ionogenic end where the ionic charge is located. The polar end restricts the solubility in the oil phase, the non-polar part favours a rather low solubility in the aqueous phase. At the interface, however, where the ion can be oriented in such a way that the polar end

[2]) VERWEY, E. J. W. and J. TH. G. OVERBEEK, Theory of the Stability of Lyophobic Colloids, Amsterdam, 133 (1948).

is embedded into the water phase and the non-polar part has the oil phase as its immediate surroundings, a maximal amount of interaction is possible, and accordingly the surface active ion has there a much lower energy than in the bulk of either of the two phases. Hence the capillary active ion will accumulate almost completely in the interface.

In those cases where the second phase has still a rather high dielectric constant or a more or less polar character (e.g. in the system water/nitrobenzene) the assumption of complete adsorption of the surface active ion may be less justified. Hence in such a case the theory given in the present paper can only be applied if sufficient information is available about the distribution equilibrium of the capillary active electrolyte and about the electrolytic dissociation of this electrolyte in both phases.

The value of the double layer potential of the original double layer depends on the nature of the electrolytes present in the system. Also the inactive ion of the capillary active electrolyte will take part in the distribution equilibrium and accordingly may alter this value of the double layer potential. However, the conditions at the interface are changed so radically by the presence of the capillary active ions that a possible change of the total double layer potential by the addition of the capillary active electrolyte becomes of minor importance in the final picture. Hence, for the sake of simplicity, we shall neglect this influence of the second ion, and assume that the capillary active electrolyte does not change appreciably the double layer potential of the original double layer.

Following the notation in our previous paper, also summarized in Fig. 1, the double layer potential considered above will be given by

$$\Delta = \frac{e(\varphi_1 - \varphi_2)}{kT}$$

and the partial potential drops in the phases 1 and 2 by

$$-z_1 = -\frac{v_1 e \psi_1(0)}{kT} \quad \text{and} \quad z_2 = \frac{v_2 e \psi_2(0)}{kT}$$

so that

$$\Delta = -z_1 + z_2$$

If the surface charge is denoted by σ, the condition of electrical neutrality yields the basic equation:

$$-\varepsilon_1 \int_{-\infty}^{0} \frac{\mathrm{d}^2 \psi_1}{\mathrm{d}x^2}\, \mathrm{d}x + 4\pi\sigma - \varepsilon_2 \int_{0}^{\infty} \frac{\mathrm{d}^2 \psi_2}{\mathrm{d}x^2}\, \mathrm{d}x = 0.$$

Integration leads to

$$-\varepsilon_1 \varkappa_1 \frac{kT}{e}\left(e^{z_1/2} - e^{-z_1/2}\right) + 4\pi\sigma - \varepsilon_2 \varkappa_2 \frac{kT}{e}\left(e^{z_2/2} - e^{-z_2/2}\right) = 0$$

or

$$e^{z_1} - \frac{e}{kT} \cdot \frac{4\pi\sigma}{\varepsilon_1 \varkappa_1 + \varepsilon_2 \varkappa_2 \cdot e^{\Delta/2}} \cdot e^{z_1/2} - \frac{\varepsilon_1 \varkappa_1 + \varepsilon_2 \varkappa_2 e^{-\Delta/2}}{\varepsilon_1 \varkappa_1 + \varepsilon_2 \varkappa_2 e^{\Delta/2}} = 0$$

with the solution

$$(1) \qquad e^{z_1/2} = \frac{e}{kT} \cdot \frac{2\pi\sigma}{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{\Delta/2}} \pm \left\{ \left(\frac{e}{kT} \cdot \frac{2\pi\sigma}{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 \cdot e^{\Delta/2}} \right)^2 + \frac{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{-\Delta/2}}{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{\Delta/2}} \right\}^{1/2}.$$

As $e^{z_1/2}$ cannot be negative we need only consider the positive root and can omit the negative sign in eq. (1). For $\sigma = 0$ this equation reduces to

$$(2) \qquad e^{z_1^0} = \frac{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{-\Delta/2}}{\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{\Delta/2}},$$

which was already derived in reference 1.

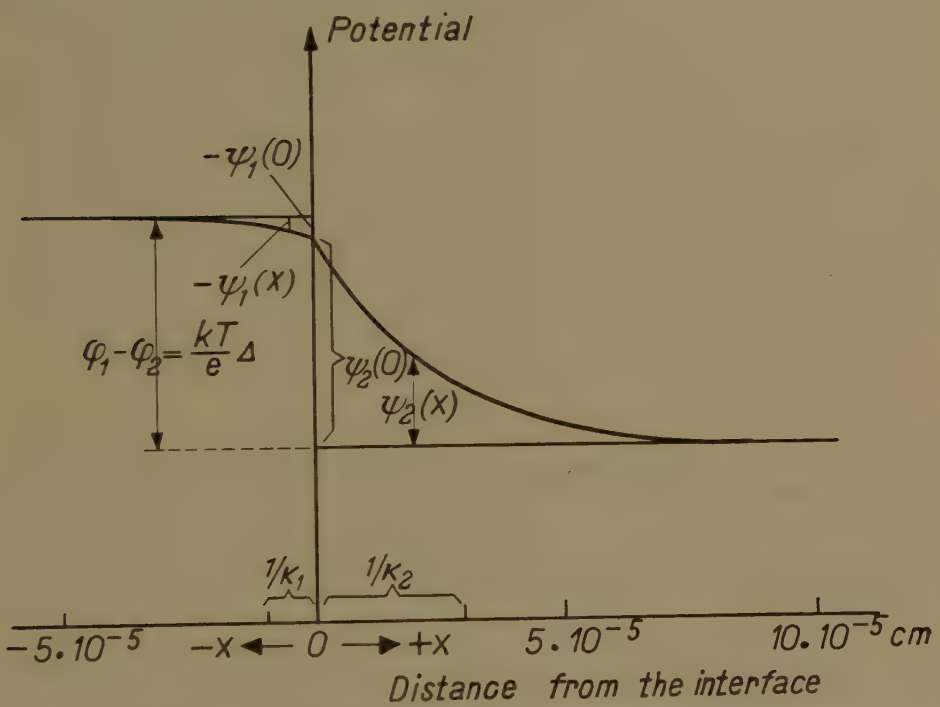

Fig. 1. Electrical potential function associated with the electrolytical double layer at the interface of two liquids for:

$\varepsilon_1 = 81$ $\quad \varkappa_1 = 10^5$ $\quad \varepsilon_1\varkappa_1 = 27\ \varepsilon_2\varkappa_2$
$\varepsilon_2 = 9$ $\quad \varkappa_2 = \frac{1}{3}.10^5$ $\quad n_1 = 81\ n_2$
$\varphi_1 - \varphi_2 = 115$ mV or $\Delta = 4.6$

Partial potential drops as calculated for this case with the aid of eq. (2):

$-\psi_1(0) = 7.8$ mV or $-z_1 = 0.31$;
$\psi_2(0) = 107.2$ mV or $z_2 = 4.29$

Equation (1) can be modified by introducing as a variable the quantity $z_1 - z_1^0$, i.e. the electrical potential at the interface relative to that for $\sigma = 0$. It can be brought into the form

$$(1a) \qquad e^{\frac{z_1 - z_1^0}{2}} = A + \sqrt{1 + A^2}$$

where

$$A = \frac{e}{kT} \frac{2\pi\sigma}{\{(\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{\triangle/2})(\varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 e^{-\triangle/2})\}^{1/2}}.$$

This equation expresses directly that $z_1 - z_1^0$ is positive for positive values of σ and negative for negative σ. It shows, too, that the absolute value of $z_1 - z_1^0$ is independent of the sign of the surface charge, as

$$e^{\frac{z_1 - z_1^0}{2}} = \{A + \sqrt{1 + A^2}\}^{-1} = -A + \sqrt{1 + A^2}.$$

2. *Discussion.*

Eq. (1) or (1a) give us immediately the partition of the electrical potential between both phases as a function of the surface charge.

The problem is symmetrical in phases 1 and 2 and in positive and negative values of the potential and we will select arbitrarily the case

$$\varepsilon_1\varkappa_1 > \varepsilon_2\varkappa_2 \text{ and } \triangle = \text{positive}.$$

By considering then the two possibilities

$$\sigma = \text{positive and } \sigma = \text{negative}$$

we will have covered all eight possible permutations, comprising two groups of four mutually equivalent cases.

(a) $\varepsilon_1\varkappa_1 > \varepsilon_2\varkappa_2$, $\triangle$ = positive, σ = positive.

This case is representative for the more general case that the surface charge at the interface has the same sign as the charge in the aqueous phase. We will investigate how the electrical potential function at the interface of the two liquids is changed by a gradually increasing surface charge.

We start from a situation as depicted in Fig. 1, showing the electrical potential as a function of the distance perpendicular to the interface for $\sigma = 0$. The aqueous phase (= phase 1) carries a positive space charge, the oil phase a negative one. The interface has a weak negative potential with reference to a point far in the aqueous solution (z_1 = negative), a comparatively large positive potential with respect to the bulk of the oil phase (z_2 = positive); the potential drop extends farther into the oil phase than into the water phase ($\varkappa_1 > \varkappa_2$). It has been assumed that also $\varepsilon_1 > \varepsilon_2$ and $\varepsilon_1\varkappa_1 \gg \varepsilon_2\varkappa_2$.

Now eq. (1) shows that, with increasing positive values of σ, the quantity $e^{z_1/2}$, starting with a value between 0 and 1, increases continually. Hence the value of $-z_1$ decreases and z_2 increases. Initially the space charge in the water phase and the surface charge are positive; together they are balanced by a negative charge in the oil phase. For a certain value of σ the quantity $e^{z_1/2}$ reaches 1, or $z_1 = 0$. In this point the potential drop and the charge in the aqueous phase are both zero, and we have only a positive surface charge balanced by a negative space charge in

the oil phase. For still larger values of σ we enter finally the region $e^{z_1/2} > 1$, or z_1 positive. The space charge and the potential drop in phase 1 are now inversed in sign, and the electrical potential function reaches a peak value at the interface. Hence we have now obtained a true *triple layer*: a positive surface charge surrounded at two sides by negative space charges. In an oil-in-water emulsion of corresponding properties the positive capillary active ions will convert the charge of the oil droplets (including the adsorbed layer) from a negative into a positive one. Also the ζ-potential of the oil droplets, the sign of which is determined by the sign of the potential gradient in the aqueous phase, will be converted from a negative into a positive value at this "inversion point" [3]).

Fig. 2 shows a number of electrical potential curves for different magnitudes of the surface charge.

Qualitatively, these results are not surprising. It is of interest, however, to investigate what factors determine the amount of adsorbed ions in the inversion point, the form of the electrical potential curve, etc.

The inversion point is reached for $e^{z_1/2} = 1$.

Substituting this into (1), or more simply into the quadratic equation of which (1) is the solution, we find the following relation for σ ($z_1 = 0$) or σ_0:

$$4\pi\sigma_0 \cdot \frac{e}{kT} = \varepsilon_2 \varkappa_2 \, (e^{\Delta/2} - e^{-\Delta/2}). \tag{3}$$

This result may be illustrated by a numerical example. We will assume that the electrolyte is one-one valent ($v = 1$); that the oil phase contains only 10^{-10} eq./l of dissociated electrolyte and that the dielectric constant of this phase is of the order of one tenth that of water, hence $\varepsilon_2 \varkappa_2 = 10^4$ cm^{-1}. Supposing now that $\varphi_1 - \varphi_2 = 460$ mV (or $e^{\Delta/2} = 10^4$) we find a value of σ_0 corresponding to 1.7×10^{12} elementary charges per cm^2. For $\varphi_1 - \varphi_2 = 230$ mV (or $e^{\Delta/2} = 10^2$) we find accordingly a value of only 1.7×10^{10} charges/cm^2. Especially the latter value corresponds to a relatively thin packing density of the capillary active ions in the interface, considering that the maximal amount to be adsorbed for this type of ions is of the order of 10^{14} ions/cm^2.

Such low values for $\varepsilon_2 \varkappa_2 e^{\Delta/2}$, and therefore for σ_0, will only be found when the "oil" phase is completely non-polar and accordingly its ionic concentration in equilibrium with an "aqueous" phase containing small amounts of electrolyte is extremely low. In many cases, however, the chemical difference between the two immiscible liquids may be less, or the system may contain some electrolyte, so that $\varepsilon_2 \varkappa_2$ is not so small. It may also be, in some cases, that the double layer potential is rather

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[3]) Hence we use the concept "inversion" here in the limited sense of inversion of potential or charge. This inversion should not be confused with the inversion of an emulsion, being the transformation of for instance an oil-in-water system into a water-in-oil emulsion.

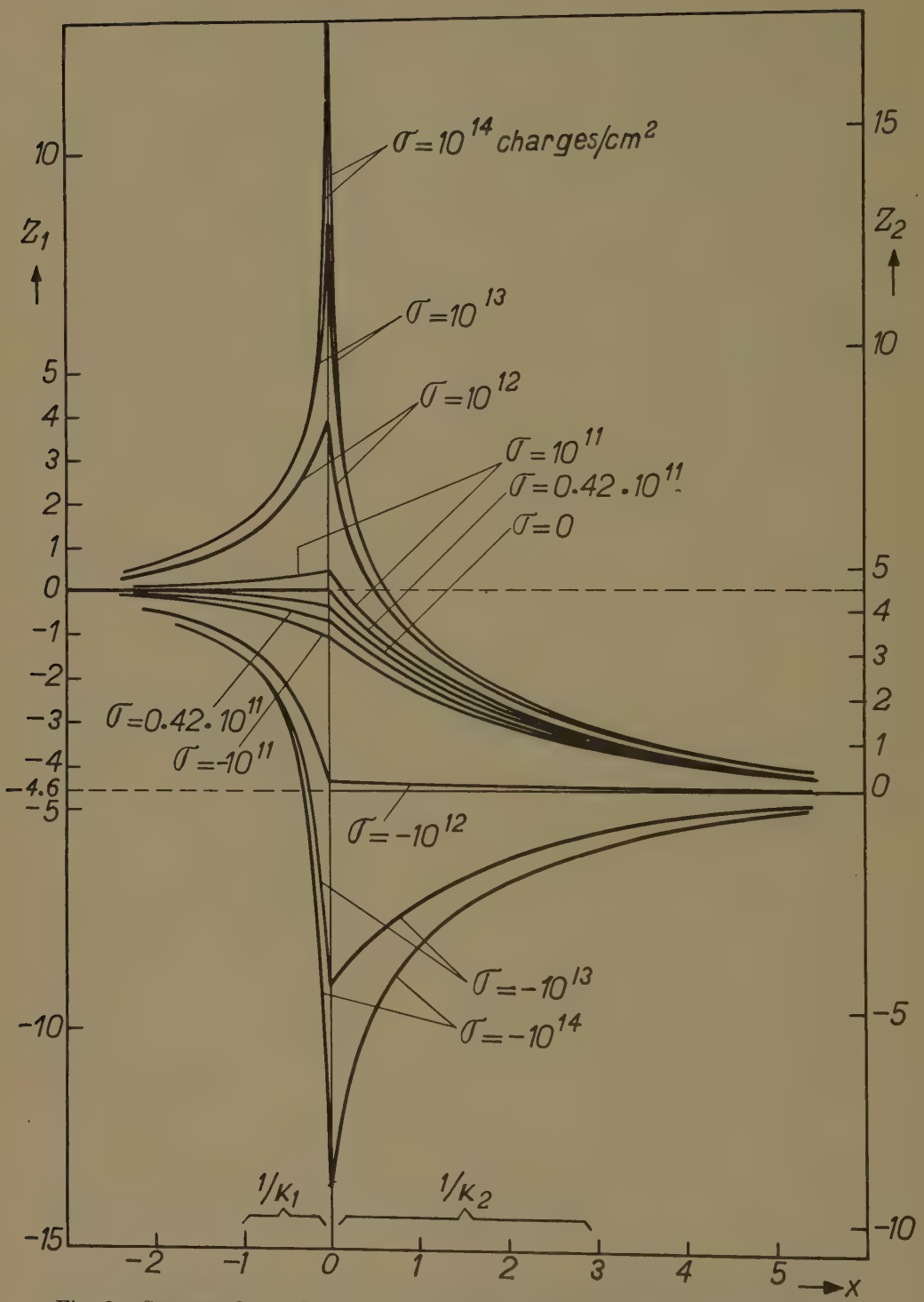

Fig. 2. Same as fig. 1, for the case that the interface carries a surface charge σ of the magnitude as indicated with each curve (σ in elementary charges per cm²; electrical potential in units $kT/e = 25$ millivolts).

high (or that the valency of the electrolyte is high). The double layer potential is greatly determined by differences between the solvation energies of positive and negative ions in both phases [4]) and may easily reach values of several hundreds of millivolts. For such cases we must expect that much larger quantities of capillary active ions are necessary to neutralize the original double layer and reach the inversion point. It may even be that they would exceed saturation quantities and cannot be realized at all. Actually different two-liquid systems show marked differences in their liability to be converted into the form of an emulsion with the aid of an emulsifier and also in the stability of the emulsions thus obtained. Our equations enable, however, to distinguish between the different factors determining these phenomena. We also see the importance of conductivity measurements and of experimental determination of double layer potentials for different sets of liquids as preliminaries to the study of emulsification by capillary active ions.

For relatively large values of σ, i.e. for the case that the triple layer is well developed, we can easily derive approximate equations for the charge distribution and the electrical potential at the interface. As soon as the first term under the square root becomes much larger than the second one equation (1) approaches

$$e^{z_1/2} = \frac{e}{kT} \cdot \frac{4\pi\sigma}{\varepsilon_1 \varkappa_1 + \varepsilon_2 \varkappa_2 e^{\Delta/2}} . \tag{4}$$

Under these conditions z_1 reaches appreciable values so that also $e^{-z_2/2}$ can be neglected against $e^{z_1/2}$, and a fortiori $e^{-z_2/2}$ against $e^{z_2/2}$. The ratio of the space charges in both phases

$$R = Q_1/Q_2,$$

which according to the derivation in section 1, equals

$$R = \frac{\varepsilon_1 \varkappa_1 (e^{z_1/2} - e^{-z_1/2})}{\varepsilon_2 \varkappa_2 (e^{z_2/2} - e^{-z_2/2})},$$

approaches then the simple equation

$$R = \frac{\varepsilon_1 \varkappa_1}{\varepsilon_2 \varkappa_2 e^{\Delta/2}} . \tag{5}$$

Both equations (4) and (5) show that the relative importance of the values $\varepsilon_1\varkappa_1$ and $\varepsilon_2\varkappa_2 \cdot e^{\Delta/2}$ greatly determines the properties of the triple layer. In the extreme case that

$$\varepsilon_1 \varkappa_1 \gg \varepsilon_2 \varkappa_2 \cdot e^{\Delta/2}$$

we will find that the space charge balancing the surface charge due to the adsorbed layer is predominantly found in the aqueous phase. Simultaneously eq. (4) approaches the equation for the electrical potential

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[4]) VERWEY, E. J. W., Rec. trav. chim., **61**, 127 (1942).
Rec. trav. chim., **61**, 564 (1942).

drop in a double layer of the GOUY-type (surface charge balanced at one side by a diffuse space charge), which, for large values of z, reads

$$e^{z/2} = \frac{e}{kT} \cdot \frac{4\pi\sigma}{\varepsilon\varkappa}.$$

How for $\varepsilon_1\varkappa_1 > \varepsilon_2\varkappa_2 e^{\triangle/2}$ the electrical potential function at the aqueous side of the triple layer approaches that of a simple GOUY layer may again be illustrated by a numerical example (1 — 1 valent electrolyte), assuming that $\varepsilon_1\varkappa_1 = 10^7$ cm^{-1} (roughly corresponding to distilled water), $\varepsilon_2\varkappa_2 = 10^4$ cm^{-1}, and $\sigma = 10^{14}$ adsorbed ions per cm^2:

$\varphi_1 - \varphi_2 = 230$ mV, $\varepsilon_2\varkappa_2 \cdot e^{\triangle/2} = 10^6$ cm^{-1}, $z_1 = 13.0$ or $\psi_1(0) = 325$ mV

$\varphi_1 - \varphi_2 = 460$ mV, $\varepsilon_2\varkappa_2 \cdot e^{\triangle/2} = 10^8$ cm^{-1}, $z_1 = 8.4$ or $\psi_1(0) = 210$ mV

Hence we see that in the second case the peak potential is appreciably lowered by the presence of the positive space charge in the "oil" phase. In the first case the potential is almost equal to that for a GOUY layer with a charge of 10^{14} charges/cm^2, viz. 330 mV.

(b) $\varepsilon_1\varkappa_1 > \varepsilon_2\varkappa_2$, $\triangle$ = positive, σ = negative.

If the surface charge has a sign opposite to that of the charge of the aqueous phase, the phenomena are more or less the inversal of those in case (a). With increasing negative σ the potential at the interface with respect to the bulk of the aqueous solution will be lowered and the potential drop in this phase increases therefore continually. It is now the oil phase in which the charge will be inversed in sign. Quantitatively, however, the changes induced by this surface charge will be different, corresponding to the fact that we start from an unequal distribution of the potential drop. Actually, the surface charge for which z_2 goes through zero (σ_{00}) is given by the equation (equivalent to eq. (3)):

$$4\pi\sigma_{00} \cdot \frac{e}{kT} = \varepsilon_1\varkappa_1 (e^{-\triangle/2} - e^{\triangle/2})$$

and, because $\varepsilon_1\varkappa_1 > \varepsilon_2\varkappa_2$, $|\sigma_{00}|$ will always be larger than $|\sigma_0|$.

For still larger values of $-\sigma_{00}$ the potential z_2 becomes negative and we have again a triple layer in which now a negative surface charge is balanced by positive space charges at either side of the interface.

Assuming again 1 — 1 valent electrolyte, and $\triangle = 9.2$, we find following values for σ_{00} for different values of $\varepsilon_1\varkappa_1$:

$\varepsilon_1\varkappa_1 = 10^7$ cm^{-1} (distilled water): $-4\pi\sigma_{00} \cdot \frac{e}{kT} = 10^9$ or $\sigma_{00} = 1.7 \times 10^{13}$ ionic charges/cm^2

$\varepsilon_1\varkappa_1 = 10^8$ cm^{-1} ($\backsim 10^{-3}$ n. electrolyte): $\sigma_{00} = 1.7 \times 10^{14}$ ionic charges/cm^2.

The latter value is already about the maximal amount of capillary active ions which can be adsorbed per cm^2. Hence saturation may easily be attained in this case before this second inversion point is reached, especially if, for instance, $\varphi_1 - \varphi_2$ is still larger than 230 mV.

Both cases (a) and (b) are summarized in the set of curves of Fig. 2, representing the electrical potential function in the neighbourhood of the interface for different values of σ. In this graph, holding again for 1 — 1 valent electrolyte, the physical constants have been chosen such that

$$\varepsilon_1\varkappa_2 > \varepsilon_2\varkappa_2 \cdot e^{\Delta/2}.$$

In order to enable a better illustration of the properties of these curves both the chemical differences between the two media and the value of the double layer potential were assumed to be rather small ($\varepsilon_1\varkappa_1 = 27 \cdot \varepsilon_2\varkappa_2$; $e^{\Delta/2} = 10$). Also the ratio R does not differ very much from 1 (for large values of $|\sigma|$, see eq. (5), $R = 2.7$). Accordingly for large values of the surface charge the space charges at both sides of the interface are roughly of the same order of magnitude (27/37 part of the total charge in the aqueous phase, 10/37 in the oil phase).

Some data for z_1 and z_2, corresponding to the case of Fig. 2 are given in Table I. The values of z_1 and z_2 are symmetrical with respect to the values z_1^0 and z_2^0, for equal absolute values of the surface charge, as expressed by eq. (1a).

TABLE I

σ/e	z_1	z_2	σ/e	z_1	z_2
0	— 0.31	4.29	0	— 0.31	4.29
10^{10}	— 0.24	4.36	-10^{10}	— 0.38	4.22
0.42×10^{11}	0	4.6	-0.42×10^{11}	— 0.62	3.98
10^{11}	0.43	5.03	-10^{11}	— 1.05	3.55
10^{12}	3.77	8.37	-10^{12}	— 4.36	0.24
—	—	—	-1.11×10^{12}	— 4.6	0
10^{13}	8.34	12.94	-10^{13}	— 8.96	— 4.36
10^{14}	12.95	17.55	-10^{14}	— 13.57	— 8.97

In conclusion, the author wishes to thank Prof. H. EILERS and Dr M. VAN DER WAARDEN of the Koninklijke / Shell Laboratory, Amsterdam, for a number of valuable remarks.

Summary.

The electrical potential function and the distribution of the charges at the interface of two inmiscible liquids have been calculated, with the aid of a GOUY-CHAPMAN type of theory, for the case that in addition to the original double layer a surface charge is present at the interface.

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A RANK-INVARIANT METHOD OF LINEAR AND POLYNOMIAL REGRESSION ANALYSIS

I

BY

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0. INTRODUCTION

0.0 Regression analysis is usually carried out under the hypothesis that one of the variables is normally distributed with constant variance, its mean being a function of the other variables. This assumption is not always satisfied, and in most cases difficult to ascertain.

In recent years attention has been paid to problems of estimating the parameters of regression equations under more general conditions (see the references at the end of this paper: A. WALD (1940), K. R. NAIR and M. P. SHRIVASTAVA (1942), K. R. NAIR and K. S. BANERJEE (1942), G. W. HOUSNER and J. F. BRENNAN (1948) and M. S. BARTLETT (1949)). Confidence regions, however, were obtained under the assumption of normality only; to obtain these without this assumption will be the main object of this paper.

0.1. In section 1. confidence regions will be given for the parameters of linear regression equations in two variables. In the sequel of this paper we hope to deal with equations in more variables, polynomial equations, systems of equations and problems of prediction.

1. CONFIDENCE REGIONS FOR THE PARAMETERS OF LINEAR REGRESSION EQUATIONS IN TWO VARIABLES

The probability set.

1.0. Throughout this section the probability set Γ ("Wahrscheinlichkeitsfeld" in the sense of A. KOLMOGOROFF) underlying the probability statements will be the $3n$-dimensional Cartesian space R_{3n} with coordinates $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n$. Every random variable mentioned is supposed to be defined on this probability set.

In the first place we suppose 3 n random variables $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$ $(i = 1, \ldots, n)$ [1]) to be defined on Γ, i.e. we suppose $\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{w}_i$ to have a simultaneous probability distribution on Γ.

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[1]) The distinction between a stochastic variable and the value it takes in a given observation (or system of observations) will be indicated by bold type for the former one.

If we now put:

$$\left.\begin{array}{ll} (1) & \theta_i = \alpha_0 + \alpha_1 \xi_i \\ (2) & \eta_i = \theta_i + \mathbf{w}_i \\ (3) & \mathbf{x}_i = \xi_i + \mathbf{u}_i \\ (4) & \mathbf{y}_i = \eta_i + \mathbf{v}_i \end{array}\right\} \; i = 1, \ldots, n$$

then, for any set of values of the $(n+2)$ parameters ξ_i, α_0 and α_1, the varables $\mathbf{x}_i$ and $\mathbf{y}_i$ have a simultaneous distribution on Γ, and are therefore random variables.

We shall call ξ_i the parameter values of the variable ξ. The equation (1) is the regression equation; this equation contains no stochastic variables. Furthermore we shall call $\mathbf{w}_i$ the "true deviations from linearity"; hence the variable η is a linear function of ξ, but for the deviations $\mathbf{w}$. Finally $\mathbf{u}_i$ and $\mathbf{v}_i$ are called the "errors of observation" of the "true" values ξ_i and η_i respectively.

The problem then is, under certain conditions for the probability distribution of $\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{w}_i$, to determine confidence intervals for the parameters α_0 and α_1, given a sequence of observations $x_1, \ldots, x_n, y_1, \ldots, y_n$ of the random variables $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_n$.

Incomplete method: confidence interval for α_1. [2])

1.1. We suppose that the following conditions are satisfied:

Condition I: The n triples $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i)$ are stochastically independent.

Condition II: 1. Each of the errors $\mathbf{u}_i$ vanishes outside a finite interval $|\mathbf{u}_i| \leqq g_i$.

2. For each $i \neq j$ we have: $|\xi_i - \xi_j| > g_i + g_j$.

From condition II it follows that either

$$P[\mathbf{x}_i < \mathbf{x}_j] = 1 \quad \text{and} \quad \xi_i < \xi_j$$

or

$$P[\mathbf{x}_i > \mathbf{x}_j] = 1 \quad \text{and} \quad \xi_i > \xi_j.$$

This condition means that the errors $\mathbf{u}_i$ are sufficiently small in order that arrangement of the observed values x_i according to increasing magnitude be identical with the arrangement according to increasing values of ξ_i (cf. also A. WALD (1940), p. 294, seq., where a similar (weaker) condition is imposed). The arrangement of the $\mathbf{x}_i$ is therefore uniquely determined. We therefore suppose the $\mathbf{x}_i$ as well as the ξ_i to be arranged according to increasing order.

Put $n_1 = n - [\frac{1}{2} n]$; if n is odd, the observation with rank $\frac{1}{2}(n+1)$ is not used. We therefore omit this observation and write $n = 2 n_1$.

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[2]) The author is indebted to Mr J. HEMELRIJK for his constructive criticism concerning some points of this section.

We determine the following n_1 statistics:

$$\Delta(i, n_1+i) = \frac{\mathfrak{y}_{n_1+i} - \mathfrak{y}_i}{\mathfrak{x}_{n_1+i} - \mathfrak{x}_i} = \alpha_1 + \frac{\mathfrak{z}_{n_1+i} - \mathfrak{z}_i}{\mathfrak{x}_{n_1+i} - \mathfrak{x}_i},$$

in which $\mathfrak{z}_i = -\alpha_1 \mathfrak{u}_i + \mathfrak{v}_i + \mathfrak{w}_i$.

We now impose:

Condition III, which states:

$$P[\mathfrak{z}_i < \mathfrak{z}_{n_1+i}] = P[\mathfrak{z}_i > \mathfrak{z}_{n_1+i}] = \tfrac{1}{2}.$$

As all denominators $\mathfrak{x}_{n_1+i} - \mathfrak{x}_i$ are positive, it follows that

$$P[\Delta(i, n_1+i) < \alpha_1] = P[\Delta(i, n_1+i) > \alpha_1] = \tfrac{1}{2},$$

i.e. that $\Delta(i, n_1+i)$ has a median α_1 and that its distribution function is continuous in the median.

The following conditions III*a* and III*b* are each sufficient in order that $P[\mathfrak{z}_i < \mathfrak{z}_{n_1+i}] = P[\mathfrak{z}_i > \mathfrak{z}_{n_1+i}] = \frac{1}{2}$:

Condition III a: the random variables $\mathfrak{z}_i$ $(i = 1, \ldots, n)$ have the same continuous distribution function.

Condition III b: the random variables $\mathfrak{z}_i$ have continuous distribution functions which are symmetrical with equal medians med $(\mathfrak{z})$.

Proof: In case III*a* the simultaneous distribution of $\mathfrak{z}_i$ and $\mathfrak{z}_{n_1+i}$ is symmetrical about the line $z_i = z_{n_1+i}$, which proves the statement. In case III*b* it is symmetrical about the lines $z_i =$ med $(\mathfrak{z})$ and $z_{n_1+i} =$ med $(\mathfrak{z})$; hence the simultaneous distribution of $\mathfrak{z}_i -$ med $(\mathfrak{z})$ and $\mathfrak{z}_{n_1+i} -$ med $(\mathfrak{z})$ is symmetrical with respect to the origin, which proves the statement.

We now arrange the n_1 statistics $\Delta(i, n_1+i)$ in increasing order:

$$\Delta_1 < \Delta_2 < \ldots < \Delta_{n_1},$$

in which

$$\Delta_j = \Delta(i_j, n_1+i_j).$$

The probability that exactly r among the n_1 values $\Delta(i, n_1+i)$ are $< \alpha_1$, i.e. that $\Delta_r < \alpha_1 < \Delta_{r+1}$, is $2^{-n_1}\binom{n_1}{r}$ because of the conditions I and III. Hence:

$$P[\Delta_{r_1} \leqq \alpha_1 \leqq \Delta_{n_1-r_1+1} \mid \alpha_1] =$$

$$= 1 - 2^{1-n_1} \sum_{s=0}^{r_1-1} \binom{n_1}{s}$$

$$= 1 - 2\, I_{\frac{1}{2}}(r_1, n_1 - r_1 + 1)$$

in which

$$I_{\frac{1}{2}}(r_1, n_1 - r_1 + 1) = \frac{\int_0^{\frac{1}{2}} x^{r_1-1}(1-x)^{n_1-r_1}\, dx}{\int_0^1 x^{r_1-1}(1-x)^{n_1-r_1}\, dx}$$

is the incomplete Beta-function for the argument $\frac{1}{2}$.

So we have proved:

Theorem I: under conditions I, II and III a confidence interval for α_1 is given by the largest but $(r_1 - 1)$ and the smallest but $(r_1 - 1)$ among the values $\Delta\,(i,\, n_1 + i)$, the level of significance being $2\, I_{\frac{1}{2}}\,(r_1,\, n_1 - r_1 + 1)$.

We shall call this method an "incomplete method" because a limited use is made of the $\binom{n}{2}$ statistics

$$\Delta\,(ij) = \frac{\mathbf{y}_i - \mathbf{y}_j}{\mathbf{x}_i - \mathbf{x}_j} \qquad (i < j).$$

Incomplete method: confidence region for α_0 and α_1.

1.2. If the median of $\mathbf{z}_i\,(i = 1, \ldots, n)$ is numerically known, a confidence region for α_0 and α_1 can be found. We suppose that the following condition is satisfied:

Condition IV: the median of each $\mathbf{z}_i\,(i = 1, \ldots, n)$ is zero:

$$P\,[\mathbf{y}_i - \alpha_1 \mathbf{x}_i > \alpha_0] = P\,[\mathbf{y}_i - \alpha_1 \mathbf{x}_i < \alpha_0] = \tfrac{1}{2}.$$

For any value of α_1 we can arrange the n quantities $Z_i = y_i - \alpha_1 x_i$ according to increasing magnitude:

$$Z_1\,(\alpha_1) < Z_2\,(\alpha_1) < \ldots < Z_n\,(\alpha_1).$$

Under the condition that α_1 has the value used in this arrangement, we can state that

$$P\,[\alpha_0 \in (\boldsymbol{Z}_{r_0}\,(\alpha_1),\ \boldsymbol{Z}_{n-r_0+1}\,(\alpha_1))\,|\,\alpha_0, \alpha_1] =$$
$$= 1 - 2\, I_{\frac{1}{2}}\,(r_0,\, n - r_0 + 1) = 1 - \varepsilon_0.$$

On the other hand, if we write $\boldsymbol{I}_1$ for the interval $(\Delta_{r_1},\ \Delta_{n_1 - r_1 + 1})$, we can state:

$$P\,[\alpha_1 \in \boldsymbol{I}_1 \,|\, \alpha_1] =$$
$$= 1 - 2\, I_{\frac{1}{2}}\,(r_1,\, n_1 - r_1 + 1) = 1 - \varepsilon_1.$$

If we denote by $\boldsymbol{I}_0$ the interval bounded by the lowest of the values $\boldsymbol{Z}_{r_0}\,(\alpha_1)$ if α_1 varies through $\boldsymbol{I}_1$ and by the largest of the values $\boldsymbol{Z}_{n-r_0+1}\,(\alpha_1)$ if α_1 varies through $\boldsymbol{I}_1$ we have

$$P\,[\alpha_0 \in \boldsymbol{I}_0 \wedge \alpha_1 \in \boldsymbol{I}_1 \,|\, \alpha_0, \alpha_1] \geqq (1 - \varepsilon_0)\,(1 - \varepsilon_1).$$

So we have proved:

Theorem 2: under conditions I, II, III and IV a rectangular confidence region in the α_0, α_1 — plane is given by the intervals $\alpha_0 \in \boldsymbol{I}_0$ and $\alpha_1 \in \boldsymbol{I}_1$, the level of significance being $\leqslant \varepsilon_0 + \varepsilon_1 - \varepsilon_0\, \varepsilon_1$.

If all observed points $(x_i,\, y_i)$ obey the inequality $x_i \geqslant 0$, all quantities $y_i - \alpha_1 x_i$ are decreasing functions of α_1. It follows that I_0 is bounded by $Z_{r_0}\,(\Delta_{n_1 - r_1 + 1})$ and by $Z_{n-r_0+1}\,(\Delta_{r_1})$. The converse holds if every point satisfies the inequality $x_i \leqq 0$.

Complete method.

1.3. We suppose that the conditions I, II and IIIa are satisfied and consider two arrangements of the points (x_i, y_i): the arrangement according to increasing values of x and that according to $z = y - \alpha_0 - \alpha_1 x$.

The arrangement according to z is possible for any assumed value of α_1. The hypothesis that this value is the true one is rejected if and only if there is a significant rank correlation between the arrangements.

Consider the statistics

$$\Delta(ij) = \frac{y_i - y_j}{x_i - x_j} = \alpha_1 + \frac{z_i - z_j}{x_i - x_j},$$

in which $i < j$, so that (if the ordering is according to x) $x_i < x_j$ and $\xi_i < \xi_j$. It follows that $\Delta(ij) > \alpha_1$, if and only if $z_i < z_j$.

Now, under the null hypothesis that the arrangements of the points according to x and according to z are independent, the distribution of Kendall's "*rank correlation coefficient*"

$$\frac{S}{\binom{n}{2}}$$

is known, in which S is the number of cases in which the ordering according to z is the same as the ordering according to x ($z_k < z_{k'}$, and $x_k < x_{k'}$) minus the number of cases in which the ordering according to z is the inverse as compared with the one according to x ($z_k > z_{k'}$, while $x_k < x_{k'}$).

For any value of α_1 the number of cases $z_i > z_j$ can be found. Suppose this to be q; it will be clear that

$$S = \binom{n}{2} - 2q\,.$$

The distribution function of S for any value of n has been given by M. G. Kendall (see M. G. Kendall (1947), p. 403—407 and (1948), p. 55—62) by means of a recurrence formula. So the probability $P[q|n]$ that $q' \leqq q$ cases $z_i > z_j$ are found can be determined. If this probability is below the level of significance chosen, we reject the hypothesis that α_1 has the value used in the arrangement according to z.

Hence, if we arrange the statistics $\Delta(ij)$ in increasing order:

$$\Delta_1 < \Delta_2 < \ldots < \Delta_{\binom{n}{2}}$$

we find by symmetry

$$P\left[\Delta_q \leqq \alpha_1 \leqq \Delta_{\binom{n}{2}-q+1} \mid \alpha_1\right] = 1 - 2P[q-1 \mid n]$$

so that we have proved:

Theorem 3: under conditions I, II and IIIa a confidence interval for α_1 is given by the largest but $(q-1)$ and the smallest but $(q-1)$ among the values $\Delta(ij)$, the level of significance being $2P[q-1|n]$.

The method of 1. 2. can be applied here to find a simultaneous confidence region for α_0 and α_1, I_1 now being the interval $(\Delta_q, \Delta_{\binom{n}{2}-q+1})$.

A comparison.

1. 4. The second method may be called a "complete method", because all statistics $\Delta(i\,j)$ are used. It requires only 5 points in order to reach the level of significance 0,05 whereas the limited method needs 12 points. However, if the number of points is large, the computational labor of the complete method is considerably greater than that of the incomplete method. Moreover, the conditions under which the complete method is valid are more stringent: the fact that the set of conditions I, II and III is sufficient for the incomplete method is important in view of the general occurrence of "heteroscedastic" distribution, i.e. distributions in which the variance (if finite) of η is larger for higher values of ξ than for lower ones if $\alpha_1 > 0$ and conversely if $\alpha_1 < 0$.

Testing linearity.

1. 5. Suppose that the set of conditions I, II and III*a* is valid. Then the hypothesis that the regression curve for two variables is linear can be tested against the alternative composite hypothesis that it is either positive- or negative-convex,[3]) i.e. in the set of equations (1), (2), (3), (4) the equation $\theta_i = \alpha_0 + \alpha_1 \xi_i$ is tested against any equation $\theta_i = \theta(\xi_i)$ with

either $$\frac{d^2\theta}{d\xi^2} > 0 \quad \text{for all } \xi$$

or $$\frac{d^2\theta}{d\xi^2} < 0 \quad \text{for all } \xi,$$

the equations (2), (3), (4) remaining unchanged.

Consider the n_1 statistics

$$\Delta(1, n_1+1), \ldots, \Delta(n_1, 2n_1)$$

in this arrangement. If this ordering has a significant rank correlation with the ordering of these statistics according to increasing magnitude, we reject the hypothesis that the regression curve is linear.

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[3]) A function $f(x)$ is positive-convex (cf. e.g. D. VAN DANTZIG, 93—94 (1947)) in an interval if for every x_1 and x_2 of this interval and for every real positive number $a < 1$ the following inequality is satisfied

$$a f(x_1) + (1-a) f(x_2) > f(a x_1 + \overline{1-a}\, x_2).$$

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CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. II *)

BY

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11. *Functions occurring in the statistical description of the system.* — The object of the following sections is to obtain expressions for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$. For this purpose it will be necessary to introduce certain statistical functions connected with the system. At first sight one might suppose that there would exist an analogy between the problems to be considered here and some investigations by KAMPÉ DE FÉRIET on correlation coefficients associated with stationary random functions [4]). On closer inspection, however, it will be seen that the present system is far more complicated than the cases treated by KAMPÉ DE FÉRIET. The complication is due to the nature of the relations between the λ_i and τ_i. The values of these quantities are derived from two sets of increasing numbers, ξ_i for the λ_i, σ_i for the τ_i, in such a way that every λ_i is associated with both τ_{i-1} and τ_i; and every τ_i with both λ_i and λ_{i+1}. This makes it impossible to define simple independent elements out of which the system can be built up, and thus far I have not succeeded in finding a convenient and precise description of the total ensemble of states possible for the system, together with an indication of the way in which all particular cases out of this ensemble can be obtained and counted.

In view of this difficulty the following way of attack has been chosen. We assume that the system has been given at a certain instant (all deductions of this section and the next ones will refer to a single instant of time only; problems of development in the course of time will not come up before section 17). Since the system is of infinite extent, we can assume that in itself it will already contain all situations compatible with a given set of statistical features, so that it is not necessary to introduce a collection of systems, to be treated simultaneously. We can then, by measuring and counting, determine various statistical properties of this given system.

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*) Continued from these Proceedings, p. 247—260.

4) See J. KAMPÉ DE FÉRIET, Les fonctions aléatoires stationnaires et la théorie statistique de la turbulence homogène, Ann. de la Soc. Scientifique de Bruxelles, **59**, serie I, 145 (1939).

In the first place, starting from an arbitrary value of i, we make a list of the lengths of the consecutive segments:

$$\lambda_i,\ \lambda_{i+1},\ \lambda_{i+2}, \ldots, \lambda_{i+k}, \ldots$$

Such a list can be made beginning from any value of i.

If we give attention to the length of a *single* segment, we can determine a distribution function $f_1(\lambda)\,d\lambda$, defined in such a way that in a group of N single segments, chosen either consecutively or at random (without reference to their lengths), there will be $N f_1(\lambda)\,d\lambda$ segments with lengths between λ and $\lambda + d\lambda$, provided N is sufficiently large. The assumption concerning the statistical homogeneity of the system ensures that such a function $f_1(\lambda)$ will exist. Evidently we shall have:

$$\int_0^\infty f_1(\lambda)\,d\lambda = 1 \quad ; \quad \int_0^\infty f_1(\lambda)\,\lambda\,d\lambda = \overline{\lambda_i} = l \tag{36}$$

We shall further introduce distribution functions $f_k(\Lambda_k)\,d\Lambda_k$ for the total length

$$\Lambda_k = \lambda_{i+1} + \lambda_{i+2} + \ldots + \lambda_{i+k} = \xi_{i+k} - \xi_i \tag{37}$$

of *a set of k consecutive segments*. The function f_k is defined in such a way that in a series of N sets of k consecutive segments there will be $N f_k(\Lambda_k)\,d\Lambda_k$ sets with total lengths between Λ_k and $\Lambda_k + d\Lambda_k$. — When no ambiguity is to be feared we shall often write $f_k(\lambda)\,d\lambda$ instead of $f_k(\Lambda_k)\,d\Lambda_k$. The functions $f_k(\lambda)$ will satisfy the relations:

$$\int_0^\infty f_k(\lambda)\,d\lambda = 1 \quad ; \quad \int_0^\infty f_k(\lambda)\,\lambda\,d\lambda = \overline{\Lambda_k} = kl \tag{38}$$

We must expect that there will exist a connection between the functions f_k and f_1, which could be formulated mathematically if we should know the statistical relations existing between the lengths of consecutive segments. Without explicit calculation it can be observed that when the quantity

$$f_0 = \lim f_1(\lambda) \text{ for } \lambda \to 0$$

has a finite value different from zero, the function $f_k(\lambda)$ will be of order λ^{k-1} for $\lambda \to 0$. It must be kept in mind that f_0 is of the dimension: (length)$^{-1}$.

It seems probable that all functions $f_k(\Lambda_k)$ will decrease to zero exponentially when Λ_k increases without limit. We shall introduce the assumption that

$$\sum_{k=1}^{\infty} f_k(\eta) \quad \text{converges for every finite value of } \eta \tag{39}$$

We can even assume that

$$\sum_{k=1}^{\infty} f_k(\eta) \cong 1/l \tag{39a}$$

provided η is sufficiently large. For $\Sigma f_k(\eta)\, d\eta$ is the probability to find a vertical segment at a point ξ_{i+h} (where h is not known a priori) satisfying the condition:

$$\xi_i + \eta < \xi_{i+h} < \xi_i + \eta + d\eta.$$

This probability must become independent of η when η is large enough.

The assumption (39) entails that

$$\sum_{k=1}^{\infty} \int_0^{\eta} f_k(\eta)\, \eta^m\, d\eta \tag{40}$$

will be convergent for every finite value of η, m being an arbitrary positive quantity (independent of k). With $m = 0$ we shall have

$$\sum_{k=1}^{\infty} \int_0^{\eta} f_k(\eta)\, d\eta = \frac{\eta}{l} - \text{constant} \tag{40a}$$

for large η, in those cases where (39a) holds.

A distribution function $F_k(\lambda_i, \Lambda_k)\, d\lambda_i\, d\Lambda_k$ for the simultaneous values of λ_i and $\Lambda_k = \lambda_{i-1} + \ldots + \lambda_{i-k}$ in a set of $(k+1)$ consecutive segments, will occur as an auxiliary function at a certain point of the deductions.

As far as the arrangement of the segments λ_i is concerned, the most important statistical features of the system are characterised by the functions f_k and F_k. These functions may differ from one type of system to another. For a given system they may also vary in the course of time.

12. Thus far no attention has been given to the values of the τ_i or ζ_i. We cannot simply associate a value τ_i with every λ_i, since the statistical connection between τ_i and λ_i cannot be different from that between τ_{i-1} and λ_i. To associate with every λ_i a half sum $\frac{1}{2}(\tau_{i-1} + \tau_i)$ would be possible in principle, but makes it difficult to find the values of the τ_i separately. (It will be attempted to make use of this idea in section 27.)

What we can do, however, for any given system, is to calculate mean values of quantities like τ_i, τ_i^2, $\tau_{i-1}\tau_i$ etc. *exclusively for segments λ_i of a given length* (more exactly: with a length between given limits λ and $\lambda + d\lambda$). Such "restricted" mean values (which, the same as ordinary mean values of the type indicated by $\overline{}$, are based on a process of counting), will be denoted by $\overline{\tau_i}^*$ etc. They are functions of the length λ assumed for the segments λ_i. It will be evident that $\overline{\tau_{i-1}}^* = \overline{\tau_i}^*$.

From the definition it follows that

$$\int_0^{\infty} f_1(\lambda)\, \overline{\tau_i}^*\, d\lambda = \overline{\tau_i} \tag{41}$$

Indeed, when both sides of the equation are multiplied by N, the right hand side gives the total of all τ_i for a group of N segments (either consecutive or chosen at random); on the left hand side the same sum appears

split up in partial sums, each referring to a particular length λ and each having its proper number of terms $N f_1(\lambda)\, d\lambda$.

In a similar way we shall introduce restricted mean values referring to a fixed value of the length $\Lambda_k = \xi_{i+k} - \xi_i$. We shall be concerned in particular with $\overline{\tau_i \tau_{i+k}}^*$. Again we shall have the relation:

$$\int_0^\infty f_k(\Lambda_k)\, \overline{\tau_i \tau_{i+k}}^*\, d\Lambda_k = \overline{\tau_i \tau_{i+k}} \tag{42}$$

Restricted mean values of the type $\overline{\tau_i \tau_{i+k}}^*$ will form a further set of quantities which serve to characterise the statistical features of a given system.

There remains to consider the ζ_i. As has been mentioned in section 8, when one ζ_i has been given all ζ_{i+k} can be calculated. The relation between these quantities follows from eq. (26) and can be put into the form:

$$\zeta_{i+k} = \zeta_i + \sum_{h=1}^{h=k} \lambda_{i+h} - \tfrac{1}{2}\tau_i - \tau_{i+1} - \ldots - \tau_{i+k-1} - \tfrac{1}{2}\tau_{i+k} \tag{43}$$

The assumption that mean values like $\overline{\zeta_i^2}$ can be defined, puts a certain restriction on the values of the λ_i and τ_i. Indeed, while the assumption (27) is sufficient to ensure that $\overline{\zeta_{i+k}} = 0$ when $\overline{\zeta_i} = 0$, there is a danger that even with a given value of $\overline{\zeta_i^2}$ over a certain limited domain of values of i, the mean value of $\overline{\zeta_{i+k}^2}$ for very large k might appear to be much larger, in consequence of uncorrelated fluctuations of the λ_i and τ_i. It is difficult to put the relevant condition into a concise mathematical form. It can be split up into two separate conditions, one referring to the λ_i, the other one to the τ_i, but it may be that in this way an unnecessary restriction of possibilities is introduced. The condition will turn up at a later point of the deductions in a somewhat disguised form (section 21). Provisionally it is sufficient to work on the basis of the assumption that $\overline{\zeta_i^2}$ exists.

It might be imagined that the difficulty concerning the ζ_i could be evaded by assigning, to each segment λ_i, a definite hinge point. However, although the position of the hinge point assigned to a segment may be chosen arbitrarily with regard to the centre of that segment, in a series of consecutive segments $\lambda_i, \lambda_{i+1}, \lambda_{i+2}, \ldots$, the positions of the consecutive hinge points must satisfy the relation $\ldots \sigma_{i-1} < \sigma_i < \sigma_{i+1} < \sigma_{i+2} < \ldots$. This condition entails awkward relations to be fulfilled by the consecutive segments λ_i.

The calculations of $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$, to be carried out in the following sections, will show in which way the various statistical quantities enter into the deductions.

We add that the invariant relations of section 8 (statistical homogeneity with respect to a shift in the starting point for i; and invariance

with respect to a change of the direction in which i is counted, together with a change of sign of ζ_i) apply likewise to the restricted mean values.

13. *Calculation of* $\overline{v_1 v_2}$. — Provisionally we assume the jumps in the curve for $v(y)$ to be discontinuities. The correction to be introduced for the "rounding off" will be considered in section **15**. We write:

$$v_2 = v_1 + \beta\eta - \triangle,$$

where $\triangle$ is the amount to be subtracted in connection with the number of vertical segments between the points $y+\eta$ and y. We assume that y is situated in the segment λ_i, so that $\xi_{i-1} \leqslant y \leqslant \xi_i$. If $y+\eta$ is situated between ξ_{i+k} and ξ_{i+k+1}, the amount to be subtracted is $\triangle = \beta(\tau_i + \tau_{i+1} + \ldots + \tau_{i+k})$ (compare fig. 6).

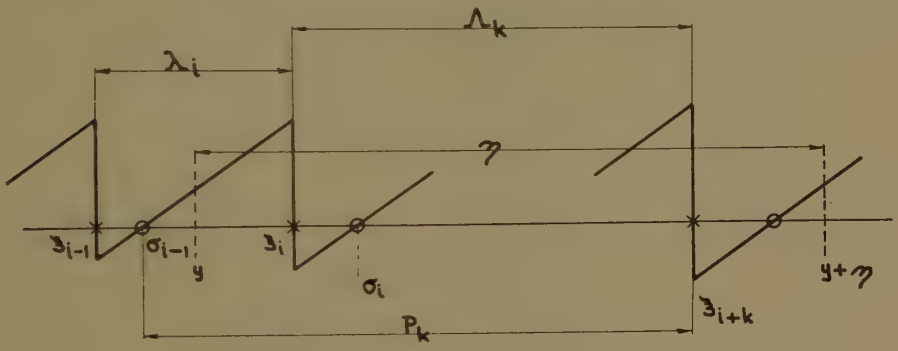

Fig. 6.

When we calculate

$$\int_{\xi_{i-1}}^{\xi_i} dy\, v_1 (v_1 + \beta\eta - \triangle),$$

the first term is the same as occurs in the calculation of $\overline{v^2}$ and its mean value can be obtained from section **9**; the second one, apart from the constant factor $\beta\eta$, occurs in the calculation of $\overline{v}$ and consequently gives zero in the final result.

Coming to the amount $\triangle$, its first term $\beta\tau_i$ must be subtracted as soon as $y+\eta > \xi_i$. When $\lambda_i > \eta$, this will be the case for values of y within the range

$$\xi_i - \eta \leqslant y \leqslant \xi_i;$$

when $\lambda_i < \eta$, it will be the case for all values of y within the range

$$\xi_{i-1} \text{ (or, } \xi_i - \lambda_i) \leqslant y \leqslant \xi_i.$$

Giving attention to the values of v corresponding to the limiting values of y, the contribution to the integral is found to be:

$$-\tfrac{1}{2}\beta^2 \tau_i \{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^2\} \quad \text{for} \quad \lambda_i > \eta;$$
$$-\tfrac{1}{2}\beta^2 \tau_i \{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_i + \tfrac{1}{2}\tau_i - \lambda_i)^2\} \quad \text{for} \quad \lambda_i < \eta.$$

Since the probabilities or frequencies of the two cases depend on the distribution function $f_1(\lambda)\,d\lambda$ for the lengths λ_i, the mean contribution to the integral takes the form:

$$-\tfrac{1}{2}\beta^2\int_{\eta}^{\infty} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}}^{*} -$$

$$-\tfrac{1}{2}\beta^2\int_{0}^{\eta} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\lambda)^2\}}^{*},$$

where use has been made of the notation introduced for restricted mean values referring to a particular value of λ_i. The expression can be transformed into:

$$-\tfrac{1}{2}\beta^2\int_{0}^{\infty} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}}^{*} -$$

$$-\tfrac{1}{2}\beta^2\int_{0}^{\eta} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\lambda)^2\}}^{*}.$$

We consider the first integral in particular. According to the general relation exemplified in (41) it is equal to:

$$-\tfrac{1}{2}\beta^2\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}},$$

which after some simple reductions yields the result:

$$\text{(I)}\qquad -\tfrac{1}{2}\beta^2(\overline{\tau_i^2}\,\eta-\overline{\tau_i}\,\eta^2)=-\tfrac{1}{2}\beta^2 l^3\left\{(1+\omega)\frac{\eta}{l}-\frac{\eta^2}{l^2}\right\}$$

It is not necessary to calculate the second integral, since that quantity will be included automatically in an expression to be considered presently.

14. *Continuation.* — We turn to the consideration of the term $\beta\tau_{i+k}$, which must be subtracted as soon as $y+\eta>\xi_{i+k}$. Since $y\leqslant\xi_i$, this can only occur provided $\Lambda_k<\eta$. Again we distinguish between two cases: if $\lambda_i+\Lambda_k>\eta$, the subtraction is to be made only for values of y within the range $\xi_{i+k}-\eta<y<\xi_i$; if $\lambda_i+\Lambda_k<\eta$, so that $\xi_{i+k}-\eta<$ $<\xi_{i-1}$, subtraction must be made for all values of y within the range $\xi_{i-1}<y<\xi_i$.

We introduce the notations:

$$\text{(44)}\qquad T_k=\tfrac{1}{2}\tau_i+\tau_{i+1}+\tau_{i+2}+\ldots+\tau_{i+k-1}+\tfrac{1}{2}\tau_{i+k}$$

$$\text{(44}a\text{)}\qquad P_k=\xi_{i+k}-\sigma_{i-1}=T_k+\tfrac{1}{2}\tau_i+\zeta_{i+k}$$

The values of v at the limits of the integration interval can then be written:

in the first case ..... $\beta(P_k-\eta)$, $\beta(P_k-\Lambda_k)$;

in the second case .... $\beta(P_k-\Lambda_k-\lambda_i)$, $\beta(P_k-\Lambda_k)$.

We now need a distribution function $F_k(\lambda_i, \Lambda_k)\, d\lambda_i\, d\Lambda_k$ for the simultaneous values of λ_i and Λ_k. The terms dependent on τ_{i+k} can appear only if $\Lambda_k < \eta$; the limits to be used depend on the difference between λ and $\eta - \Lambda_k$. Hence the contribution to the integral, after a slight re-arrangement, can be written:

$$\text{(II)} \quad \left\{ \begin{aligned} &-\tfrac{1}{2}\beta^2 \int_0^{\eta} d\Lambda_k \int_0^{\infty} d\lambda_i\, F_k(\lambda_i, \Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\Lambda_k)^2-(P_k-\eta)^2\}}^* - \\ &-\tfrac{1}{2}\beta^2 \int_0^{\eta} d\Lambda_k \int_0^{\eta-\Lambda_k} d\lambda_i\, F_k(\lambda_i, \Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\eta)^2-(P_k-\Lambda_k-\lambda_i)^2\}}^* \end{aligned} \right.$$

where the restricted mean values now refer to fixed values of both λ_i and Λ_k. Giving attention to the definition of the function $F_k(\lambda_i, \Lambda_k)$ we see that its integral with respect to λ_i from 0 to ∞ yields the function $f_k(\Lambda_k)$. The first integral occurring in (II) conseqently transforms into:

$$\text{(III)} \qquad -\tfrac{1}{2}\beta^2 \int_0^{\eta} d\Lambda_k\, f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\Lambda_k)^2-(P_k-\eta)^2\}}^*$$

where the restricted mean value simply refers to a fixed value of Λ_k, without reference to a particular value of λ_i.

In consequence of (40) the series obtained by summing (III) with respect to k will be convergent for every finite value of η. It follows that we may treat both parts of the expression (II) separately.

To reduce the second integral to a more convenient form we decrease k by one unit. At the same time we replace i by $i+1$, which can be done without affecting the mean value. By this process P_k (which first became P_{k-1}) is changed into $P_k - \tau_i$. Further Λ_k is changed into $\Lambda_{k-1} = \lambda_{i+2} + \ldots + \lambda_{i+k}$; and $\Lambda_k + \lambda_i$ becomes $\Lambda_{k-1} + \lambda_{i+1} = \lambda_{i+1} + \lambda_{i+2} + \ldots + \lambda_{i+k} = \Lambda_k$. The expression to be calculated consequently takes the form:

$$-\tfrac{1}{2}\beta^2 \int_0^{\eta} d\Lambda_{k-1} \int_0^{\eta-\Lambda_{k-1}} d\lambda_{i+1}\, F_{k-1}(\lambda_{i+1}, \Lambda_{k-1})\, \overline{\tau_{i+k}\{(P_k-\tau_i-\eta)^2-(P_k-\tau_i-\Lambda_k)^2\}}^*.$$

It will be recognised that in this integral we combine all cases in which $\Lambda_{k-1} + \lambda_{i+1} = \Lambda_k$ takes values from 0 to η, with the proper probability function. Hence the integral can be transformed into:

$$\text{(IV)} \qquad -\tfrac{1}{2}\beta^2 \int_0^{\eta} d\Lambda_k\, f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\tau_i-\eta)^2-(P_k-\tau_i-\Lambda_k)^2\}}^*$$

We now combine the integrals (III) and (IV). When the expressions between the { } are worked out, the result of the combination is:

$$\text{(V)} \qquad +\beta^2 \int_0^{\eta} d\Lambda_k\, f_k(\Lambda_k)\, (\Lambda_k - \eta)\, \overline{\tau_i \tau_{i+k}}^*$$

With $k=1$ the integral (IV) reduces to

$$-\tfrac{1}{2}\beta^2 \int_0^\eta d\lambda_{i+1} f_1(\lambda_{i+1}) \overline{\tau_{i+1}\{(\zeta_{i+1}+\tfrac{1}{2}\tau_{i+1}-\eta)^2-(\zeta_{i+1}+\tfrac{1}{2}\tau_{i+1}-\lambda_{i+1})^2\}}^*,$$

which, by changing $i+1$ into i, will be seen to be precisely the quantity we had left aside at the end of the preceding section, when we considered the contributions derived from $\beta\tau_i$. Hence, when the expression (IV) is summed with respect to k from $k=1$ onward, that quantity is automatically included. The convergence of the sum again follows from (40).

We introduce the functions:

$$\varphi_k(\eta) = l^{-2} \int_0^\eta d\lambda\, f_k(\lambda) \overline{\tau_i \tau_{i+k}}^* \tag{45}$$

$$\chi_k(\eta) = l^{-3} \int_0^\eta d\lambda\, f_k(\lambda)\, \lambda \overline{\tau_i \tau_{i+k}}^* \tag{46}$$

The final expression for $\overline{v_1 v_2}$ is obtained by combining the expression (I) of section **11** with (V) as found above. We must divide by l to pass from $\overline{}$ mean values to mean values per unit length of the y-axis. Making use of (45) and (46) this gives:

$$\overline{v_1 v_2} = \overline{v^2} - \beta^2 l^2 \left\{ \frac{1+\omega}{2}\frac{\eta}{l} - \frac{\eta^2}{2l^2} + \sum_1^\infty \left(\frac{\eta}{l}\varphi_k - \chi_k \right) \right\} \tag{47}$$

The value of $\overline{v^2}$ must be taken from (30). — We mention the derivative:

$$\frac{\partial}{\partial\eta}\overline{v_1 v_2} = -\beta^2 l \left\{ \frac{1+\omega}{2} - \frac{\eta}{l} + \sum_1^\infty \varphi_k \right\} \tag{48}$$

(the derivatives of φ_k and χ_k drop out).

Since $f_k(\lambda)$ is of order λ^{k-1} for small λ, it follows that, for small η, the function φ_k is of order η^k and the function χ_k of order η^{k+1}. Hence the sum in (47) begins with a term in η^2 and that in (48) with a term in η.

The expressions (47) and (48) must satisfy the condition that $\overline{v_1 v_2}$ and $\partial(\overline{v_1 v_2})/\partial\eta$ shall vanish for infinite values of η. In order to prove this we must investigate the sums $\Sigma\varphi_k$ and $\Sigma\chi_k$. We will come back to this point in section **20**.

15. *Correction for the rounding off of the jumps in the function $v(y)$.* — The expression (47) is not valid for values of η of the order $\nu/(v_- - v_+)$, or in the present notation, $\nu/\beta\tau_i$. It cannot be continued analytically down to $\eta=0$ and through $\eta=0$ to negative values. Indeed (47) is not an even function of η; when it should be used both for positive and negative values, η must be replaced by its absolute value $|\eta|$.

A correction to (47), leading to a formula valid for very small values (positive or negative) of η, can be found without difficulty if we may assume that the distance between two consecutive jumps in the great

majority of cases is large compared with $\nu/\beta\tau_i$. The frequency of cases where this is otherwise will be determined by $f_0\nu/\beta\tau_i$; since f_0 can be expected to be of order $1/l$, this frequency will be of the order $\nu/\beta l^2 = \nu t/l^2$ [5]). We assume that this quantity is small compared with unity.

The correction to be applied to (47) can be found by making use of the hyperbolic tangent function indicated in (24). For simplicity we take the origin for y at the point ξ_i: in the neighbourhood of this point we write:

$$v = \beta\,(y - \zeta_i) - \tfrac{1}{2}\,\beta\,\tau_i \tanh\,(\beta\,\tau_i\,y/4\nu),$$

where ζ_i is a constant. We calculate the integral of $v_1 v_2$ (in which η is supposed to be small) over the interval $-L < y < +L$, L being a length of order l, but less than both λ_i and λ_{i+1}. The difference between the value of the integral for finite ν and its value for $\nu = 0$, is the correction to be applied in connection with the jump at ξ_i. We pass over the details of the calculation; when we may assume that $\beta\tau_i(L - |\eta|)/\nu = \tau_i(L - |\eta|)/\nu t$ is sufficiently large (say greater than 10), the result is:

$$\tfrac{1}{2}\,\beta^2\,\eta\,\tau_i^2\,(1 - \mathrm{ctnh}\,\beta\,\tau_i\,\eta/4\nu) \tag{49}$$

The correction to formula (47) is the mean value of this expression:

$$\frac{\beta^2\,\eta}{2\,l}\,\overline{\tau_i^2\,(1 - \mathrm{ctnh}\,\beta\,\tau_i\,\eta/4\nu)} \tag{49a}$$

For values of η small in comparison with $4\nu t/l$ we may develop the ctnh function, which gives:

$$\frac{\beta^2\,\eta}{2\,l}\,\overline{\tau_i^2} - \frac{2\,\beta\,\nu}{l}\,\overline{\tau_i} - \frac{\beta^3\,\eta^2}{24\,l\nu}\,\overline{\tau_i^3} \ldots,$$

or, making use of eqs. (27) and (31) and re-arranging:

$$-2\,\beta\,\nu + \beta^2\,l^2\,\frac{1+\omega}{2}\,\frac{\eta}{l} - \frac{\beta^3\,l^4}{\nu}\,\frac{1+\omega^*}{24}\,\frac{\eta^2}{l^2} \ldots \tag{49b}$$

There is thus a small correction to the value of $\overline{v^2}$ given in (30); while if (49b) is added to (47) the following expression is obtained for $\overline{v_1 v_2}$, valid for $\eta \ll 4\nu t/l$:

$$\overline{v_1\,v_2} = \overline{(v^2)}_{\text{uncorrected}} - 2\,\beta\,\nu - \frac{\beta^3\,l^4}{\nu}\,\frac{1+\omega^*}{24}\,\frac{^2}{l^2} \tag{50}$$

The term with η^2 in (47) and the terms depending on the functions φ_k and χ_k have been omitted, since, for the values of η considered here, they are insignificant in comparison with the last term of (50).

It is now possible to apply eqs. (8) and (9). This gives the result:

$$\varepsilon = \tfrac{1}{12}\,\beta^3\,l^2\,(1 + \omega^*),$$

which is in conformity with (33).

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[5]) Very small values of τ_i will be an exception, since the τ can only increase by coalescence. Hence for estimating orders of magnitude we may replace τ_i by the mean value l.

16. *Calculation of* $\overline{v_1^2 v_2}$. — We follow the same procedure as applied in sections 13 and 14 and calculate

$$\int_{\xi_{i-1}}^{\xi_i} dy\, v_1^2\,(v_1 + \beta\eta - \Delta).$$

The first term leads to the mean value of v^3, which is zero. The second term leads to

$$\beta\eta\,\overline{v^2} = \beta^3 l^3 \{\tilde{\omega} + \tfrac{1}{12}(1+\omega^*)\}\frac{\eta}{l} \tag{VI}$$

[compare (30) and (31)].

The first term in Δ, *viz.* $\beta\tau_i$, gives rise to two integrals of a type similar to those occurring in section 13. The first one of these has the form

$$-\tfrac{1}{3}\beta^3 \int_0^\infty d\lambda\, f_1(\lambda)\, \overline{\tau_i \{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^3\}}^*,$$

which according to (41) is equal to:

$$-\tfrac{1}{3}\beta^3 \overline{\tau_i \{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^3\}}.$$

After a few reductions this yields the result:

$$\left.\begin{aligned} &-\beta^3 \{(\overline{\tau_i \zeta_i^2} + \tfrac{1}{4}\overline{\tau_i^3})\eta - \tfrac{1}{2}\overline{\tau_i^2}\eta^2 + \tfrac{1}{3}\overline{\tau_i}\eta^3\} = \\ &\qquad = \beta^3 l^4 \left\{-\left(\tilde{\omega} + \frac{1+\omega^*}{4}\right)\frac{\eta}{l} + \frac{1+\omega}{2}\frac{\eta^2}{l^2} - \frac{1}{3}\frac{\eta^3}{l^3}\right\} \end{aligned}\right. \tag{VII}$$

The contribution depending on $\beta\tau_{i+k}$ can be transformed in a similar way as was followed in section 14. We pass over the details; the result is

$$\left.\begin{aligned} &-\tfrac{1}{3}\beta^3 \int_0^\eta d\Lambda_k f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k - \Lambda_k)^3 - (P_k - \eta)^3\}}^* - \\ &\qquad -\tfrac{1}{3}\beta^3 \int_0^\eta d\Lambda_k f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k - \tau_i - \eta)^3 - (P_k - \tau_i - \Lambda_k)^3\}}^* = \\ &= \beta^3 \int_0^\eta d\Lambda_k f_k(\Lambda_k)\{2(\Lambda_k - \eta)\,\overline{\tau_i \tau_{i+k}(P_k - \tfrac{1}{2}\tau_i)}^* - (\Lambda_k^2 - \eta^2)\,\overline{\tau_i \tau_{i+k}}^*\} \end{aligned}\right. \tag{VIII}$$

The first two expressions correspond to the integrals (III) and (IV), respectively, of section 14; the final expression corresponds to (V).

According to (44*a*) $P_k - \frac{1}{2}\tau_i = T_k + \zeta_{i+k}$. Making use of the second invariant relation, we can replace ζ_{i+k} by $-\zeta_i$, which will not affect the mean value. It is thus possible to replace $2(P_k - \frac{1}{2}\tau_i)$ by $2T_k + \zeta_{i+k} - \zeta_i = T_k + \Lambda_k$ [compare (43)]. In this way (VIII) is transformed into

$$\beta^3 \int_0^\eta d\Lambda_k f_k(\Lambda_k)\{(\Lambda_k - \eta)\,\overline{T_k \tau_i \tau_{i+k}}^* - (\Lambda_k \eta - \eta^2)\,\overline{\tau_i \tau_{i+k}}^*\} \tag{IX}$$

We put:

$$\Phi_k(\eta) = l^{-3} \int_0^\eta d\lambda\, f_k(\lambda)\, \overline{T_k \tau_i \tau_{i+k}}^* \tag{51}$$

$$X_k(\eta) = l^{-4} \int_0^\eta d\lambda\, f_k(\lambda)\, \lambda\, \overline{T_k \tau_i \tau_{i+k}}^* \tag{52}$$

Combining the expression (VI) with (VII) and (IX) divided by l, we arrive at the following formula for $\overline{v_1^2 v_2}$:

$$\overline{v_1^2 v_2} = -\beta^3 l^3 \left[\frac{1+\omega^*}{6}\frac{\eta}{l} - \frac{1+\omega}{2}\frac{\eta^2}{l^2} + \frac{1}{3}\frac{\eta^3}{l^3} - \right. \\ \left. - \sum_1^\infty \left\{\frac{\eta^2}{l^2}\varphi_k - \frac{\eta}{l}(\Phi_k + \chi_k) + X_k\right\}\right] \tag{53}$$

From (53) we derive:

$$\frac{\partial}{\partial \eta}(\overline{v_1^2 v_2}) = -\beta^3 l^2 \left[\frac{1+\omega^*}{6} - (1+\omega)\frac{\eta}{l} + \frac{\eta^2}{l^2} - \sum_1^\infty \left\{\frac{2\eta}{l}\varphi_k - \Phi_k - \chi_k\right\}\right] \tag{54}$$

For small η (small compared with l, but not so small as $4vt/l$) the function Φ_k is of order η^k and the function X_k of order η^{k+1}. It follows that the expression between $\{\}$ in (53) begins with a term in η^2 and in (54) with a term in η. Formulae (53) and (54) are not valid down to $\eta = 0$ and cannot be continued analytically to negative values of η.

17. *Application of the fundamental equation* (12). — Now that we have obtained expressions for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ it is possible to make use of equation (12) to investigate the change in time of some of the statistical quantities we have encountered.

As was mentioned in section 2, equation (12), when applied to $\eta = 0$, brings us back to eq. (4). With E as given by (32), ε by (33) and $\beta = 1/t$, we obtain (after multiplication by 2):

$$\frac{d}{dt}\left[\frac{l^2}{t^2}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right)\right] = -\frac{l^2}{t^3}\frac{1+\omega^*}{6} \tag{55}$$

When $\eta \gg 4vt/l$, we can use (12a) instead of eq. (12) and apply expressions (47) and (53) for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ respectively. This gives

$$\frac{\partial}{\partial t}\left[\frac{l^2}{t^2}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right) - \frac{l\eta}{t^2}\frac{1+\omega}{2} + \ldots\right] = \\ = -\frac{l^2}{t^3}\frac{1+\omega^*}{6} + \frac{l\eta}{t^3}(1+\omega) - \frac{l^2\Phi_1}{t^3} + \ldots \tag{56}$$

where terms not written out are at least of order η^2. Comparison of the terms of zero order brings us back to eq. (55), which thus also follows from expression (53) for $\overline{v_1^2 v_2}$. Comparison of the first order terms gives a new relation:

$$\frac{d}{dt}\left[\frac{l}{t^2}\frac{1+\omega}{2}\right] = -\frac{l}{t^3}(1+\omega) + \frac{l^2}{t^3}\lim_{\eta\to 0}\frac{\Phi_1}{\eta} \tag{57}$$

Having regard to (51) and noting that $T_1 = \frac{1}{2}(\tau_i + \tau_{i+1})$, we find:

$$\lim_{\eta \to 0} (\Phi_1/\eta) = (f_0/2\,l^3)\ \overline{(\tau_i + \tau_{i+1})\,\tau_i \tau_{i+1}}^{*} \tag{58}$$

the restricted mean value referring to $\lambda_i \to 0$.

By eliminating t^{-2} from the left hand members we can simplify eqs. (55), (57) to:

$$\frac{d}{dt}\left[l^2\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right)\right] = \frac{2\,l^2\,\tilde{\omega}}{t} \tag{59}$$

$$\frac{d}{dt}\,[l\,(1+\omega)] = \frac{f_0}{l\,t}\,\overline{(\tau_i + \tau_{i+1})\,\tau_i \tau_{i+1}}^{*} \tag{60}$$

18. *Direct derivation of eqs.* (59), (60). — It is of interest to deduce these equations in a different way, which will throw some light on the meaning of the terms occurring in them.

We begin with eq. (60). According to (31) the expression between brackets on the left hand side is equal to $\overline{\tau_i^2}/l$. If we multiply by a large length S which is independent of the time, we obtain $N\,\overline{\tau_i^2}$ (with $N = S/l$). This is equal to the sum of all τ_i^2 to be found in the length S. Now in consequence of the laws of motion, section 7 (V), the τ_i do not change in the course of time, unless two consecutive τ's coalesce. When this happens to τ_i and τ_{i+1}, the sum of the squares increases by the amount:

$$(\tau_i + \tau_{i+1})^2 - \tau_i^2 - \tau_{i+1}^2 = 2\,\tau_i\,\tau_{i+1}.$$

In order to find the frequency of this process we must determine the number of segments λ_{i+1} which decrease to zero in the element of time dt. Since for small λ we have $d\lambda_{i+1}/dt = -(\tau_i + \tau_{i+1})/2t$, the magnitude of λ_{i+1} may be at most $dt \cdot (\tau_i + \tau_{i+1})/2t$ and the number of such segments in the length S will be given by:

$$N\,f_0\,dt\,(\tau_i + \tau_{i+1})/2\,t \tag{61}$$

Hence the rate of increase of the sum of all τ_i^2 in S is given by the mean value $N\,f_0\,\overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{*}/t$ (restricted mean value for $\lambda_i \to 0$), and we obtain:

$$d\,(N\,\overline{\tau_i^2})/dt = N\,f_0\,\overline{(\tau_i + \tau_i)\,\tau_i\,\tau_{i+1}}^{*}/t.$$

Division by the constant length S brings us back to (60).

To prove (59) we transcribe it in the same way and consider:

$$d\,[N\,(\overline{\tau_i\,\zeta_i^2} + \tfrac{1}{12}\,\overline{\tau_i^3})]/dt = 2\,N\,\overline{\tau_i\,\zeta_i^2}/t.$$

From section 7 (III), we have $d\zeta_i/dt = \zeta_i/t$. Hence:

$$d\,(\tau_i\,\zeta_i^2)/dt = 2\,\tau_i\,\zeta_i^2/t.$$

This already gives us the term on the right hand side.

Further, when τ_i and τ_{i+1} coalesce, τ_i and τ_{i+1} simply add up; while ζ_i and ζ_{i+1} vanish, being replaced by a single quantity equal to:

$$\zeta^0 = \zeta_i - \tfrac{1}{2}\tau_{i+1} = \zeta_{i+1} + \tfrac{1}{2}\tau_i$$

(compare the end of section 7). It follows that the sum increases by an amount which can be written

$$\tau_i(\zeta_i - \tfrac{1}{2}\tau_{i+1})^2 + \tau_{i+1}(\zeta_{i+1} + \tfrac{1}{2}\tau_i)^2 + \tfrac{1}{12}(\tau_i + \tau_{i+1})^3 - \\ - \tau_i\zeta_i^2 - \tau_{i+1}\zeta_{i+1}^2 - \tfrac{1}{12}\tau_i^3 - \tfrac{1}{12}\tau_{i+1}^3 = \\ = \tau_i\tau_{i+1}(\tfrac{1}{2}\tau_i + \tfrac{1}{2}\tau_{i+1} + \zeta_{i+1} - \zeta_i) = 0,$$

since the expression between ( ) vanishes at the moment of coalescence. Hence the coalescence of segments is without influence on the value of the sum, so that the relation we were considering is proved.

The same method can be applied to other quantities.

In the first place we refer to LOITSIANSKY's invariant J_0 for which an expression was obtained in section 10. We consider formula (35) for J_0 and note that according to the laws of motion $\beta\zeta_i = \zeta_i/t$ is constant so long as coalescence does not take place. Hence the regular development of the system will not influence the value of J_0. To prove that J_0 neither changes through the effect of coalescence, we take J_0 in the form:

$$\tfrac{1}{4}\beta^2\overline{\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2\}\,\Sigma\,\tau_{i+k}\zeta_{i+k}}.$$

It is not difficult to show that the sum $\Sigma\tau_{i+k}\zeta_{i+k}$ does not change upon coalescence of any two segments. As regards the factor $\{(\zeta_i + \frac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \frac{1}{2}\zeta_{i-1})^2\}$, it does certainly not change upon coalescence of τ_{k-1} and τ_k when $k < i-1$ or $k > i+1$; neither does it change when $k = i-1$ or $k = i+1$, since in the first case the value of $\zeta_{i-1} - \frac{1}{2}\tau_{i-1}$, and in the second case the value of $\zeta_i + \frac{1}{2}\tau_i$ is unaffected. Finally, when $k = i$, the factor will disappear from the sum already just before coalescence has taken place, on account of the relation $\zeta_i + \frac{1}{2}\tau_i = \zeta_{i-1} - \frac{1}{2}\tau_{i-1}$, which is valid for $\lambda_i \to 0$: while after coalescence it is to be dropped altogether [6]).

We finally apply the method to find $d\varepsilon/dt$. If we multiply by S, we have:

$$S\,\varepsilon = N\,\overline{\tau_i^3}/12\,t^3.$$

In the case of coalescence of τ_i and τ_{i+1}, the sum of all τ_i^3 increases by $3(\tau_i^2\tau_{i-1} + \tau_i\tau_{i+1}^2)$. Since the frequency of this process is determined by the expression (61), it is easily found that

$$\frac{d\varepsilon}{dt} = -\frac{3\varepsilon}{t} + \frac{f_0}{8\,l\,t^4}\overline{(\tau_i + \tau_{i+1})^2\,\tau_i\tau_{i+1}}^* \tag{62}$$

the restricted mean value again referring to $\lambda_i \to 0$. An equivalent form is:

$$\frac{d}{dt}[l^2(1+\omega^*)] = \frac{3\,f_0}{2\,l\,t}\overline{(\tau_i + \tau_{i+1})^2\,\tau_i\tau_{i+1}}^* \tag{63}$$

[6]) Another proof of the invariance of J_0 will be given in the third part of this paper (section 26).

19. *Value of dl/dt.* — The expression (61), giving the number of cases in which segments coalesce, can be used to find the rate of decrease of N, which is given by:

$$dN/dt = -N f_0 \overline{(\tau_i + \tau_{i+1})}^* / 2t \tag{64}$$

In view of what has been observed in section 12 (end of second paragraph), we may replace $\frac{1}{2}\overline{(\tau_i+\tau_{i+1})}^*$ by $\overline{\tau_i}^*$. The symmetrical notation used in (64), however, has been retained on account of its analogy with (60) and (63).

Since $N = S/l$, S being a constant length, we find:

$$\frac{dl}{dt} = \frac{f_0 l}{2t} \overline{\tau_i + \tau_{i+1}}^* \tag{65}$$

The restricted mean value refers to $\lambda_i \to 0$.

We can apply this result to eliminate dl/dt from equations (60), (63) and (59). This gives

$$\frac{d}{dt}(1+\omega) = \frac{f_0}{t}\left\{ \frac{\overline{(\tau_i+\tau_{i+1})\,\tau_i\tau_{i+1}}^*}{l^2} - (1+\omega)\frac{\overline{\tau_i+\tau_{i+1}}^*}{2} \right\} \tag{66}$$

$$\frac{d}{dt}(1+\omega^*) = \frac{f_0}{t}\left\{ \frac{3\,\overline{(\tau_i+\tau_{i+1})^2\,\tau_i\tau_{i+1}}^*}{2\,l^3} - (1+\omega^*)\,\overline{(\tau_i+\tau_{i+1})}^* \right\} \tag{67}$$

$$\frac{d}{dt}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right) = \frac{2\tilde{\omega}}{t} - \frac{f_0}{t}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right)\overline{(\tau_i+\tau_{i+1})}^* \tag{68}$$

These equations bring to light a few of the features of the system which determine the change with time of its statistical character. For instance, if we should start with a case in which all τ_i originally had the same value, equal to l (so that ω would be zero), while the length of the segments λ_i was distributed at random, equation (66) would give:

$$d(1+\omega)/dt = f_0\, l/t.$$

This would mean that an increase of the ratio $\overline{\tau_i^2}/l^2$ would immediately set in, which is, of course, to be expected.

It may be supposed that a statistical state can be reached with a pattern independent of the time. This supposition is made by those authors who speak of "self-preserving" correlation functions. For such a state eqs. (66) — (68) should have zero right hand members. We shall come back to this subject in section 24.

(*To be continued*).

INFLUENCE OF ORGANIC COMPOUNDS ON SOAP- AND PHOSPHATIDE COACERVATES — XII [1])

THE ACTION OF ALKANES, ALKENES AND ALKYNES ON OLEATE COACERVATES

BY

H. L. BOOIJ, J. C. LYCKLAMA *) AND C. J. VOGELSANG *) [2]) [3])

(Communicated by Prof. H. G. BUNGENBERG DE JONG at the meeting of March 25, 1950)

1. *Introduction.*

From theoretical considerations one of us (H. L. BOOIJ) concluded that substances without polar groups might be found in soap micelles in two places, viz. parallel to the soap molecules and between the CH_3-end groups of these molecules (see H. L. BOOIJ, 1949, 1). This hypothesis was substantiated by an investigation of the influence of some organic ring systems with and without polar groups (BOOIJ, 1949, 2). From this point of view it seemed interesting to investigate the action of a homologous series of alkanes. KOETS and BUNGENBERG DE JONG (1938) had already found that the influence of *n*-octane, *n*-heptane, *n*-hexame and *n*-pentane did not fit into the scheme of the action of organic substances as laid down by BUNGENBERG DE JONG, SAUBERT and BOOIJ (1938). As a general rule it may be said that an organic substance with a certain number of carbon atoms has a stronger action than a substance of the same nature but with less carbon atoms. For the alkanes, however, the condensing action increased in the series:

$$n\text{-octane} < n\text{-heptane} < n\text{-hexane} - n\text{-pentane}.$$

These experiments were performed in the following manner. As it is excluded to dissolve the substances in water, they were mixed with oleic acid in the ratio wanted. Then a solution of KOH and water are

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[1]) Publication no. XI of this series will be found in Proc. Kon. Ned. Akad. v. Wetensch. Amst. **53**, 299 (1950).

[2]) This investigation has been made possible thanks to a generous gift of Na-oleate from The Rockefeller Foundation. We are also much indebted to the Koninklijke/Shell Laboratorium, Amsterdam, which sent us the paraffins for our experiments.

[3]) Publication no. 3 of the Team for Fundamental Biochemical Research (under the direction of H. G. BUNGENBERG DE JONG, E. HAVINGA and H. L. BOOIJ).

added, which gives a solution of *K*-oleate (the alkanes remaining in the soap micelles). This solution was coacervated with the aid of KCl in the usual manner. In our experiments we used another method in order to investigate a large number of hydrocarbons.

2. *Method.*

When we add *n*-propylalcohol to a coacervate we get in low concentration a condensing action on the coacervate (the volume of the coacervate decreases), while in higher concentration (> 0.15 mol/1) we observe the opposite result. At approximately 150 mmol/1 the alcohol has no influence on the volume of the coacervate. It is possible to make a solution of the substance in an oleate sol by adding a measured quantity of oleate solution to the propylalcohol in which we dissolve the substance to be investigated. We thus get clear solutions of compounds practically completely insoluble in water.

In the usual way (Booij and Bungenberg de Jong, 1949) the oleate solutions have been coacervated with the aid of KCl. At a constant temperature of 25° C the following mixtures are made in tubes with ground glass stoppers:

$$\left\{\begin{array}{ll} x & \text{ml KCl } 3.8\ n \\ (6.5 - x) & \text{ml } H_2O \\ 3 & \text{ml oleate solution.} \end{array}\right.$$

To prevent changes in the coacervate volume by the action of carbondioxyde from the air (the greatest drawback of our former experiments, Bungenberg de Jong et al., 1937, 1938) the oleate solution has been made as follows:

$$\left\{\begin{array}{l} 980 \text{ ml } H_2O \\ 120 \text{ ml KOH } 2\ n \\ 20 \text{ g Na-oleate.} \end{array}\right.$$

The alkane to be investigated was weighed in an erlenmeyer flask, 0.9 ml *n*-propylalcohol was added and after the dissolution of the substance a quantity of oleate solution (25 ml) was introduced.

The tubes were left in the thermostate overnight, after which the volumes of the coacervate layers were measured. The volume percentage of the layers is plotted against the KCl-concentration, while the shift in KCl concentration (at a coacervate volume of 50 %, comparing the blank and the oleate solution with a known quantity of added alkane) is read off the graphs. This shift may go into two directions, we say that a substance has a condensing action if the concentration of KCl needed for coacervation is less than that of the blank, an opening action shows the reverse characteristic.

3. *The influence of the normal alkanes.*

Using the method described in the preceding section we could ascertain

the results obtained by BUNGENBERG DE JONG and KOETS to a certain extent. *n*-Pentane and *n*-hexane exert a condensing action only (that is to say in the concentrations measured), *n*-heptane and *n*-octane show an opening action at higher, a condensing action at lower concentrations (fig. 1). The longer alkanes form a continuous series, whereby the condensing action grows ever less pronounced.

So it is likely that here occur two opposing factors, and it is not far stretched to suppose that these molecules without polar groups may in principle be found in two different places in the oleate micelle. When we consider the usual schemes for oleate micelles (see e.g. HESS, KIESIG and PHILIPPOFF, 1941 or the "Groszmizelle" of STAUFF, 1941) we see at once that for these compounds two places are present (fig. 2 *A* and *B*). Our experiments with alcohols (BOOIJ, VOGELSANG and LYCKLAMA, 1950) show that the situation 2 *A* will result in a condensing action of the added substances. An increase of the London-Van der Waals-forces between the parallel carbon chains is in this case not counteracted by the repelling action of the charged carboxyl groups. From this argument we would conclude that a hydrocarbon when present in situation 2 *A* would also exert a condensing action.

Then it would follow that the situation 2 *B* causes an opening action on the coacervate. This too is not in conflict with the ideas developed before (BOOIJ et al. 1950). Every factor which makes the micelles less flat will lead to an increase of the KCl concentration needed for coacervation, in other words has "an opening action". Concentration of hydrocarbons within the micelles would, of course, make the micelles less flat, thus hydrocarbons might have an opening action.

When considering fig. 1 in this light we might conclude that short alkanes prefer situation 2 *A*, while longer ones are concentrated in

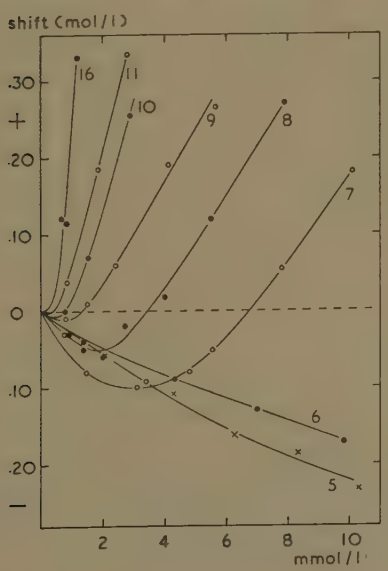

Fig. 1. The action of normal paraffines on an oleate coacervate. This action is expressed in the shift of the KCl-concentration needed for a coacervate of 50 % (5, 6, 7 etc. denote the number of carbon atoms in the chain). Abcissa = concentration of the alkanes added. Ordinate: + = opening action, − = condensing action.

situation 2 *B*. The first supposition, however, does not seem very likely at first sight. Why should a short hydrocarbon prefer the situation 2 *A*, while such a hydrocarbon does not fit well into the place between the long soap chains? Perhaps still another possibility should be taken into account. Every micelle has not only a flat surface, but curved surfaces at the end of the micelles as well (fig. 3). Such a micelle is to be seen as a dynamic structure, the composing molecules being in equilibrium with

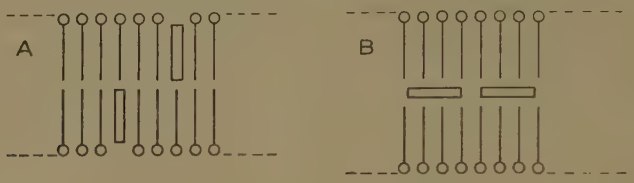

Fig. 2. Possible situation of added non-polar substances (A = parallel to the carbon chains, B = between the CH_3-groups).

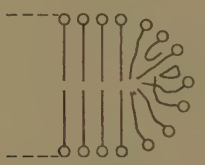

Fig. 3. At the curved end of the micelle small non-polar substances might find a place.

the separate molecules in solution. The weakest spot of the micelle is undoubtedly the curved annular part. Here the carbon chains are closely packed together at one end (the CH_3-group), while the COO-groups are more or less far apart. Here a short hydrocarbon might be introduced, which would result in a (statistical) stabilisation of the structure. Less KCl would then be needed, which would mean that a short hydrocarbon would be a condensing substance. Of course, the space left vacant between the COO-ends of the soap ions would not be a vacuum. In the original micelle we must therefore suppose that water molecules enter this space, which would be a factor tending to disintegrate the micelle. Replacement of these water molecules for hydrocarbons would mean more stable micelles.

So we arrive at the following picture. Short hydrocarbons prefer place 3, giving a condensing action. As we lengthen the carbon chain of the hydrocarbon the equilibrium between the molecules found in place 3 and in place 2 *B* shifts more and more in the direction of the latter situation, resulting in an opening action. We must not forget, however, that situation 2 *A* might be prefered in certain cases. In general there will be for a certain substance an equilibrium between the amounts found at the three different places.

4. *Isomers of n-hexane, n-heptane and n-octane.*

If the hypothesis developed in the preceding section is true, it would be expected that isomers of normal alkanes have another influence on an oleate coacervate than the parent substances. This proved to be true for a number of isomers which we investigated. The method used was exactly the same as for the normal alkanes (fig. 4, 5 and 6). It is

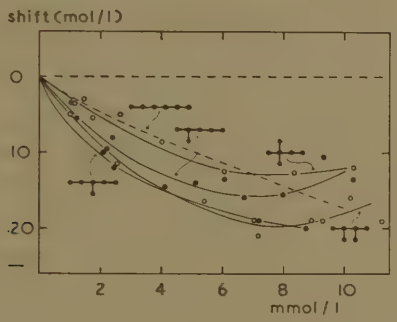

Fig. 4. The influence of isomers of hexane.

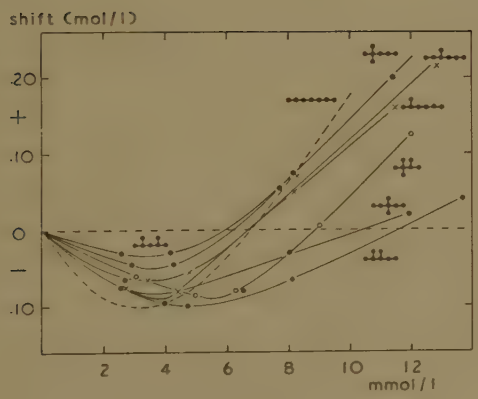

Fig. 5. The action of isomers of heptane.

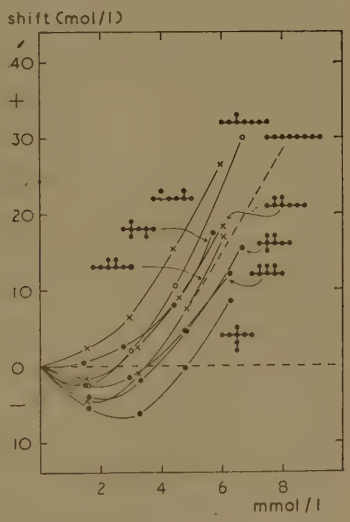

Fig. 6. The influence of isomers of octane.

really remarkable that in this simple physico-chemical system the influence of the isomers differs so much. It is not yet possible to see what rules govern the actions of the substances investigated.

We see from fig. 4 that at low concentrations *n*-hexane has a slighter condensing action than its isomers. The reverse holds for *n*-heptane (fig. 5) and (with one exception) for *n*-octane (fig. 6). When we see that *n*-pentane has a stronger condensing action than *n*-hexane (fig. 1) we might suppose that *n*-hexane is already somewhat too long for place 3.

We will not try to explain the differences between the various isomers, especially as it became clear during our experiments that the propyl-alcohol which we used for dissolving the alkanes has a significant influence on the structure of the micelles. A more thorough study is in progress.

5. *The influence of double bonds in the hydrocarbons.*

In several investigations it is seen that the introduction of a double bond into a biologically active molecule changes its activity profoundly. This is the reason why we investigated some alkenes and one alkyn. It was expected that here too the introduction of a double bond into a hydrocarbon would change its activity. As this group is more polar than the single bond between two carbon atoms we supposed that a like substance would show some preference for situation 2*A*, as the polar group would be more or less anchored to the watery medium (cf. Booij, 1949, 2). Thus the substance with the double bond would have a stronger action than the parent hydrocarbon.

Figs. 7 and 8 show this to be the case indeed. There is a very marked difference between 3-methylheptane and 3-methylheptene-2. Even when

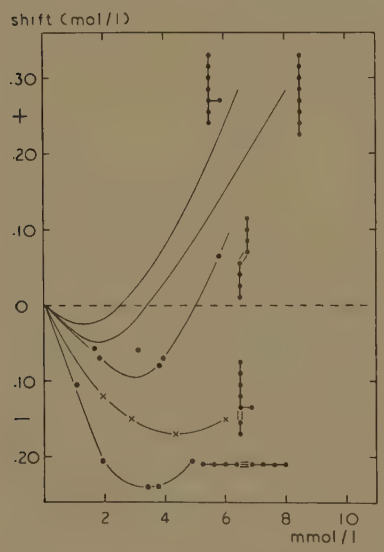

Fig. 7. The influence of the introduction of a double bond. Compare 3-methylheptane with 3-methylheptene-2 and octane with trans-octene-4 and octyn-4.

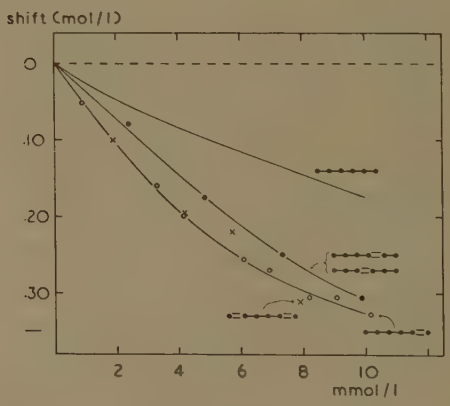

Fig. 8. With hexane too the introduction of one or two double bonds will increase the condensing action.

the double bond is situated in the middle of the molecule (trans-octene-4) a change of activity is observed, becoming more marked when a triple bond is introduced (octyn-4). This is exactly what would be expected, when the double bond is situated toward the end of the molecule, the

change will be greatest. An experiment with some derivatives of hexane shows the same phenomenon, hexene-1 being a stronger condensing substance than a mixture of hexene-2 and hexene-3. These experiments fit very well into the hypothesis developed in a preceding publication.

Summary.

1. The influence of alkanes on oleate coacervates may be explained by assuming two different actions on the soap micelles.

2. These different actions may be the result of the fact that the substances added will go to different places in the soap micelles.

3. Three places are theoretically possible:

a) parallel to the carbon chains of the soap molecules,

b) between the CH_3-groups of the soap molecules,

c) at the circular ends of the micelles, between the outer ends of the carbon chains.

4. A substance going to place *a*) and *c*) will have a condensing action, at *b*) the opposite influence will be found.

5. Short hydrocarbons prefer place *c*), longer ones prefer *b*) (while it is not excluded that *a*) enters the picture too).

6. Isomers of hydrocarbons may have a different action, which shows that minor variations in these molecules will be reflected even in their action on this simple physico-chemical system.

7. Introduction of a double bond has a profound influence; a condensing type of action is the result of this change in the molecule.

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RADIATIVE ELECTRON INTERACTION. I

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(Communicated by Prof. F. A. VENING MEINESZ at the meeting of Febr. 25, 1950)

1. RADIATIVE TRANSITIONS.

1. 1 *Dualistic and unitary theories.* In a previous paper [1]) (refered to by 2) we have asked how a unitary charge theory with interaction at a distance, if possible at all, would have to look like in its general aspects. In the present paper (refered to by 3) we shall see how such a theory might describe some processes, which in the common dualistic theory of electric charges on one hand and of photons (the electromagnetic field) on the other hand are considered as typical radiative processes.

1. 2 *Radiative processes.* Such processes are

R_1 spontaneous and induced emission and absorption of photons;
R_2 interference and diffraction of electromagnetic waves.

1. 21 *Emission and absorption.* The emission and absorption processes are characteristic for the particle aspect of the electromagnetic field. The relation between the spontaneous and induced processes is furthermore characteristic for the statistics of the photons. In case of FERMI-DIRAC statistic the presence of a photon in a certain state would block this state against other photons (exclusion principle). The induced emission probability due to the presence of the photon would thus be equal to the opposite of the spontaneous emission probability into the same state. In case of BOSE-EINSTEIN statistics on the other hand the presence of a photon in a certain state decoys more photons into this state. Then every present photon adds an induced emission probability which is equal to the spontaneous emission probability into the same state. It is obviously the case of BOSE-EINSTEIN statistics which fits in with the observations.

The emission and absorption processes are accompanied by respectively downward and upward transitions of electrons from one stationary energy state to another one (radiative transitions). A unitary charge theory would have to derive these transitions entirely from the electron interaction at a distance. That might show in a different aspect the relation between what in dualistic theory are spontaneous and induced processes. At the same time it might throw some light on the statistical properties of photons if the latter are introduced into the theory afterhand.

1. 22 *Interference and diffraction.* The interference and diffraction effects are characteristic for the periodic wave aspect of photons. They are finally observed by means of electron transitions. In a unitary charge theory these final effects would have to be derived entirely from properties of electrons (presumably their periodic wave aspect). That might throw some light on the periodic wave aspect of photons afterhand.

1. 3 *Transition equations.* The unitary equations of motion tentatively suggested in 2 have there been put in the form of transition equations (275), which are particularly suited for dealing with radiative transitions.

1. 31 *Relativistic invariance.* In 2 the relativistic invariance of the description has not yet been rigorously established. We insert here a proof following (though in quite a different representation) the methods of TOMONAGA [2]) and SCHWINGER [3]), which meanwhile have become available. The flat 3-dimensional spaces $t =$ constant are replaced by more general curved space-like 3-dimensional surfaces $\sigma(x) =$ constant. It can be checked that then all inner products of the form

$$\int (\vec{dx})\, \chi^{\dagger}\, |xt)\, (xt|\, \psi \tag{1}$$

can be replaced by

$$\int d\sigma^{a}\, \chi^{\dagger}\, |x)\, \alpha_{a}\, (x|\, \psi, \tag{2}$$

where $\sigma(x) =$ constant takes the place of the parameter t and the 4-vector $d\sigma^{a}$ denotes the 3-dimensional volume element. The transition equations (275) are then generalized into

$$\left.\begin{aligned} &(\mu_1\, \sigma_1, \ldots \mu_n\, \sigma_n\, |\, \Psi_{\sigma_1 = \ldots \sigma_{r-1} = \sigma_{r+1} = \ldots \sigma_n \to +\infty} \\ &= \exp\left[\frac{ie^2}{\hbar c}\, \frac{1}{2} \sum_{k,l} (\alpha_k^{a}\, \alpha_l^{a'}\, W_{aa'}\, (x_k, x_l))\, (\sigma_k, \sigma_k'; \sigma_l, \sigma_l')\, \{\mu_k, \mu_l\}\right] \\ &(\mu_1\, \sigma_1', \ldots \mu_n\, \sigma_n'\, |\, \Psi_{\sigma_1' = \ldots \sigma_{r-1}' = \sigma_{r+1}' = \ldots \sigma_n' \to -\infty} \end{aligned}\right\} \quad (r = 1, \ldots n) \tag{3}$$

with

$$\left.\begin{aligned} &Q\, (\sigma_k, \sigma_k'; \sigma_l, \sigma_l')\, \{\mu_k, \mu_l\}\, (\mu_1\, \sigma_1', \ldots \mu_n\, \sigma_n'\, |\, \Psi \\ &= \sum_{\mu_k', \mu_l'} (\mu_k\, \sigma_k, \mu_l\, \sigma_l\, |\, Q\, \{x_k, x_l\}\, |\, \mu_k'\, \sigma_k', \mu_l'\, \sigma_l')\, (\mu_1\, \sigma_1', \ldots \mu_k\, \sigma_k', \ldots \mu_l\, \sigma_l', \ldots \mu_n\, \sigma_n'\, |\, \Psi \end{aligned}\right. \tag{4}$$

and

$$\left.\begin{aligned} &(\mu_k\, \sigma_k, \mu_l\, \sigma_l\, |\, Q\, \{x_k, x_l\}\, |\, \mu_k'\, \sigma_k', \mu_l'\, \sigma_l') \\ &= \int_{\sigma_k'}^{\sigma_k} \int_{\sigma_l'}^{\sigma_l} (dx_k)^4\, (dx_l)^4\, (\mu_l\, |\, \psi_l^{\dagger}\, |\, x_l)\, (\mu_k\, |\, \psi_k^{\dagger}\, |\, x_k)\, \mathfrak{Q}\, \{x_k, x_l\}\, (x_k\, |\, \psi_k\, |\, \mu_k')\, (x_l\, |\, \psi_l\, |\, \mu')\,. \end{aligned}\right. \tag{5}$$

This form directly shows the invariance of the processes described.

1. 32 *Reduced equations.* For practical purposes we shall use the transition equations in their original form (275) with flat spaces $t =$ constant and especially in the reduced form (275*). In (275*) the interaction has been split up relative to an arbitrarily chosen time-like direction j^a into transverse radiative interaction and instantaneous Coulomb interaction. The latter will be already accounted for in the unperturbed electron states in a way which will be explained in 1. 4. The reduced equations then take the form

$$(6)\quad \begin{cases} (\mu_1 t_1, \ldots \mu_n t_n \,|\, \Psi_{t_1 = \ldots t_{r-1} = t_{r+1} = \ldots t_n + \infty} \\ \mathbf{U}(t_1, \ldots t_n; t_1', \ldots t_n')\,\{\mu_1, \ldots \mu_n\}\,(\mu_1' t_1', \ldots \mu_n' t_n' \,|\, \Psi_{t_1' = \ldots t_{r-1}' = t_{r+1}' = t_n \to -\infty} \end{cases}$$

with the operator

$$(7)\quad \begin{cases} \mathbf{U}(t_1, \ldots t_n; t_1', \ldots t_n')\,\{\mu_1, \ldots \mu_n\} \\ = \exp\left[-\frac{ie^2}{\hbar c}\frac{1}{2}\sum_{k,l=1}^{n}\int_{t_k'}^{t_k}\int_{t_l'}^{t_l} c dt_k'' c dt_l'' \alpha_k^a\{\mu_k t_k''\}\,\alpha_l^{a'}\{\mu_l t_l''\}\,W_{aa'}\{\mu_k t_k'', \mu_l t_l''\}\right] \end{cases}$$

(in the exponential in (275) some omissions have occurred).

1. 33 *Transition probability.* If we consider the electron r selected in (6), then the probability that in the time interval $t_r' t_r$ it makes a transition from the state μ_r' to the state μ_r together with transitions of the other electrons from the states μ_k' to the states μ_k ($k = 1, \ldots r-1, r+1, \ldots n$) in the infinite time intervals $-\infty + \infty$ is given by

$$(8)\qquad |(\mu_1 t_1, \ldots \mu_n t_n \,|\, \mathbf{U} \,|\, \mu_1' t_1', \ldots \mu_n' t_n')|^2$$

for $t_1 = \ldots t_{r-1} = t_{r-1} = \ldots t_n \to +\infty$, $t_1' = \ldots t_{r-1}' = t_{r+1}' = \ldots t_n' \to -\infty$.

1. 4 *Model.* The unperturbed states have to contain already the Coulomb interaction. In order to avoid all kinds of unessential complications we choose a greatly simplified artificial model, which still just shows the effects we are looking after. For this purpose we have to make quite a lot of assumptions.

We assume that the whole universe we are dealing with consists of a large number of separate parts, which are fixed suitably far apart. For convenience let us call them atoms. There may be various kinds of them. We assume them all to be electrically neutral. Then the only effective instantaneous Coulomb interaction of an electron is with other charges of its own atom. The contributions of the charges (inclusive the heavy nuclei) of other atoms cancel. The wave function can then be built up from atomic wave funtions.

Further we assume that in no atom there is more than 1 electron which is liable to partake in the transitions with which we shall be concerned and that the unperturbed states of such electrons can be described by

1-particle wave functions $(x_k|\psi_k^0|\mu_k)$ ($k = 1, 2, \ldots$ for the various atoms or electrons). Those atoms which contain no such an electron can be left out of consideration. In the remaining atoms, which do contain such an electron, the latter will be denoted as *the* electron of the atom. We assume the rest of the atom to remain unaffected by the transitions of its electron between its unperturbed states, so that the wave functions $(x|\psi^0|\mu)$ of the latter form a complete orthonormal system. The unperturbed states can further be assumed stationary so that their wave functions have the form

$$(9) \qquad (x_k\,|\,\psi_k^0|\,\mu_k) = (\vec{x_k}\,|\,\psi_k^s|\,\mu_k)\,\exp\left[\frac{i}{\hbar}\,\xi_{\mu_k}\,t_k\right]$$

with ψ^s independent of time. The energy of such a state is ξ_{μ_k}. The effective number n of atoms is treated as finite, but may in the limit tend to ∞.

The assumptions made so far make that whereas an electron has effective instantaneous Coulomb interaction only with the rest of its own atom, it has effective radiative interaction only with the electrons of the other atoms. They further avoid the occurrence of unessential complications like Auger effects, Lorentz-Lorenz corrections and others. In particular they make that electron exchange can be neglected. We have already prematurely anticipated the latter simplification by speaking of individual electrons belonging to definite atoms instead of more circumstantially speaking in terms of atomic states.

We shall be especially interested in the transitions of the electron of one particular atom, which we select for this purpose. We consider this atom in a kind of *Hohlraum* formed by the rest of the universe. We further simplify the derivation by assuming that the average constitution of the surrounding universe is spheric symmetric around the selected atom. It may sufficiently slowly vary with the distance from the latter.

Further details of the model will be specified during the course of its application. It is rather artificial and crude and in some aspects perhaps not even entirely consistent, but it may suffice for the purpose of our present demonstration.

It is obvious that in our model of resting atoms the idea of relativistic invariance has been given up. The 4-vector j^a will be chosen in the direction of the positive time axis throughout.

2. INTERACTION MATRIX.

2. 1 *Interaction functions.* Of the cases *Dd*, *Dh*, *Hd* and *Hh* of interaction functions (222) the cases *Dh* and *Hh* have already been rejected in 26. 1. We shall now restrict the derivations to the case *Hd*, from which the case *Dd* can be easily derived at each stage (but not inversely). In 2. 4 we shall see that *Dd* does not satisfy the asymmetry conditions, so that finally *Hd* will be left as the only suitable interaction function.

With j^a in the direction of the positive time axis we find for the scalar,

longitudinal and transverse parts of the interaction function (249)

$$(10)\qquad \begin{cases} \frac{i}{\hbar c}\, W_{00\ sc} = -\, D^{\sigma}_{+}, \\ \frac{i}{\hbar c}\, W_{pq\ ln} = G^{\sigma}_{+pq}, \qquad\qquad (p, q = 1, 2, 3) \\ \frac{i}{\hbar c}\, W_{pq\ tr} = g_{pq}\, D^{\sigma}_{+} - G^{\sigma}_{+pq}, \end{cases}$$

all other components are zero.

Using the representation corresponding to (127) from [4]) (refered to by 1) for $D^{\sigma}_{+}(x)$ and the relation (245) for $G^{\sigma}_{+\,pq}(x)$ we get

$$(11)\qquad \begin{cases} \frac{i}{\hbar c}\, W_{pq\ tr}(x) = \int \frac{(d\vec{\xi})}{(hc)^3}\, \frac{1}{\xi} \left(g_{pq} - \frac{\xi_p\, \xi_q}{\xi^2}\right) \exp\left[-\frac{i}{\hbar c}\, \xi\, |t|\right] \exp\left[\frac{i}{\hbar c}\, \vec{\xi}\, \vec{x}\right] \\ = \int \frac{(d\vec{\xi})}{(hc)^3} \int\limits_{-\infty}^{\infty} d\xi_0 \left(g_{pq} - \frac{\xi_p\, \xi_q}{\xi^2}\right) \left\{ \delta(\xi^2 - \xi_0^2) + \frac{1}{\pi i\, (\xi^2 - \xi_0^2)} \right\} \exp\left[-\frac{i}{\hbar}\, \xi_0 t - \frac{i}{\hbar c}\, \vec{\xi}\, \vec{x}\right] \\ = \frac{-i}{(hc)^3} \int\limits_{-\infty}^{\infty} d\xi_0 \exp\left[-\frac{i}{\hbar}\, \xi_0 t\right] \left\{ \left(g_{pq} - \frac{x_p\, x_q}{x^2}\right) + \left(g_{pq} - 3\, \frac{x_p\, x_q}{x^2}\right) \left(\frac{i\hbar c}{|\xi_0|\, x} - \left(\frac{\hbar c}{\xi_0\, x}\right)^2 \right) \right\} \frac{\exp\left[-\frac{i}{\hbar c} |\xi_0| x\right.}{x} \end{cases}$$

Within the curled brackets only the first g_{pq} originates from $g_{pq}\, D^{\sigma}_{+}$, all the rest from $-\, G^{\sigma}_{+\,pq}$.

2. 2 *Matrix elements.* With the help of (9) and (11) we obtain for the matrix elements corresponding to the operator in the integrant in (7)

$$(12)\qquad \begin{cases} (\varkappa t_k, \lambda t_l \,|\, \alpha_k^{\varrho}\, \alpha_l^{q}\, W_{pq\ tr}(x_k - x_l) \,|\, \varkappa' t_k, \lambda' t_l) \\ = -\frac{1}{hc} \int\limits_{-\infty}^{\infty} d\xi_0 \exp\left[-\frac{i}{\hbar}\, (\xi_{\varkappa\varkappa'} + \xi_0)\, t_k - \frac{i}{\hbar}\, (\xi_{\lambda\lambda'} - \xi_0)\, t_l\right] \\ \int\int (d\vec{x_k})\, (d\vec{x_l})\, (\varkappa \,|\, \psi_k^{s\dagger} \,|\, \vec{x_k})\, (\lambda \,|\, \psi_l^{s\dagger} \,|\, \vec{x_l}) \left\{ \left((\vec{\alpha_k}\, \vec{\alpha_l}) - \frac{(\vec{\alpha_k}\, \vec{x_{kl}})\, (\vec{\alpha_l}\, \vec{x_{kl}})}{x_{kl}^2} \right) \right. \\ \left. + \left((\vec{\alpha_k}\, \vec{\alpha_l}) - 3\, \frac{(\vec{\alpha_k}\, \vec{x_{kl}})\, (\vec{\alpha_l}\, \vec{x_{kl}})}{x_{kl}^2} \right) \left(\frac{i\hbar c}{|\xi_0|\, x_{kl}} - \left(\frac{\hbar c}{\xi_0\, x_{kl}}\right)^2 \right) \right\} \frac{\exp\left[-\frac{i}{\hbar c} |\xi_0|\, x_{kl}\right]}{x_{kl}} (\vec{x_l} \,|\, \psi_l^{s} \,|\, \lambda')\, (\vec{x_k} \,|\, \psi_k^{s} \,|\, \varkappa' \end{cases}$$

where μ_k has been replaced by $\varkappa$ and where $\xi_{\mu\mu'} = \xi_{\mu} - \xi_{\mu'}$ and $\vec{x}_{kl} = \vec{x}_k - \vec{x}_l$.

In the further derivations there will appear to be only an appreciable contribution to the integral over ξ_0 from those values of ξ_0 which lie within a small resonance region about $-\, \xi_{\mu\mu'}$ or/and $\xi_{\lambda\lambda'}$. To this corresponds a wavelength $hc/|\xi_0|$, which we shall assume to be large compared with the dimensions of the orbit of an electron. Then everywhere in (12) x_{kl} may approximately be replaced by its average value over the orbits of the electrons k and l, so that it can be put outside the integrals over

x_k and x_l. Then we are left with integrals of the type

$$\int (d\vec{x_k})\,(\varkappa|\psi_k^{s\dagger}|\vec{x_k})\,\vec{\alpha_k}\,(\vec{x_k}|\psi_k^{s}|\varkappa') = (\varkappa|\vec{\alpha}^{s}|\varkappa'), \tag{13}$$

for which we shortly write $\vec{\alpha}_{\varkappa\varkappa'}$, and similarly for l.

2. 3 *Mutilated interaction.* The calculations of the desired transition probabilities based on the matrix elements (12) would become rather involved. The purpose of the present paper is not to perform the correct calculations, but to demonstrate certain aspects of the connection between the unitary and dualistic descriptions of radiative processes. We shall facilitate this demonstration by a further radical simplification. Instead of using the supposedly correct interaction function W_{tr}, we shall take $W_{mt} = W_{tr} + W_{ln}$. This mutilated unitary theory has to be compared with a likewise mutilated dualistic theory of radiation in which the radiative interaction of the electrons is not only with the two transverse modes of the electromagnetic field, but also with the one longitudinal mode. If the insight in the connection between the mutilated descriptions has become sufficiently clear, it can in principle be transfered to the connection between the (supposedly) correct descriptions afterhand.

The mutilated matrix elements become

$$\left.\begin{aligned}&(\varkappa t_k, \lambda t_l|\,\alpha_k^p \alpha_l^q\, W_{pq\,mt}(x_k - x_l)\,|\varkappa' t_k, \lambda' t_l)\\ &= -\frac{1}{hc}\int_{-\infty}^{\infty} d\xi_0 \exp\Big[-\frac{i}{\hbar}(\xi_{\varkappa\varkappa'}+\xi_0)t_k - \frac{i}{\hbar}(\xi_{\lambda\lambda'}-\xi_0)t_l\Big]\,(\vec{\alpha}_{\varkappa\varkappa'}\,\vec{\alpha}_{\lambda\lambda'})\,\frac{\exp\Big[-\frac{i}{\hbar c}|\xi_0|x_{kl}\Big]}{x_{kl}}.\end{aligned}\right\} \tag{14}$$

2. 4 *Effective (mutilated) interaction.* The form of (7) makes that (14) will always be needed integrated over intervals of t_k and t_l. It can already be predicted with the help of the method of stationary phase that an important contribution will only arise if $-\xi_{\varkappa\varkappa'}$, $\xi_{\lambda\lambda'}$ and ξ_0 lie close together. In fact we shall in 4. 11, assuming that $-\xi_{\varkappa\varkappa'}$ and $\xi_{\lambda\lambda'}$ are sufficiently different from zero and with the same sign, approximately replace $|\xi_0|$ by $\pm\xi_0$ with the corresponding sign. That gives for (14)

$$\left.\begin{aligned}&(\varkappa t_k, \lambda t_l|\,\alpha_k^p \alpha^q\, W_{pq\,mt}(x_k - x_l)\,|\varkappa' t_k, \lambda' t)\\ &= -\frac{1}{hc}\,(\vec{\alpha}_{\varkappa\varkappa'}\cdot\vec{\alpha}_{\lambda\lambda'})\,\frac{\delta(ct_k - ct_l \pm x_{kl})}{x_{kl}}\exp\Big[-\frac{i}{\hbar}\xi_{\varkappa\varkappa'}t_k - \frac{i}{\hbar}\xi_{\lambda\lambda'}t_l\Big],\end{aligned}\right\} \tag{15}$$

with the upper or lower sign as $-\xi_{\varkappa\varkappa'}$ and $\xi_{\lambda\lambda'}$ are positive or negative.

(15) shows that the interaction has effectively a time lag corresponding to a propagation velocity c with the restriction that the downward transition alway occurs earlier than the upward one. This is just the restriction required by the asymmetry condition of 21. 42. If in the interaction function (10) D_+^σ would be replaced by D_-^σ (212) (and consequently G_+^σ by G_-^σ), the down transition would always be later than the up one. With D_+^σ replaced by D_a^σ up and down transitions would occur in both orders. That is why in 23. 36 the cases *Dd* and *Dh* have been rejected

and why in 2. 1 *Hd* has ultimately been accepted as the interaction function.

The asymmetry in the connection between the sense of the paired transitions and their order in time will be discussed somewhat further in 7.

Meanwhile the δ-function in (15) reduces the double integral in the exponent of (7) effectively to a single integral. That greatly simplifies the situation.

3, COMPOSITE TRANSITIONS.

3. 1 *Expansion.* Let us in (7) put e^2 behind the sum sign and write it as e_{kl}^2 in order to indicate to which electron pair it is due. Then expand the exponential into powers of $e_{kl}^2 dt_k dt_l$ for the various pair combinations k, l and the various times t_k and t_l.

We choose the lenght $t_r - t_r'$ of the time interval $t_r' t_r$ small of 1st order. If necessary it would afterhand always be possible to integrate over larger time intervals. For the other electrons we must originally take infinite time intervals. The reconnoitring of 2. 4 suggests that in the expansion a large number of terms will be ineffective. Partially anticipating this simplification we neglect all terms which contain any e_{kl}^2 in higher than the 1st power. If these terms were treated afterhand, they would appear small of higher order.

3. 2 *Transition diagrams.* Each term containing no e_{kl}^2 more than once can be split up into a product of subproducts of ordered factors so that two successive factors have one of their two electrons in common. The common electron occurs in the left factor not at an earlier time than in the right one. A factor can in 4-dimensional time-space schematically be represented by a straight line (link) connecting the world line intervals dt_k and dt_l. A subproduct is represented by a series of links connecting successive world lines. For many purposes it is convenient to consider the projection on 3-dimensional space [5]). In the latter a subproduct is represented by a nonbifurcated broken line (chain). To 1 3-dimensional chain correspond a large number of terms all with different time intervals dt. They can be taken together in integral expressions. The latter can be regarded as the terms of the expansion of (7) into powers of combinations of e_{kl}^2 (up to the 1st power in each e_{kl}^2). These terms are represented by the 3-dimensional diagram, but as yet not by the 4-dimensional one.

Each term in the expansion of the transition probability $|\mathbf{U}|^2$ (8) is a product of a term of **U** and a complex conjugate term of **U**. Because these two factors correspond to the same initial and final states the chains of their transition diagrams fit together into closed loops. Subproducts are represented by separate loops without connection. They describe processes which are independent of each other and only the one containing the selected electron r will have to be taken into account.

If an electron makes an intermediate transition from which it returns to another state than the initial one, it must occur within both the chains which form a loop and it knots them together into a network. If it does return to its initial state (virtual transition) it needs not occur in both chains. If yet it happens to do, it still does not knot them together, there is only an inessential crossing. We shall mainly be concerned with loops without knots.

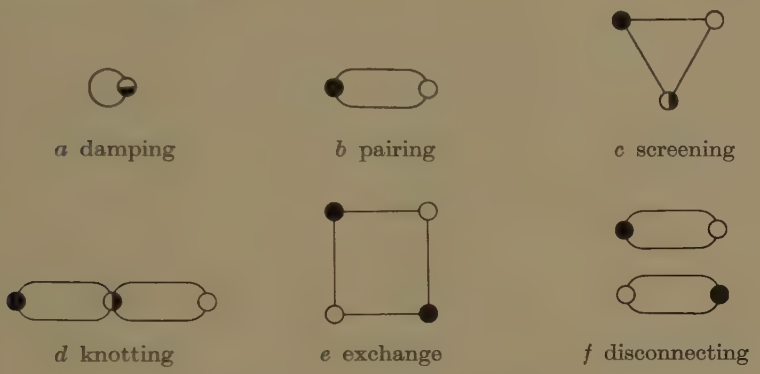

Fig. 1. Transition diagrams.

The δ-function in (15) will make that in the 4-dimensional diagram the effective links all have the directions of light paths. It will further appear that only such terms in the expansion of $|\mathbf{U}|^2$ can be effective, for which also in the 4-dimensional diagram the links form a closed loop.

3. 3 *Exclusion principle.* Strictly the antisymmetry of the wave functions has to be observed for the initial, final and all intermediate states. This could explicitely be done with the help of the wave operators ψ, $\psi^\dagger$ (110), which in μ-representation are determined by

$$(16)\quad \begin{cases} (\nu s|\psi^\dagger|\mu)(\mu_1 t_1,\ldots\mu_n t_n|\Psi = n^{1/2}\mathbf{S}_n(\mu_1 t_1,\ldots\mu_{n-1}t_{n-1};\nu s|\Psi, \\ (\mu t|\psi|\nu s)(\mu_1 t_1,\ldots\mu_{n-1}t_{n-1};\nu s|\Psi = n^{1/2}\mathbf{S}_n\delta_{\mu_n\nu}(\mu_1 t_1,\ldots\mu_{n-1}t_{n-1};\nu s|\Psi \end{cases}$$

with $s = t_n$. An infinitesimal term of the exponent of (7) could e.g. be written

$$(17)\quad \begin{cases} -\frac{ie^2}{\hbar c}\frac{1}{2}\sum_{\nu_1,\nu_2}(\mu t|\psi^\dagger|\nu_1 s_1')(\mu t|\psi^\dagger|\nu_2 s_2'')\,cds_1'\,cds_2''\,\alpha_1^a\{\nu_1 s_1'\}\alpha_2^{a'}\{\nu_2 s_2''\} \\ \qquad W_{aa'\,mt}\{\nu_1 s_1', \nu_2 s_2''\}(\nu_2 s_2|\psi|\mu t\}\nu_1 s_1|\psi|\mu t\}. \end{cases}$$

The wave operators ψ and $\psi^\dagger$ contribute in a loop a factor $\psi^\dagger\psi$ for every transition out of a state, a factor $\psi\psi^\dagger$ for every transition into a state, a factor $\psi^\dagger\psi$ for every virtual transition out of and back into a state and a factor $\psi\psi^\dagger$ for every virtual transition into a state and out of it again. The expectation values of $\psi^\dagger\psi$ and $\psi\psi^\dagger$ (and of their powers, which would appear in higher order terms) are respectively z_μ and $1 - z_\mu$, where z_μ is the occupation number of the concerning state.

According to the assumptions of 1. 4 exchange effects can be neglected. The antisymmetry of the wave functions has only to be taken into account with respect to the other electrons of the home atom as far as PAULI's exclusion principle is concerned. The factors z_μ and $1 - z_\mu$ just take care of this task. If they are inserted in the transition probability one needs not trouble anymore about antisymmetrizing the wave functions.

3. 4 *Low and high order transitions.* Counted in powers of e the order of a term of $\mathbf{U}$ or $|\mathbf{U}|^2$ is twice the number of links in its diagram or twice the number of partaking electrons (counted twice at crossings of the loop). The high order terms can successively be built up from the lower order terms. At the base are the elementary paired transitions of 2 electrons. The loop consists of a double link (fig. 1*b*). The addition of 1 more electron in the loop leads to a new effect (screening), the addition of 2 electrons also to another new effect (exchange). The effect of adding still more electrons shows no new features, it can be composed of screening and exchange. The description cannot be complete before all (or at least a very large number of) electrons in the universe are taken into account. Therefore the derivation falls into two parts. First in 4 we deal with transitions with 2, 3 and 4-electron loops. Then in 5 we consider the composition of the effects of many-electron loops out of the elementary effects of paired transitions, screening and exchange.

An electron which does not partake in the chain or loop contributes a factor $\delta_{\mu\mu'}$ in the terms of $\mathbf{U}$ or $|\mathbf{U}|^2$. We omit them as well as those factors contributed by subproducts or disconnected loops, which do not contain the central electron r.

4. LOW ORDER TRANSITIONS.

4. 1 *2-electron transitions.* 4. 11 *Transition matrix.* For elementary paired transitions of 2 electrons (fig. 1*b*) we find with (14) for the matrix elements of the terms of $\mathbf{U}$

$$
\begin{aligned}
&(\varrho t_r, \varkappa t_k | \, \mathbf{U}^{(2)} \, | \varrho' t'_r, \varkappa' t'_k) \\
&= -\frac{ie^2}{\hbar c} \int_{t'_r}^{t_r} c dt^*_r \int_{t'_k}^{t_k} c dt^*_k \, (\varrho t^*_r, \varkappa t^*_k | \, a^p_r a^q_k \, W_{pq\, mt}(x_r - x_k) \, | \varrho' t_r, \varkappa' t^*_k) \\
&= -\frac{ie^2}{2\pi} \frac{(\vec{a}_{\varrho\varrho'} \cdot \vec{a}_{\varkappa\varkappa'})}{x_{rk}} \int_{-\infty}^{\infty} d\xi_0 \frac{\exp\left[-\frac{i}{\hbar}(\xi_{\varrho\varrho'} + \xi_0) t_r\right] - \exp\left[-\frac{i}{\hbar}(\xi_{\varrho\varrho'} + \xi_0) t'_r\right]}{\xi_{\varrho\varrho'} + \xi_0} \\
&\quad \frac{\exp\left[-\frac{i}{\hbar}(\xi_{\varkappa\varkappa'} - \xi_0) t_k\right] - \exp\left[-\frac{i}{\hbar}(\xi_{\varkappa\varkappa'} - \xi_0) t'_k\right]}{\xi_{\varkappa\varkappa'} - \xi_0} \exp\left[-\frac{i}{\hbar c} |\xi_0| x_{rk}\right].
\end{aligned}
\tag{18}
$$

The main contributions to the integral over ξ_0 come from the neighbourhood of the poles at $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$. Provided $\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$ are suffi-

ciently large, $|\xi_0|$ may in each contribution be replaced by $\pm \xi_0$ with the sign $\pm$ the same as that of the corresponding pole. The integral can now be evaluated by complex integration in the ξ_0 plane. It vanishes along the CAUCHY path in fig. 2. If the radius of the large circle tends to ∞, the contribution of the term with t_r and t_k tends to 0 along the

Fig. 2. Complex ξ_0-plane.

upper or lower half circle respectively for $-t_r + t_k \mp x_{rk}/c > 0$ or < 0 and similar for the terms with t_r' and t_k, t_r and t_k' and t_r' and t_k'. We shall see that in all applications the effective values of $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$ are approximately the same (resonance), so that the sign $\pm$ is the same for the two poles. If we restrict ourselves directly to that case we obtain for (18)

$$(19)\quad \begin{cases} (\varrho t_r, \varkappa t_k | \mathbf{U}^{(2)} | \varrho' t_r', \varkappa' t_k') \\ = e^2 \frac{(\vec{a}_{\varrho\varrho'} \cdot \vec{a}_{\varkappa\varkappa'})}{x_{rk}} \frac{\exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'} t_r'' - \frac{i}{\hbar}\xi_{\varkappa\varkappa'} t_k''\right] - \exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'} t_r''' - \frac{i}{\hbar}\xi_{\varkappa\varkappa'} t_k'''\right]}{\xi_{\varrho\varrho'} + \xi_{\varkappa\varkappa'}}, \end{cases}$$

where $t_r''' t_r''$ is the overlapping part of $t_r' t_r$ and $t_k' \mp x_{rk}/c\ t_k \mp x_{rk}/c$ (for the latter interval we further simply write $t_k' t_k$; $\mp x_{rk}$), $t_k''' t_k''$ the overlapping part of $t_k' t_k$ and $t_r' t_r$; $\pm x_{rk}$. The upper or lower sign holds as $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'} \gtrless 0$. If there is no overlapping the right hand member of (19) has to read 0. These results entirely confirm (15) from which they could directly be derived.

4. 12 *Transition probability.* Because we are interested in the transitions of the electron r, we have according to (8) to take $t_k' \to -\infty$, $t_k \to +\infty$. That makes $t_r''' t_r'' = t_r' t_r$ and $t_k''' t_k'' = t_r' t_r$; $\pm x_{rk}$. The probability of a transition of r from ϱ' to ϱ during the time interval $t_r' t_r$ together with a partner transition of k from $\varkappa'$ to $\varkappa$ at any time is then with (19)

$$(20)\quad \begin{cases} z_{\varrho'}(1-z_\varrho)\, z_{\varkappa'}(1-z_\varkappa)\, e^4 \frac{(\vec{a}_{\varrho\varrho'} \cdot \vec{a}_{\varkappa\varkappa'})\ (\vec{a}_{\varkappa'\varkappa} \cdot \vec{a}_{\varrho'\varrho})}{x_{rk}^2} \\ \frac{2 - \exp\left[-\frac{i}{\hbar}(\xi_{\varrho\varrho'} + \xi_{\varkappa\varkappa'})(t_r - t_r')\right] - \exp\left[-\frac{i}{\hbar}(\xi_{\varrho\varrho'} + \xi_{\varkappa\varkappa'})(t_r - t_r')\right]}{(\xi_{\varrho\varrho'} + \xi_{\varkappa\varkappa'})^2}. \end{cases}$$

We have made use of $\overline{(\varrho|\mathbf{u}|\varrho')} = (\varrho'|\mathbf{u}|\varrho)$ for a HERMITIAN operator $\mathbf{u}$. The initial and final occupations have been accounted for according to 3. 3. The value of (20) is independent of the asymmetry condition and could have been obtained with D^{σ}_{-} or D^{σ}_{a} instead of D^{σ}_{+} as well. The effect of D^{σ}_{+} is that the partner transition of k occurs in the time interval $t'_r t_r$; $\pm x_{rk}$ for $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'} \gtrless 0$, i.e. just with the sign required by the asymmetry condition. The strong and troublesome condition $t'_k \to -\infty$, $t_k \to +\infty$ is greatly mitigated by the special form of (15), by which the infinite interval is reduced to one which is small of 1st order.

4. 13 *Small region.* Now take a 3-dimensional volume element $d\tau_k$ whose dimensions are small compared with the wave length $hc/|\xi_{\varkappa\varkappa'}|$, but which still contains a large number of atoms. Then for the electrons k in $d\tau_k$ x_{rk} may approximately be replaced by its average over the region $d\tau_k$. In the present approximation the probabilities of transitions of the electron r paired with transitions of the electrons k in $d\tau_k$ can simply be summed over the latter.

Averaging over the random orientations of the atoms k in $d\tau_k$ we get

$$\{(\vec{a}_{\varrho\varrho'} \cdot \vec{a}_{\varkappa\varkappa'})\,(\vec{a}_{\varkappa'\varkappa} \cdot \vec{a}_{\varrho'\varrho})\}_{av} = \tfrac{1}{3}\,|a_{\varrho\varrho'}|^2\,|a_{\varkappa\varkappa'}|^2 = \tfrac{1}{3}\,f_{\varrho\varrho'}\,f_{\varkappa\varkappa'}. \tag{21}$$

$f_{\varrho\varrho'}$ is a measure for the oscillatorstrength of radiation theory for the transition $\varrho' \to \varrho$. For the sum of $z_{\varkappa'}(1-z_{\varkappa})\,f_{\varkappa\varkappa'}$ over all transitions with an energy jump $\xi_{\varkappa\varkappa'}$ in an interval $\xi\ \xi + d\xi$ for all atoms k in $d\tau_k$ we write

$$d\tau_k\,d\xi\,\varphi(\vec{x}_k;\,\xi). \tag{22}$$

The distribution function $\varphi(\vec{x}_k;\,\xi)$ is a measure for the oscillator density.

The probability of a transition of r from ϱ' to ϱ during $t'_r t_r$ together with any partner transition in $d\tau_k$ (which then occurs in the time interval $t'_r t_r$; $\pm x_{rk}$) is

$$\begin{cases} z_{\varrho'}(1-z_{\varrho})\,\dfrac{e^4}{3}\,\dfrac{f_{\varrho\varrho'}}{x_{rk}^2}\,d\tau_k \displaystyle\int_{-\infty}^{\infty} d\xi\,\varphi(\vec{x}_k;\,\xi) \\ \dfrac{\left(1-\exp\left[-\dfrac{i}{\hbar}\,(\xi_{\varrho\varrho'}+\xi)\,(t_r-t'_r)\right]\right)+\left(1-\exp\left[+\dfrac{i}{\hbar}\,(\xi_{\varrho\varrho'}+\xi)\,(t_r-t'_r)\right]\right)}{(\xi_{\varrho\varrho'}+\xi)^2}. \end{cases} \tag{23}$$

The integral can again be evaluated by complex integration in the ξ-plane, one term in the upper half, one in the lower. The remaining contribution from the pole at $\xi = -\xi_{\varrho\varrho'}$ is

$$z_{\varrho'}(1-z_{\varrho})\,\frac{2\pi e^4}{3h}\,\frac{f_{\varrho\varrho'}}{x_{rk}^2}\,d\tau_k\,\varphi(\vec{x}_k;\,-\xi_{\varrho\varrho'})\,|t_r-t'_r|. \tag{24}$$

Here the resonance between $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$ has occurred indeed.

4. 14 *Thin shell.* Then put the atom r inside a concentrical homogeneous shell with radius r_k and thickness dr_k, which is small compared

with $hc/|\xi_{xx'}|$, but so large that there are still a large number of atoms in a volume element of the order of $(dr_k)^3$. The probability of a transition $\varrho' \to \varrho$ during $t'_r t_r$ together with any partner transition in the shell (during $t'_r t_r$; $\pm x_{rk}$) is in the present approximation the sum of (24) over all small regions into which the shell can be divided. The result can (for later purposes) be written in the form

$$\gamma_{\varrho\varrho'}\, dr_k\, \chi(r_k; -\xi_{\varrho\varrho'})\, |t_r - t'_r| \tag{25}$$

with

$$\gamma_{\varrho\varrho'} = z_{\varrho'}\,(1-z_\varrho)\,\frac{2e^2}{\hbar c}\, f_{\varrho\varrho'}\, |\xi_{\varrho\varrho'}| \tag{26}$$

and

$$\chi(r;\xi) = \frac{4\pi^2}{3}\,\frac{e^2}{|\xi|}\,\varphi(r;\xi). \tag{27}$$

4. 15 *Universe.* Ultimately we shall have to consider the atom r in the centre of a spheric symmetric extended oscillator distribution. In the present approximation the probability of a transition $\varrho' \to \varrho$ during $t'_r t_r$ together with any partner transition of any other electron would be given by the sum of (25) over all thin shells into which the distribution can be divided. But here the present approximation becomes insufficient. Whereas in a small region or even in a thin shell the probability of transitions in which more than 1 electrons partake can be neglected (as it contains higher orders of $d\tau_k$ or dr_k), this can no longer be done in the extended distribution. There we have to account for transitions of arbitrarily many electrons, all in different thin shells.

4. 2 *3-electron transitions (screening).* 4.21 *Transition matrix.* For transitions in which 3 electrons take a part (fig. 1*c*, *d*) we get in addition to (18) terms of the type

$$\begin{aligned}
&(\varrho\, t_r, \mu\, t_m, \varkappa\, t_k \,|\, \mathbf{U}^{(4)} \,|\, \varrho'\, t'_r, \mu'\, t'_m, \varkappa'\, t'_k) \\
&= \Big(-\frac{ie^2}{\hbar c}\Big)^2 \int_{t'_r}^{t_r} c\,dt''_r \int_{t'_m}^{t_m} c\,dt''_m \int_{t'_m}^{t''_m} c\,dt'''_m \int_{t'_k}^{t_k} c\,dt''_k\, (\varrho t''_r, \mu t''_m \,|\, \alpha_r^p \alpha_m^q\, W_{pq\,mt}(x_r - x_m) \,|\, \varrho'\, t''_r, \mu''\, t'''_m) \\
&(\mu''\, t'''_m, \varkappa t''_k \,|\, \alpha_m^p\, \alpha_k^q\, W_{pq\,mt}(x_m - x_k) \,|\, \mu'\, t'''_m, \varkappa'\, t''_k) + \ldots .
\end{aligned} \tag{28}$$

The term which has been written down in full describes a process in which the transition of m from μ' to μ is over the intermediate state μ'' in such a way that the first transition $\mu' \to \mu''$ is paired with $\varkappa' \to \varkappa$ and the second one $\mu'' \to \mu$ with $\varrho' \to \varrho$. There are 6 terms together, which can all be found by taking the 6 permutations of r, m and k. We particularly consider the terms in which it is m which makes an intermediate transition.

The paired transitions of m with k and of r with m out of which the first term of (28) is built up are only dependent of each other in the time

intervals $t_r'''t_r'$, $t_m'''t_m''$ and $t_k'''t_k''$, where now $t_r'''t_r'$ is the overlapping part of the 3 intervals $t_r't_r$, $t_m't_m$; $\mp x_{mr}$ and $t_k't_k$; $\mp x_{km} \mp x_{mr}$, $t_m'''t_m''$ is the overlapping part of $t_r't_r$; $\pm x_{rm}$, $t_m't_m$ and $t_k't_k$; $\mp x_{km}$ and still similar for $t_k'''t_k''$. The upper or lower sign of x_{rm} correspond to the cases $-\xi_{\varrho\varrho'}$ and $\xi_{\mu\mu''} \gtrless 0$, that of x_{mk} to $-\xi_{\mu''\mu'}$ and $\xi_{\varkappa\varkappa'} \gtrless 0$. For these intervals (28) becomes

$$
(29)\quad
\begin{cases}
(\varrho t_r'', \mu t_m'', \varkappa t_k'' \,|\, \mathbf{U}^{(4)} \,|\, \varrho' t_r''', \mu' t_m''', \varkappa' t_k''') \\
= e^4 \dfrac{(\vec{\alpha}_{\varrho\varrho'} \cdot \vec{\alpha}_{\mu\mu''})\ (\vec{\alpha}_{\mu''\mu'} \cdot \vec{\alpha}_{\varkappa\varkappa'})}{x_{rm}\qquad x_{mk}} \dfrac{\exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'} t_r'' - \frac{i}{\hbar}\xi_{\mu\mu'} t_m'' - \frac{i}{\hbar}\xi_{\varkappa\varkappa'} t_k''\right]}{(\xi_{\varrho\varrho'}+\xi_{\mu\mu'}+\xi_{\varkappa\varkappa'})\,(\xi_{\mu'\mu''}+\xi_{\varkappa\varkappa'})} \\
- \dfrac{\exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'} t_r''' - \frac{i}{\hbar}\xi_{\mu\mu''} t_m'' - \frac{i}{\hbar}\xi_{\mu''\mu'} t_m''' - \frac{i}{\hbar}\xi_{\varkappa\varkappa'} t_k'''\right]}{(\xi_{\varrho\varrho'}+\xi_{\mu\mu''})\,(\xi_{\mu''\mu'}+\xi_{\varkappa\varkappa'})} \\
+ \dfrac{\exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'} t_r''' - \frac{i}{\hbar}\xi_{\mu\mu'} t_m''' - \frac{i}{\hbar}\xi_{\varkappa\varkappa'} t_k'''\right]}{(\xi_{\varrho\varrho'}+\xi_{\mu\mu'}+\xi_{\varkappa\varkappa'})\,(\xi_{\varrho\varrho'}+\xi_{\mu\mu'})}.
\end{cases}
$$

Outside these intervals the paired transitions of m with k are independent of those of r with m as far as they are possible at all. In that case they can only affect the initial or final distribution of m over its states μ. Because we use statistical distributions of the electrons over their atomic states this effect may be considered to be accounted for so that we further need not bother about these intervals.

To (29) a term has to be added with r and k interchanged. In the practical case that $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$ have the same sign the time intervals are the same in the two terms.

4. 22 *Transition probability.* The terms in the transition probability $|\mathbf{U}|^2$ (8) corresponding to fig. 1c are of the type of (29) times the complex conjugate of (19) plus the complex conjugate of (29) times (19). Such terms only occur if $\mu = \mu'$, i.e. if m makes a virtual transition. $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'}$ will then have the same sign and the time intervals in (29) and the term with r and k interchanged are the same. According to 3. 3 a factor $z_\mu\,(1 - z_{\mu''})$ has to be inserted for the virtual transitions between μ and μ''.

We have to take $t_m', t_k' \to -\infty$, $t_m, t_k \to +\infty$. Then $t_r''t_r'' = t_r't_r$ and in (29) $t_m'''t_m'' = t_r't_r$; $\pm x_{rm}$ and $t_k'''t_k'' = t_r't_r$; $\pm x_{rm} \pm x_{mk}$. The latter interval is in general different from the corresponding one in (19).

4. 23 *Small region.* Now as in 4. 13 take a small region $d\tau_m$ containing a large number of atoms m. Averaging over the random orientations of the atoms m we get

$$
(30)\qquad \{(\vec{\alpha}_{\varrho\varrho'} \cdot \vec{\alpha}_{\mu\mu''})\ (\vec{\alpha}_{\mu''\mu'} \cdot \vec{\alpha}_{\varkappa\varkappa'})\}_{av} = \tfrac{1}{3}\,(\vec{\alpha}_{\varrho\varrho'} \cdot \vec{\alpha}_{\varkappa\varkappa'})\, f_{\mu\mu''}.
$$

Summing (29) with $\mu = \mu'$ and the term with r and k interchanged over all virtual transitions with energy jumps ξ running from $-\infty$ to $+\infty$ of all electrons m in $d\tau_m$, accounting for the factor $z_\mu\,(1 - z_{\mu''})$ and using

the distribution function (22), we obtain

$$(31)\quad \begin{cases} -\dfrac{\pi i e^4}{3}\dfrac{(\vec{a}_{\varrho\varrho'}\cdot\vec{a}_{\varkappa\varkappa'})}{x_{rm}x_{mk}}\dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t_r''-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t_k''\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}}d\tau_m\{\varphi(\vec{x}_m;\xi_{\varkappa\varkappa'})+\varphi(\vec{x}_m;\xi_{\varrho\varrho'})\} \\ -\dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t_r''-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t_k''\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}}d\tau_m\{\varphi(\vec{x}_m;-\xi_{\varrho\varrho'})+\varphi(\vec{x}_m;-\xi_{\varkappa\varkappa'})\}. \end{cases}$$

4. 24 *Thin shell.* Then as in 4. 14 take a homogeneous thin shell with radius r_m and thickness dr_m containing a large number of atoms m. x_{rm} is then equal to r_m. In taking the sum of (31) over all small regions into which the shell can be divided, t_k'' and t_k' (expressed in t_r' and t_r) saddle us with a factor

$$\frac{\exp\left[-\dfrac{i}{\hbar c}|\xi_{\varkappa\varkappa'}|\,x_{mk}\right]}{x_{mk}},$$

which has to be integrated over the shell. This is done with the expansions in BESSEL functions [6])

$$(32)\quad \begin{cases} \dfrac{\sin\eta\,x_{mk}}{x_{mk}}=\pi\sum\limits_{l=0}^{\infty}(l+1/2)\dfrac{J_{l-1/2}(\eta\,r_m)}{r_m^{1/2}}\dfrac{J_{l-1/2}(\eta\,x_{rk})}{x_{rk}^{1/2}}P_l(\cos\theta), \\ \dfrac{\cos\eta\,x_{mk}}{x_{mk}}=\pi\sum\limits_{l=0}^{\infty}(l+1/2)\dfrac{J_{\pm(l+1/2)}(\eta\,r_m)}{r_m^{1/2}}\dfrac{J_{\mp(l+1/2)}(\eta\,x_{rk})}{x_{rk}^{1/2}}P_l(\cos\theta)\quad\text{for}\quad r_m\lessgtr x_{rk}. \end{cases}$$

Θ means the angle under which $\vec{x}_{mk}$ is seen from the central atom r. The average over the shell vanishes for all spherical harmonics except for the constant one $P_0(\cos\Theta)=1$. The average of (32) is

$$(33)\quad \begin{cases} \left\{\dfrac{\sin\eta\,x_{mk}}{x_{mk}}\right\}_{av}=\dfrac{1}{\eta}\dfrac{\sin\eta\,r_m}{r_m}\dfrac{\sin\eta\,x_{rk}}{x_{rk}}, \\ \left\{\dfrac{\cos\eta\,x_{mk}}{x_{mk}}\right\}_{av}=\dfrac{1}{\eta}\dfrac{\overset{\sin}{\cos}\eta\,r_m}{s_m}\dfrac{\overset{\cos}{\sin}\eta\,x_{rk}}{x_{rk}}\quad\text{for}\quad r_m\lessgtr x_{rk}. \end{cases}$$

That gives for the sum of (31) over all small regions of the shell

$$(34)\quad \begin{cases} \dfrac{-2\pi^2e^4\hbar c}{3|\xi_{\varkappa\varkappa'}|}\dfrac{(\vec{a}_{\varrho\varrho'}\cdot\vec{a}_{\varkappa\varkappa'})}{x_{rk}}\Bigg\{\dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t_r''-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t_k''\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}}dr_m\{\varphi(r_m;\xi_{\varkappa\varkappa'})+\varphi(r_m;\xi_{\varrho\varrho'})\} \\ \quad-\dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t_r''-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t_k''\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}}dr_m\{\varphi(r_m;-\xi_{\varrho\varrho'})+\varphi(r_m;-\xi_{\varkappa\varkappa'})\} \\ \qquad\qquad -\ldots+\ldots\Bigg\}. \end{cases}$$

The two terms which have not been written down are only distinguished from the preceeding two by the situation of the time interval $t_k''t_k''$. For

$r_m < x_{rk}$ this interval is in the first two terms given by $t'_r t_r$; $\pm 2r_m \pm x_{rk}$ and in the last two terms by $t'_r t_r$; $\pm x_{rk}$; for $r_m > x_{rk}$ respectively by $t'_r t_r$; $\pm 2r_m \pm x_{rk}$ and $t'_r t_r$; $\pm 2r_m \mp x_{rk}$ (signs according to $-\xi_{\varrho\varrho'}$ and $\xi_{\varkappa\varkappa'} \gtrless 0$). It is remarkable that the time lags of these intervals correspond to travel times (with velocity c) along the longest path ($\vec{x}_{rm}$ and $\vec{x}_{rk}$ opposite direction) and along the shortest path (same direction).

If we compare (34) with (19), the phase difference between the two matrix elements is only independent of r_m for the last two terms in the case $r_m < x_{rk}$. Therefore if we ultimately sum over a large number of thin shells this will for the other terms and also in the other case lead to approximate cancellation. So let us cancel them in anticipation. The remaining terms have the same time interval $t''_k t'_k$ as those in (19). The time lag corresponds to the travel time along a straight path ($\vec{x}_{rm}$ and $\vec{x}_{mk}$ same direction), which just only in the case $r_m < x_{rk}$ cuts the shell between r and k. The probability of this transition is now given by (34) times the complex conjugate of (19) plus the complex conjugate of (34) times (19).

Compared with the direct interaction between r and k the virtual transitions of the electrons m in a small region $d\tau_m$ give a phase shift $-\pi/2$. The interference between the phase differences along the paths from r to k via all regions of the shell gives effectively a further phase shift $-\pi/2$ if k lies outside the shell and cancellation if k lies inside. They give no effective additional time lag. The effective total phase shift $-\pi$ leads to an attenuation of the transition probability if k lies outside the shell. For this reason the effect of the virtual transitions is here indicated as screening. A similar effective phase shift $-\pi/2$ of the path differences outside and cancellation insde a spherical surface is met with in the formulation of HUYGENS' principle according to FRESNEL or KIRCHHOF, though in that case it comes about in a different way.

Finally we take a small region $d\tau_k$ and sum the transition probability which we have just obtained over all transitions of all atoms k as in 4. 13. Adding the resulting 6th order term to the 4th order term (24) has the effect of multiplying the latter by a screening factor, which is

$$1 - dr_m \{\chi(r_m; -\xi_{\varrho\varrho'}) + \chi(r_m; \xi_{\varrho\varrho'})\} \tag{35}$$

if $d\tau_k$ lies outside the thin shell $r_m; dr_m$, whereas it is equal to 1 if $d\tau_k$ lies inside. The same screening factor remains if instead of the region $d\tau_k$ we take as in 4. 14 a thin shell $r_k; dr_k$.

4. 25 *Universe.* Whereas inside a small region $d\tau_m$ or even inside a thin shell $r_m; dr_m$ virtual transitions of more than 1 electrons (giving higher orders of $d\tau_m$ or dr_m) could be neglected, this can no longer be done in a thick shell, which can be built up from thin ones. In this case the present approximation, which would give a sum over thin shells, becomes again insufficient.

4. 3 *4-electron transitions (exchange).* 4. 31 *Transition matrix.* The matrix elements for transitions in which 4 electrons partake (fig. 1*e*, *f*) are of the type

$$(36)\quad \left\{ \begin{array}{l} (\varrho t_r, \mu t_m, \nu t_n, \varkappa t_k \,|\, \mathbf{U}^{(4)} \,|\, \varrho' t'_r, \mu' t'_m, \nu' t'_n, \varkappa' t'_k) \\ = \sum\limits_C (\varrho t_r, \mu t_m \,|\, \mathbf{U}^{(2)} \,|\, \varrho' t'_r, \mu' t'_m)\ (\nu t_n, \varkappa t_k \,|\, \mathbf{U}^{(2)} \,|\, \nu' t'_n, \varkappa' t'_k). \end{array} \right.$$

with $\mathbf{U}^{(2)}$ given by (19). C denotes the 3 combinations of the 4 electrons into pairs. Apart from (36) describing 2 pairs of transitions there are also the matrix elements with 1 pair transition and 2 virtual transitions. After 4. 2 they do not lead to new aspects and at present we shall be concerned with (36).

4. 32 *Transition probability.* The corresponding transition probability is

$$(37)\quad \left\{ \begin{array}{l} \sum\limits_C |(\varrho t_r, \mu t_m \,|\, \mathbf{U}^{(2)} \,|\, \varrho' t'_r, \mu' t'_m)|^2\, |(\nu t_n, \varkappa t_k \,|\, \mathbf{U}^{(2)} \,|\, \nu' t'_n, \varkappa' t'_k)|^2 \\ + \sum\limits_D (\varrho t_r, \mu t_m \,|\, \mathbf{U}^{(2)} \,|\, \varrho' t'_r, \mu' t'_m)\, (\mu' t_m, \nu' t_n \,|\, \mathbf{U}^{(2)} \,|\, \mu t'_m, \nu t'_n)\, (\nu t_n, \varkappa t_k \,|\, \mathbf{U}^{(2)} \,|\, \nu' t'_n, \varkappa' t'_k) \\ \quad (\varkappa' t_k, \varrho' t_r \,|\, \mathbf{U}^{(2)} \,|\, \varkappa t'_k, \varrho t'_r). \end{array} \right.$$

D denotes the 6 combinations of 2 of the 3 elements of C into pairs. The quadratic terms in the right hand member of (37) are products of the transition probabilities of 2-electron transitions (fig. 1*f*). Because we are not interested in transitions which are independent of those of r, these terms do not give us anything new and can be left out of account. The cross terms describe the interdependence of the 2-electron transitions. They are somewhat similar to exchange terms in ordinary quantum mechanics and will here also be denoted as exchange terms.

In (37) resonance can only occur if of the 4 energy differences $\xi_{\varrho\varrho'}$, $\xi_{\mu\mu'}$, $\xi_{\nu\nu'}$ and $\xi_{\varkappa\varkappa'}$ 2 are positive and 2 negative. In that case 2 terms of C and 2 of D lead to resonance. Suppose the energy jump of r and n is in one sense, of m and k in the opposite sense. Then in D resonance comes from the term written in full in (37) and the one obtained by interchange of m and k. The first one gives with (19) and with the occupation factors according to 3. 3

$$(38)\quad \left\{ \begin{array}{l} z_{\varrho'}(1-z_\varrho)\, z_{\mu'}(1-z_\mu)\, z_{\nu'}(1-z_\nu)\, z_{\varkappa'}(1-z_\varkappa)\, e^8\, \dfrac{(\vec{\alpha}_{\varrho\varrho'}\cdot\vec{\alpha}_{\mu\mu'})}{x_{rm}}\, \dfrac{(\vec{\alpha}_{\mu'\mu}\cdot\vec{\alpha}_{\nu'\nu})}{x_{mn}}\, \dfrac{(\vec{\alpha}_{\nu\nu'}\cdot\vec{\alpha}_{\varkappa\varkappa'})}{x_{nk}}\, \dfrac{(\vec{\alpha}_{\varkappa'\varkappa}\cdot\vec{\alpha}_{\varrho'\varrho})}{x_{kr}} \\ \quad \dfrac{\exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'}t''_r - \frac{i}{\hbar}\xi_{\mu\mu'}t''_m\right] - \exp\left[-\frac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r - \frac{i}{\hbar}\xi_{\mu\mu'}t'''_m\right]}{\xi_{\varrho\varrho'}+\xi_{\mu\mu'}} \\ \quad \dfrac{\exp\left[+\frac{i}{\hbar}\xi_{\mu\mu'}t''_m + \frac{i}{\hbar}\xi_{\nu\nu'}t''_n\right] - \exp\left[+\frac{i}{\hbar}\xi_{\mu\mu'}t'''_m + \frac{i}{\hbar}\xi_{\nu\nu'}t'''_n\right]}{\xi_{\mu\mu'}+\xi_{\nu\nu'}} \\ \quad \dfrac{\exp\left[-\frac{i}{\hbar}\xi_{\nu\nu'}t''_n - \frac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k\right] - \exp\left[-\frac{i}{\hbar}\xi_{\nu\nu'}t'''_n - \frac{i}{\hbar}\xi_{\varkappa\varkappa'}t'''_k\right]}{\xi_{\nu\nu'}+\xi_{\varkappa\varkappa'}} \\ \quad \dfrac{\exp\left[+\frac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k + \frac{i}{\hbar}\xi_{\varrho\varrho'}t''_r\right] - \exp\left[+\frac{i}{\hbar}\xi_{\varkappa\varkappa'}t'''_k + \frac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r\right]}{\xi_{\varkappa\varkappa'}+\xi_{\varrho\varrho'}}. \end{array} \right.$$

We have to take $t'_m, t'_n, t'_k \to -\infty$, $t_m, t_n, t_k \to +\infty$. At least one of the intervals $t'''t''$ has in general to be taken different in the two different factors.

4. 33 *Small regions.* Now as in 4. 13 and 4. 23 take two small regions $d\tau_m$ and $d\tau_n$ containing large numbers of electrons m and n respectively. Summing (38) over all possible transitions of these electrons we get in the same way as in 4. 13 and 4. 23

$$
(39)\quad \left\{ \begin{array}{l} z_{\varrho'}(1-z_\varrho)\, z_{\varkappa'}(1-z_\varkappa)\, \dfrac{\pi^2 e^8}{9}\, \dfrac{(\vec{a}_{\varrho\varrho'}\cdot\vec{a}_{\varkappa\varkappa'})\,(\vec{a}_{\varkappa'\varkappa}\cdot\vec{a}_{\varrho'\varrho})}{x_{rm}\, x_{mn}\, x_{nk}\, x_{kr}} \\[2ex] \dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t''_r-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t'_k\right]-\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}} \\[2ex] \dfrac{\exp\left[+\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k+\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t''_r\right]-\exp\left[+\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t'''_k+\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r\right]}{\xi_{\varkappa\varkappa'}+\xi_{\varrho\varrho'}} \\[2ex] d\tau_m\{\varphi(x_m;-\xi_{\varrho\varrho'})+\varphi(x_m;\xi_{\varkappa\varkappa'})\}\, d\tau_n\{\varphi(x_n;\xi_{\varrho\varrho'})+\varphi(x_n;-\xi_{\varkappa\varkappa'})\}. \end{array} \right.
$$

In both factors we take $t'''_r t''_r = t'_r t_r$. Then in the first factor $t''_k t'_k = t'_r t_r$; $\pm x_{rm} \mp x_{mn} \pm x_{nk}$ and in the second factor $t'''_k t''_k = t'_r t_r$; $\pm x_{rk}$ (signs according to $-\xi_{\varrho\varrho'}$, $\xi_{\mu\mu'}$, $-\xi_{\nu\nu'}$ and $\xi_{\varkappa\varkappa'} \gtrless 0$).

4. 34 *Thin shells.* Then as in 4. 14 and 4. 24 take two homogeneous thin concentric shells $r_m; dr_m$ and $r_n; dr_n$ around the central r. In the same way as in 4. 24 summation over all small regions first of the shell $r_m; dr_m$, then of the shell $r_n; dr_n$ gives

$$
(40)\quad \left\{ \begin{array}{l} z_{\varrho'}(1-z_\varrho)\, z_{\varkappa'}(1-z_\varkappa)\, \dfrac{4\pi^4}{9}\, \dfrac{e^8\hbar^2c^2}{\xi_{\varkappa\varkappa'}}\, \dfrac{(\vec{a}_{\varrho\varrho'}\cdot\vec{a}_{\varkappa\varkappa'})\,(\vec{a}_{\varkappa'\varkappa}\cdot\vec{a}_{\varrho'\varrho})}{x^2_{rk}} \\[2ex] \left\{ \dfrac{\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t''_r-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k\right]-\exp\left[-\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r-\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t'''_k\right]}{\xi_{\varrho\varrho'}+\xi_{\varkappa\varkappa'}} - \ldots - \ldots + \ldots \right\} \\[2ex] \qquad \dfrac{\exp\left[+\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t''_k+\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t''_r\right]-\exp\left[+\dfrac{i}{\hbar}\xi_{\varkappa\varkappa'}t'''_k+\dfrac{i}{\hbar}\xi_{\varrho\varrho'}t'''_r\right]}{\xi_{\varkappa\varkappa'}+\xi_{\varrho\varrho'}} \\[2ex] dr_m\{\varphi(r_m;-\xi_{\varrho\varrho'})+\varphi(r_m;\xi_{\varkappa\varkappa'})\}\, dr_n\{\varphi(r_n;\xi_{\varrho\varrho'})+\varphi(r_n;-\xi_{\varkappa\varkappa'})\}. \end{array} \right.
$$

The three terms which have not been written down are only distinguished from the preceding one by the situation of the time interval $t'''_k t''_k$. There are 4 different cases corresponding to the combinations of $r_n \gtrless r_m$ and $r_n \gtrless x_{rk}$. All time lags correspond to travel times from r to k via the two shells along paths which are either in the same or in opposite direction. For the same reasons as in 4. 24 only those terms will hold out against later integration over thick shells, which have the same interval $t'''_k t''_k$ as the last numerator, viz $t'_r t_r$; $\pm x_{rk}$. Such a term occurs with a minus sign

twice in the case $r_n < r_m$, x_{rk} and once in the 3 other cases. The transitions occur during the time intervals $t_m''t_m'' = t_r't_r$; $\pm r_m$, $t_n''t_n'' = t_r't_r$; $\pm r_n$ (all cases) or $t_r't_r$; $\mp r_n$ (for $r_n < r_m$, x_{rk}) and $t_k''t_k'' = t_r't_r$; $\pm x_{rk}$. The travel times correspond to the shortest paths with the two down transitions occurring before the two up ones. All cases where this can happen are thus just exhausted.

Finally we take a small region $d\tau_k$ and sum the transition probability for one of the remaining 1 or 2 terms just discussed over all transitions of all atoms k as in 4. 13. Adding the resulting 8th order term to the 4th order term (24) has the effect of multiplying the latter by an exchange factor

$$1 + dr_m\, \chi\,(r_m; -\xi_{\varrho\varrho'})\, dr_n\, \chi\,(r_n; \xi_{\varrho\varrho'}). \tag{41}$$

The same exchange factor remains if instead of the region $d\tau_k$ we take as in 4. 14 a thin shell $r_k; dr_k$.

4. 35 *Universe.* For the same reasons as in 4. 15 and 4. 25 the present approximation is insufficient for thick shells.

4. 4 *3-electron transitions (knotting).* There is still another type of transitions which we ought to consider, viz those corresponding to (29) with $\mu \neq \mu'$. The transition probability is the absolute square of (29). It is an 8th order term and represented by a knotted loop (fig. 1*d*). The 2-electron transitions of m with r and of m with k are not independent of each other, but there is no resonance between the two and therefore also no coherence. We boldly assume that they have no ultimate effect on the transitions of r and disregard them without further investigation.

4. 5 *Closed loops.* All three kinds of transitions of 4. 1 (elementary paired), 4. 2 (screening) and 4. 3 (exchange) are represented by simple closed 3-dimensional loops. In their 4-dimensional diagram the links can in general not be joined into closed loops and the time intervals during which the related transitions of the other atoms occur are not always unambiguously determined. Only for those terms which we have maintained under all integrations this difficulty disappears. These effective terms can be represented by closed loops in 4-dimensional diagrams as well.

(To be continued).

QUELQUES EXPÉRIENCES SUR LA NON-LINÉARITÉ DES FILS CHAUDS

PAR

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(*Mededeling no. 66 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated by Prof. J. M. BURGERS at the meeting of March 25, 1950)

Nous comparons ici les résultats de quelques expériences avec les formules établies dans notre précédente communication "Théorie non-linéaire de l'anémomètre à fil chaud". Nous montrons que le terme non-linéaire joue un rôle aussi important que la conduction thermique dans le calcul de l'inertie thermique du fil chaud.

I. *Introduction.* — Selon notre théorie non-linéaire [1]) l'équation du fil chaud doit contenir les termes exprimant:

a) le transfert de chaleur du fil à l'air proportionnellement à la température T,

b) le transfert de chaleur proportionnellement à T^2 (non-linéarité),

c) la conduction de chaleur aux extrémités du fil,

d) l'inertie thermique due à la chaleur spécifique et la masse du metal.

Nous étudions d'abord les effets *a*), *b*) et *c*) en traitant le cas de 11 fils de petit diamètre (2 microns), avant de passer à l'effet d'inertie. Nous nous référons sans autres précisions aux formules de la théorie non-linéaire, numérotées de 1 à 73, et continuerons avec le numéro 74.

2. *Préparation des fils.* — Nous avons préparé nos fils en utilisant du fil de Wollaston, de Pt, plié en forme de U et soudé sur un support avant d'être décapé. Pour enlever l'Ag nous utilisons un jet d'une solution acide (50 % eau distillée, 50 % HNO_3) et une électrolyse avec un courant de 5 à 20 mA. La fig. 1 indique le dispositif utilisé.

Le jet ne brise pas le fil. Cependant, si le diamètre est inférieur à 5 microns, les poussières entraînées par le liquide sont dangereuses et il faut filtrer la solution avant chaque usage. Le ballon est monté sur un support que l'on peut déplacer avec précision, et la fig. 1 indique la marche des opérations, vues au microscope.

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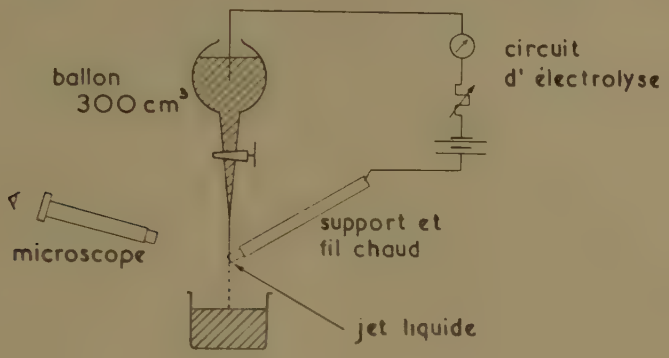
[1]) R. BETCHOV, Théorie non-linéaire de l'anémomètre à fil chaud, (Meded. 61), Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam, 52, 195—207 (1949).

On peut obtenir des fils extrêmement courts en déplaçant le jet perpendiculairement au fil, de manière à enlever l'Ag tantôt en avant, tantôt en arrière. Le fil décapé est rincé à l'eau ordinaire et porté à une



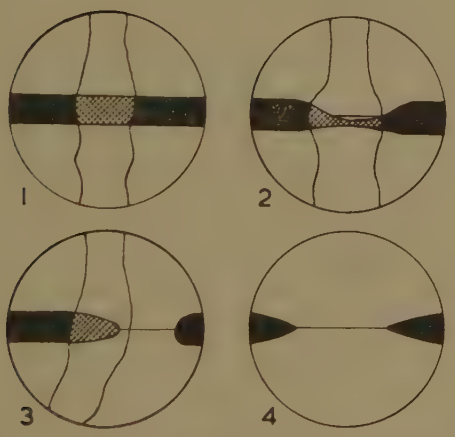


Fig. 1. Préparation des fils de Wollaston.
1) Début
2) Le Pt apparaît
3) Le jet forme une pointe
4) Fil terminé

légère incandescence pour permettre un examen au microscope. Seul sont utilisés les fils qui rougissent de manière régulière et symétrique.

3. *Etalonnage statique.* — Nous donnons ici les résultats obtenus avec 11 fils de Pt de longueurs comprises entre 0,25 et 1,6 mm et de diamètres mesurant environ 2 microns. La fig. 2 indique les résistances à froid et les longueurs, montrant l'ordre de grandeur des variations individuelles.

Chaque fil a été étalonné avec des vents de 2 à 10 m/sec et nous avons étudié la grandeur

$$H = \frac{RI^2}{R - R_0} \tag{74}$$

en fonction du rapport R/R_0. En extrapolant à partir des valeurs mesurées de H, nous avons déterminé la valeur limite H_0, correspondant au cas $R/R_0 = 1$.

Nos fils ont donné 11 valeurs de H_0 pour $V = 2$ m/sec, que nous

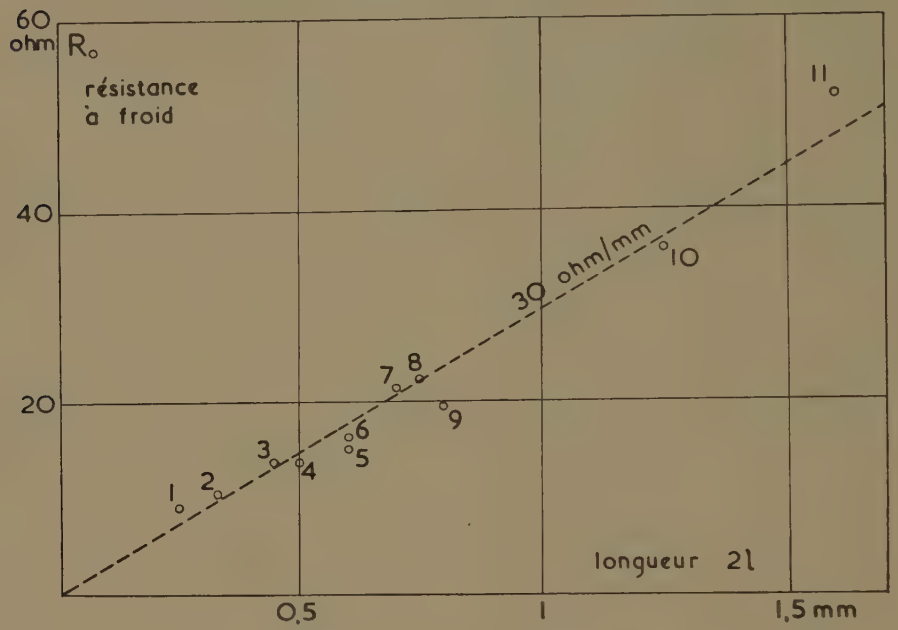


Fig. 2. Résistances à froid et longueurs des fils.

avons indiquées en fonction des longueurs des fils sur la fig. 3. Le calcul donne pour I tendant vers zéro

$$H_0 = \frac{A}{1 - Tgh\, \xi_0/\xi_0} \tag{75}$$

avec:

$$\xi_0 = l/l_0^*, \text{ où } l_0^* = \sqrt{\frac{\varkappa\, \sigma}{\alpha S_0 A}} \tag{76}$$

et nous avons tracé les courbes théoriques correspondant aux valeurs $l_0^* = 6.10^{-2}$ mm et 8.10^{-2} mm. On voit que pour l tendant vers l'infini H_0 tend vers la valeur 500 mA². La formule de King nous donne avec $d = 2{,}1$ microns, $A = 500$ mA² et nous voyons ainsi que l'effet de conduction en basse température correspond à la théorie.

Lorsque la résistance augmente, l'effet de la conduction thermique tend à faire décroître H. Nous prenons l'exemple du fil no. 8, long de 0,75 mm. La figure 4 nous donne les valeurs expérimentales de H, prises deux fois, à un jour d'intervalle. En ne tenant compte que de la conduction, on obtient pour H la courbe pointillée, selon:

$$H = A \frac{1 - (I^2/A)\, Tgh\, \xi/\xi}{1 - Tgh\, \xi/\xi}. \tag{77}$$

La courbe en trait plein a été calculée en tenant compte de la non-linéarité suivant la formule (36) avec les coefficients $\gamma = 1{,}14.10^{-3}$,

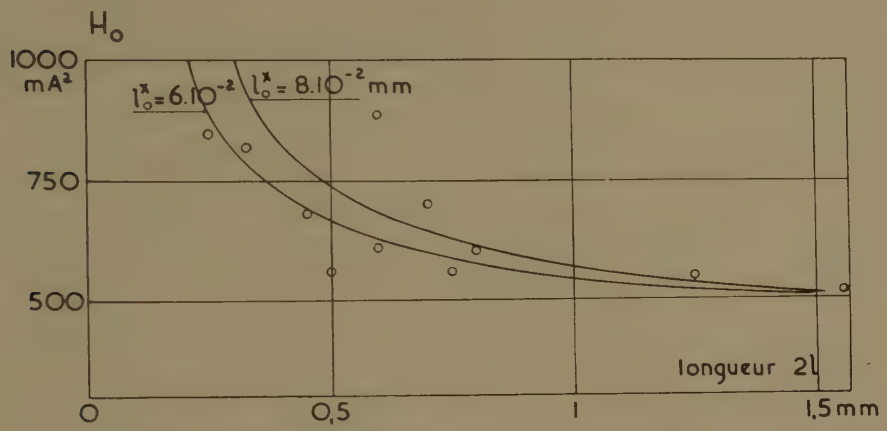


Fig. 3. Effet de conduction en basse température.

$A = 450$ mA², $l_0^* = 7.10^{-2}$ mm. Nous voyons qu'elle correspond à l'expérience du 1er jour, et qu'on a environ $H_0 = 555$. Les valeurs de H mesurées le lendemain sont plus basses.

Probablement les différences sont causées par des particules de pous-

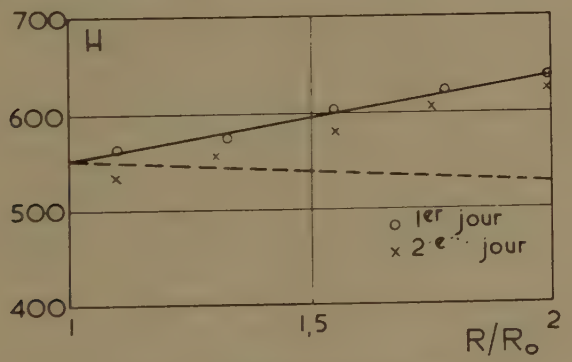


Fig. 4. Effet de non-linéarité.

sière qui se sont déposées sur le fil pendant cet intervalle de temps, et qui produisent un élargissement de la région d'air immobile autour du fil, ce qui diminue le transport de chaleur par le vent.

Nous avons étudié l'augmentation de H avec la température et nous avons mesuré pour chaque fil traité le rapport

$$h = \frac{H\,(R/R_0 = 2) - H_0}{H_0} \tag{78}$$

Ce rapport peut être calculé et la figure 5 indique les résultats expérimentaux et théoriques.

Malgré les écarts des points, il nous semble que la théorie est satisfaisante.

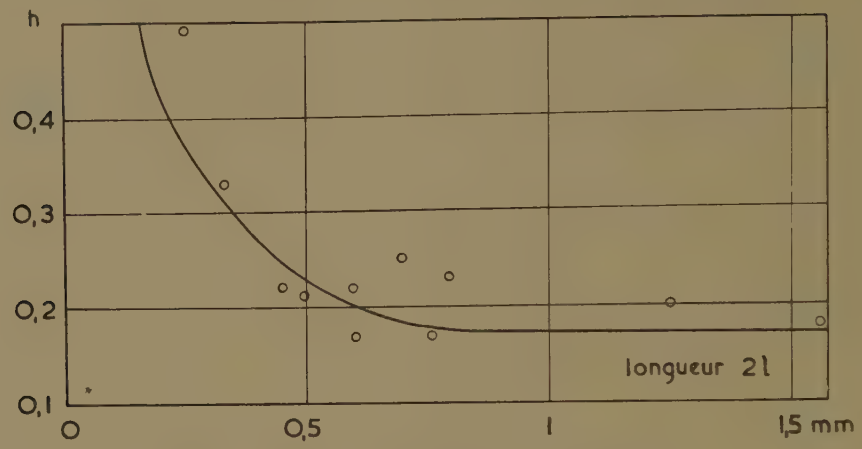


Fig. 5. Effet de la longueur sur le rapport h.

4. *Etalonnage dynamique.* — Nous avons mesuré la réponse d'un fil chaud lorsque le courant électrique fluctue. Pour cela le fil a été placé dans un pont (fig. 6) alimenté par le courant plaque d'une pentode. Le courant de chauffage peut être modulé à l'aide d'un oscillateur basse fréquence et l'intensité alternative i est indiquée par un appareil spécial. Nous avions $R_1 = 100$ ohms et $R_2 = 1000$ ohms; la self-induction L compensait à la fois la self-induction de la ligne menant au fil chaud et sa résistance ohmique. La tension du pont était appliquée à un analyseur qui ne transmettait que le signal ayant la fréquence de l'oscillateur, ce qui permettait de travailler sans être gêné par la turbulence. Le signal filtré était transmis à un oscillographe cathodique qui permettait d'obtenir l'équilibre du pont pour la fréquence utilisée.

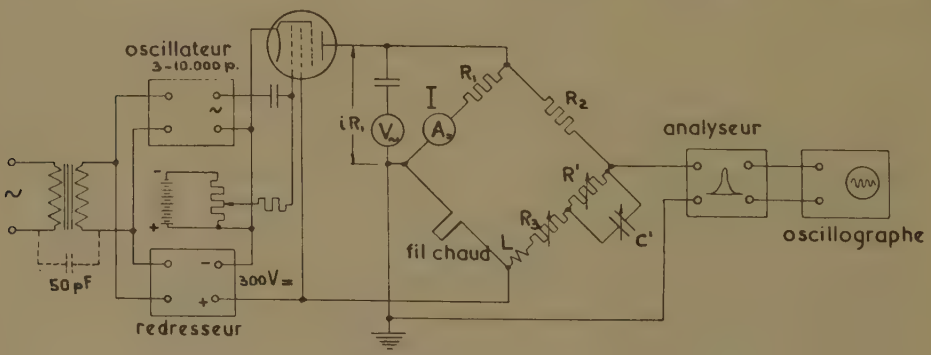


Fig. 6. Mesure de l'inertie thermique d'un fil chaud.

Comme l'analyseur et l'oscillographe étaient mis à la terre, il a été nécessaire d'isoler spécialement le système d'alimentation. En effet, le redresseur et l'oscillateur présentent normalement par rapport au

réseau alternatif une capacité d'environ 5000 cm et ce réseau est toujours mis à la terre en quelque point, ce qui introduit un élément indésirable dans le circuit. Nous avons supprimé cet inconvénient en utilisant un transformateur ayant entre le primaire et le secondaire une faible capacité.

Pour éliminer l'effet pellicullaire (skin effet), nous avons dû utiliser une ligne spéciale, menant au fil chaud. De cette façon le pont s'est montré satisfaisant de 0 à 75 kc. L'impédance du circuit $R' C'$ est destinée à compenser les fluctuations de la résistance du fil et le calcul montre que, si le pont est équilibré, la tension $r I$ est proportionelle à la tension aux bornes de R'. Les mesures ont été faites de la manière suivante:

1. Le fil est placé dans le tunnel et soumis au vent, avec $i = 0$ et $R' = 0$. On ajuste alors R_3 de manière à équilibrer le pont. On utilise pour cela un galvanomètre non-représenté sur la figure.
2. En modulant le courant avec i de l'ordre de 3 % de I, et avec une basse fréquence, on ajuste R' et C' de manière à équilibrer le pont en courant alternatif. Les valeurs de R' et C' sont notées, ainsi que la fréquence.
3. La même opération a lieu avec des fréquences f croissantes, jusqu'à environ 10.000 périodes.

Le produit $2 \pi f R' C'$ donne la tangente de l'angle de déphasage entre le courant alternatif traversant le fil et la variation de sa résistance. Ce déphasage est de 45 degrés pour une certaine valeur $f_{45°}$ de la fréquence, que nous comparons (a) avec la valeur théorique f^* pour les fils linéaires et infinis (51), (*b*) avec la valeur f^{**} de notre théorie non-linéaire et (*c*) avec la valeur donnée par DRYDEN:

$$f_{Dryden} = \frac{1}{2\pi} \frac{a S_0}{m c} \frac{R_0 I^2}{R - R_0} \tag{79}$$

déduit de (51) en remplaçant A par $RI^2/(R - R_0)$. Nous avons ici la relation générale $\omega = 2 \pi f$.

Pour calculer ces valeurs théoriques il faut connaître la constante

$$\frac{a S_0}{m c} = \frac{16 a \varrho}{\pi^2 d^4 c \delta} \tag{80}$$

On voit qu'une erreur de 5 % sur le diamètre du fil donne une erreur de 20 % sur cette constante; aussi est-il préférable de la déterminer par l'expérience. Le calcul de cette constante nous a donné dans le cas du fil no. 8 des valeurs trop basses et nous l'avons arbitrairement multipliée par un facteur 1,26, de manière à faire coincider la théorie avec l'expérience, lorsque la température du fil est basse. La figure 7 indique en fonction du rapport R/R_0 les grandeurs mesureés $f_{45°}$, ainsi que les courbes calculées.

On voit que l'expérience donne des résultats intermédiaires entre

f^{**} et f_{Dryden}. Pour expliquer ceci, il faut tenir compte de la variation avec la température, de la chaleur spécifique du métal, du produit $\alpha\,\varrho$, et de la densité. A l'aide des International Critical Tables, nous avons estimé que la constante $\alpha\,\varrho/m\,c$ diminue selon la formule approximative:

$$\left(\frac{\alpha\,\varrho}{m\,c}\right)_R = \left(\frac{\alpha\,\varrho}{m\,c}\right)_{R_0} \{1-\varepsilon\,(R/R_0-1)\} \tag{81}$$

avec $\varepsilon = 0{,}06$ dans le cas du *Pt*. Cette correction diminue les fréquences calculées et nous avons indiqué en fig. 7 la courbe obtenue à partir de

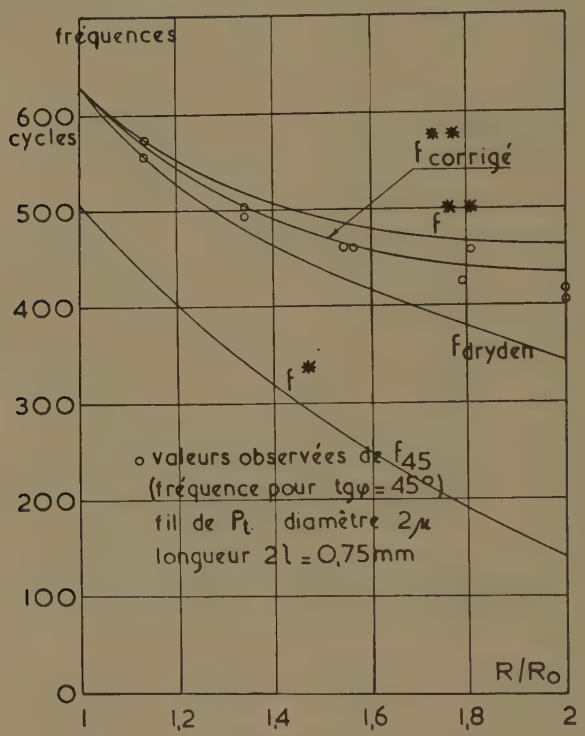


Fig. 7

f^{**}, et indiquée "f^{**} corrigé". Cette correction devrait également être appliquée à f_{Dryden}, et elle augmenterait encore les écarts constatés. D'autres fils donnent des résultats analogues et nous estimons que notre théorie non-linéaire correspond aux expériences.

Aprés avoir étudié la fréquence donnant un déphasage de 45 degrés, il nous faut examiner le comportement du fil lorsque la fréquence est plus petite ou plus grande. La tangente de l'angle de déphasage φ, égale au rapport des parties imaginaire et réelle de la tension rI, est donnée par le produit $2\,\pi\,R'\,C'\,f$, et le produit $R'\,C'$ devrait être indépendant de la fréquence, en l'absence de conduction thermique, si la formule 67 serait exacte.

Lorsque f est inférieure à f_{45}, l'amplitude correspond bien aux valeurs calculées (formule 66), ce qui est normal, puisque ce résultat ne dépend

que de la dérivée de R par rapport à I, et que la théorie donne les valeurs correctes de H.

Le produit $R'\,C'$ est bien constant, ainsi que le montre la fig. 8, tant que f est inférieur à f_{45}, mais les valeurs calculées sont légèrement inférieures aux mesures.

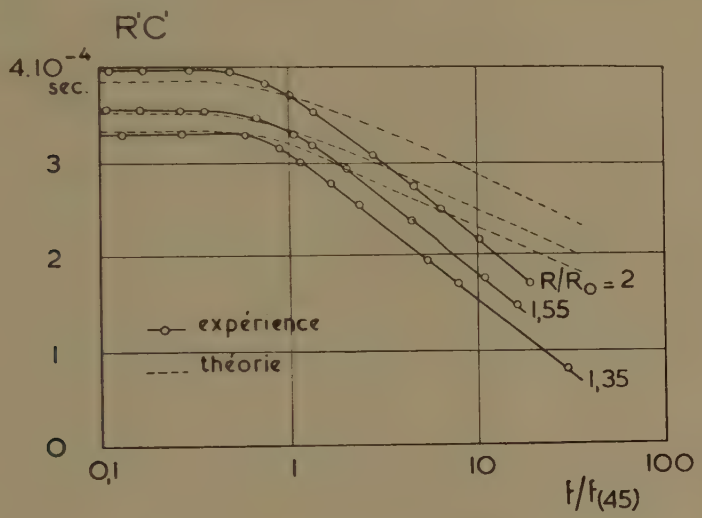


Fig. 8. Anomalies du déphasage.

Au dela de f_{45}, l'amplitude diminue selon la théorie et la formule (64) est vérifiée, mais le produit $R'\,C'$ tombe plus vite que la théorie ne le prévoit.

Les courbes pointillées de la fig. 8 ont été calculées à partir de la formule (62), et on peut démontrer que, pour des fréquences tendant vers l'infini, le déphasage tend vers:

$$(82) \qquad Tg\,\varphi = \frac{1 - \dfrac{BI^2}{A - I^2 + BI^2} \cdot \dfrac{1}{\sqrt{E}\,\xi}}{1 - \dfrac{BI^2}{A - I^2 + BI^2}}\,\xi\sqrt{2}\,\sqrt{\frac{f}{f^*}}$$

Il semble donc que ce déphasage anormal est plus prononcé encore que l'expérience ne le prévoit, mais les erreurs expérimentales peuvent affecter considérablement nos résultats, en particulier l'erreur sur R_3. Il faut aussi remarquer que si la tangente varie par exemple de 10 à 20, l'angle ne varie que de 84 à 87 degrés, ce qui diminue l'importance de cet effet.

Enfin nous avons fait quelques essais préliminaires avec des fils de tungstène et constaté que la grandeur H est remarquablement constante, quelle que soit la longueur du fil.

ON TRANSFORMATION IN QUANTUM MECHANICS

BY

E. M. BRUINS

(Communicated by Prof. L. E. J. BROUWER at the meeting of March 25, 1950)

A projective generalisation of DIRAC equations is contained in the "line and point" incidence in a projective space G_4

$$(\varphi \partial^2 \psi) + (\varphi c^2 \psi) \equiv 0 \{\varphi\}$$

which gives a projectively invariant *system* of equations. According to GRAM's theorem the equations themselves are not invariant, but the system is transformed into a system equivalent to the same equations in the new coordinates. It seems of use to show how, by some elementary relations, the transformations of coordinates in quantum mechanics fit into the general projective scheme.

Be the collineation in G_4

$$(a' x)\,(\alpha u') = 0 \qquad a'_i \alpha_k = e_{ik} \quad \text{or}$$

$$\tilde{x}_k = \sum_{i=1}^{4} e_{ik} x_i ,$$

the transformation induced in the line coordinates p_{ik} is then

$$(a' p)\,(b' p)\,(\alpha \beta \pi^2) = 0 \quad \text{or}$$

$$\tilde{p}_{ik} = \sum_{\lambda\mu} (\alpha\beta)_{ik}\,(a' b')_{\lambda\mu}\, p_{\lambda\mu} ,$$

which reads in e_{ik} putting

$$e_{ik,\,\lambda\mu} = e_{i\lambda}\, e_{k\mu} - e_{i\mu}\, e_{k\lambda}$$

$$\tilde{p}_{ik} = \sum_{(\lambda\mu)} \tfrac{1}{2}\, e_{ik,\,\lambda\mu}\, p_{\lambda\mu} .$$

Expressing the sum and difference of the PLUECKER coordinates with complementary indices we have

$$\tilde{p}_{ik} + \tilde{p}_{\overline{ik}} = \sum_{(\lambda\mu)} \tfrac{1}{2}\,(e_{ik,\,\lambda\mu} + e_{\overline{ik},\,\lambda\mu} + e_{ik,\,\overline{\lambda\mu}} + e_{\overline{ik},\,\overline{\lambda\mu}})\,(p_{\lambda\mu} + p_{\overline{\lambda\mu}}) +$$

$$\tfrac{1}{2}\,(e_{ik,\,\lambda\mu} + e_{\overline{ik},\,\lambda\mu} - e_{ik,\,\overline{\lambda\mu}} - e_{\overline{ik},\,\overline{\lambda\mu}})\,(p_{\lambda\mu} - p_{\overline{\lambda\mu}})$$

$$\tilde{p}_{ik} - \tilde{p}_{\overline{ik}} = \sum_{(\lambda\mu)} \tfrac{1}{2}\,(e_{ik,\,\lambda\mu} - e_{\overline{ik},\,\lambda\mu} + e_{ik,\,\overline{\lambda\mu}} - e_{\overline{ik},\,\overline{\lambda\mu}})\,(p_{\lambda\mu} + p_{\overline{\lambda\mu}}) +$$

$$\tfrac{1}{2}\,(e_{ik,\,\lambda\mu} - e_{\overline{ik},\,\lambda\mu} - e_{ik,\,\overline{\lambda\mu}} + e_{\overline{ik},\,\overline{\lambda\mu}})\,(p_{\lambda\mu} - p_{\overline{\lambda\mu}}) .$$

The condition that $p_{12} - p_{34}$, $p_{14} - p_{23}$ are transforming not involving the other sums and differences and vice versa gives us 16 equations between the coefficients e_{ik}, which are, putting

$$\| e_{ik} \| = \left\| \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array} \right\|$$

I: $(\alpha\beta)_{24} = (\gamma\delta)_{24}$, $(\alpha\beta)_{13} = (\gamma\delta)_{13}$, $(\alpha\beta)_{12} = (\gamma\delta)_{34}$, $(\alpha\beta)_{34} = (\gamma\delta)_{12}$, $(\alpha\beta)_{14} = (\gamma\delta)_{23}$, $(\alpha\beta_{23}) = (\gamma\delta)_{14}$,

II: $(\beta\gamma)_{24} = (\alpha\delta)_{24}$, $(\beta\gamma)_{13} = (\alpha\delta)_{13}$, $(\beta\gamma)_{12} = (\alpha\delta)_{34}$, $(\beta\gamma)_{34} = (\alpha\delta)_{12}$, $(\beta\gamma)_{14} = (\alpha\delta)_{23}$, $(\beta\gamma)_{23} = (\alpha\delta)_{14}$,

III: $(\alpha\gamma)_{23} = (\alpha\gamma)_{14}$, $(\alpha\gamma)_{12} = (\alpha\gamma)_{34}$, $(\beta\delta)_{23} = (\beta\delta)_{14}$, $(\beta\delta)_{12} = (\beta\delta)_{34}$.

Eliminating the γ from I, because of the quadratic p-relation between the $(\alpha\beta)_{ik}$, only two independent results are obtained:

$$(\alpha\beta)_{34}\,\delta_3 = (\alpha\beta)_{13}\,\delta_2 - (\alpha\beta)_{14}\,\delta_1$$
$$(\alpha\beta)_{34}\,\delta_4 = (\alpha\beta)_{23}\,\delta_2 - (\alpha\beta)_{24}\,\delta_1$$

and because of the symmetry the same relations connect γ_3, γ_4, γ_1, γ_2. Substituting these relations in III we have

$$[(\alpha\beta)_{34} - (\alpha\beta)_{12}]\,(\alpha_3\gamma_2 - \alpha_4\gamma_1) = 0\ ,\ [(\alpha\beta)_{23} - (\alpha\beta)_{14}]\,(\alpha_3\gamma_2 - \alpha_4\gamma_1) = 0,$$
$$[(\alpha\beta)_{34} - (\alpha\beta)_{12}]\,(\beta_3\delta_2 - \beta_4\delta_1) = 0\ ,\ [(\alpha\beta)_{23} - (\alpha\beta)_{14}]\,(\beta_3\delta_2 - \beta_4\delta_1) = 0,$$

and so if the two different [ ]-brackets are not vanishing simultaneously we have

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\lambda\alpha_3, \lambda\alpha_4, -\lambda\alpha_1, -\lambda\alpha_2),$$
$$(\delta_1, \delta_2, \delta_3, \delta_4) = (\mu\beta_3, \mu\beta_4, -\mu\beta_1, -\mu\beta_2).$$

Substitution of these values in I gives $\lambda\mu = 1$, in II $\lambda = \mu$, so $\lambda = \mu = \pm 1$, and the system of transformations containing the identical transformation is obtained for $\lambda = -1$ as

$$\Psi = \left\| \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -\alpha_3 & -\alpha_4 & \alpha_1 & \alpha_2 \\ -\beta_3 & -\beta_4 & \beta_1 & \beta_2 \end{array} \right\|$$

the corresponding sextarny transformation being

$$\left\| \begin{array}{cc} \|P\| & \|0\| \\ \|0\| & \|Q\| \end{array} \right\|$$

with *)

$$P \equiv \left\| \begin{array}{cc|cc} (\alpha\beta)_{12}+(\alpha\beta)_{34} & & (\alpha\beta)_{13}+(\alpha\beta)_{42} & \\ -\alpha_1\alpha_4+\alpha_2\alpha_3+\beta_1\beta_4-\beta_2\beta_3 & \tfrac{1}{2}(\alpha_1^2+\alpha_3^2-\alpha_2^2-\alpha_4^2+-\beta_1^2-\beta_3^2+\beta_2^2+\beta_4^2) & & \\ -\alpha_1\beta_4+\alpha_2\beta_3-\beta_1\alpha_4+\beta_2\alpha_3 & & \alpha_1\beta_1+\alpha_3\beta_3-\alpha_2\beta_2-\alpha_4\beta_4 & \\ -\alpha_1\alpha_4+\alpha_2\alpha_3-\beta_1\beta_4+\beta_2\beta_3 & \tfrac{1}{2}(\alpha_1^2+\alpha_3^2-\alpha_2^2-\alpha_4^2+\beta_1^2+\beta_3^2-\beta_2^2-\beta_4^2) & & \\ \hline (\alpha\beta)_{13}+(\alpha\beta)_{42} & & (\alpha\beta)_{13}-(\alpha\beta)_{42} & \\ \alpha_1\alpha_2+\alpha_3\alpha_4-\beta_1\beta_2-\beta_3\beta_4 & \tfrac{1}{2}(\alpha_1^2+\alpha_3^2+\alpha_2^2+\alpha_4^2+-\beta_1^2-\beta_3^2-\beta_2^2-\beta_4^2) & & \\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_4+\alpha_4\beta_3 & & \alpha_1\beta_1+\alpha_3\beta_3+\alpha_2\beta_2+\alpha_4\beta_4 & \\ \alpha_1\alpha_2+\alpha_3\alpha_4+\beta_1\beta_2+\beta_3\beta_4 & \tfrac{1}{2}(\alpha_1^2+\alpha_3^2+\alpha_2^2+\alpha_4^2+\beta_1^2+\beta_3^2+\beta_2^2+\beta_4^2) & & \end{array} \right\|$$

$$Q \equiv \left\| \begin{array}{cc} (\alpha\beta)_{12}-(\alpha\beta)_{34} & (\alpha\beta)_{14}-(\alpha\beta)_{23} \\ -(\alpha\beta)_{14}+(\alpha\beta)_{23} & (\alpha\beta)_{12}-(\alpha\beta)_{34} \end{array} \right\| .$$

Putting

$$p_{12}=X_4+iX_6 \qquad p_{13}=X_1-iX_2 \qquad p_{14}=-X_3+iX_5$$
$$p_{34}=X_4-iX_6 \qquad p_{42}=X_1+iX_2 \qquad p_{23}=-X_3-iX_5$$

the general projective transformation of G_4 induces an orthogonal transformation in X_k leaving invariant

$$X_1^2+X_2^2+X_3^2+X_4^2+X_5^2+X_6^2 .$$

The transformations Ψ leave both $X_1^2+X_2^2+X_3^2+X_4^2$ and $X_5^2+X_6^2$ invariant.

If we put

$$\alpha_1=A_1\cos\omega \ , \ \alpha_3=A_1\sin\omega \ , \ \alpha_2=A_2\cos\Omega \ , \ \alpha_4=A_2\sin\Omega$$
$$\beta_1=B_1\cos\varphi \ , \ \beta_3=B_1\sin\varphi \ , \ \beta_2=B_2\cos\Phi \ , \ B_4=B_2\sin\Phi$$

the elements of P are functions of the *differences* of φ, Φ, ω, Ω only, whereas the elements of Q are

$$(\alpha\beta)_{12}-(\alpha\beta)_{34}=A_1B_2\cos(\omega+\Phi)-A_2B_1\cos(\Omega+\varphi)$$
$$(\alpha\beta)_{14}-(\alpha\beta)_{23}=A_1B_2\sin(\omega+\Phi)-A_2B_1\sin(\Omega+\varphi)$$

The transformations Ψ form a group. The matrix Ψ determines P uniquely.

$$\Psi(\alpha,\beta).\Psi(a,b)=\Psi(\mathfrak{A},\mathfrak{B})$$

gives

$$(\mathfrak{A}\,\mathfrak{B})_{14}-(\mathfrak{A}\,\mathfrak{B})_{23}=[(ab)_{14}-(ab)_{23}]\,[(\alpha\beta)_{12}-(\alpha\beta)_{34}]+$$
$$+[(ab)_{12}-(ab)_{34}]\,[(\alpha\beta)_{14}-(\alpha\beta)_{23}]$$

and the transformations for which

$$(\alpha\beta)_{14}-(\alpha\beta)_{23}=0$$

form a sub-group for which Q is the identical transformation. Moreover

$$P(\alpha,\beta)\ P(a,b)=P(\mathfrak{A},\mathfrak{B}).$$

*) The last two columns of P are printed in the last four rows.

The transformations

$$I_\Delta \equiv \begin{Vmatrix} \cos\Delta & 0 & \sin\Delta & 0 \\ 0 & \cos\Delta & 0 & \sin\Delta \\ -\sin\Delta & 0 & \cos\Delta & 0 \\ 0 & -\sin\Delta & 0 & \cos\Delta \end{Vmatrix}$$

correspond to the identical transformation P. We have

$$I_\Delta \Psi = \Psi I_\Delta , \quad \text{so} \quad \Psi_1 I_\Delta \cdot \Psi_2 I_D = \Psi_1 \Psi_2 I_{\Delta+D} .$$

An orthogonal transformation of X_4, X_1, X_3, X_2

$$\begin{Vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ q_1 & q_2 & q_3 & q_4 \end{Vmatrix} \qquad l_1^2+l_2^2+l_3^2+l_4^2 = S \quad \text{etc.}$$

gives the equations for $\alpha_1, \ldots, \beta_4$:

$$\begin{aligned}
&\alpha_1^2+\alpha_3^2 = \tfrac{1}{2}[m_2+i\,m_4-i\,q_2+q_4] = \tfrac{1}{2}N, && \alpha_2^2+\alpha_4^2 = \tfrac{1}{2}[-m_2+i\,m_4+i\,q_2+q_4],\\
&\beta_1^2+\beta_3^2 = \tfrac{1}{2}[-m_2-i\,m_4-i\,q_2+q_4] && , \beta_2^2+\beta_4^2 = \tfrac{1}{2}[m_2-i\,m_4+i\,q_2+q_4],\\
&-\alpha_1\alpha_4+\alpha_2\alpha_3 = \tfrac{1}{2}[m_1-iq_1] && , \alpha_1\alpha_2+\alpha_3\alpha_4 = \tfrac{1}{2}[-m_3+iq_3] ,\\
&\beta_1\beta_4-\beta_2\beta_3 = \tfrac{1}{2}[m_1+iq_1] && , \beta_1\beta_2+\beta_3\beta_4 = \tfrac{1}{2}[m_3+iq_3] ,\\
&\alpha_1\beta_3-\alpha_3\beta_1 = \tfrac{1}{2}[l_2+il_4] && , \alpha_1\beta_2+\alpha_3\beta_4 = \tfrac{1}{2}[l_1+n_3] ,\\
&\alpha_1\beta_2-\alpha_2\beta_4 = \tfrac{1}{2}[l_2-il_4] && , \alpha_2\beta_1+\alpha_4\beta_3 = \tfrac{1}{2}[-l_1+n_3] ,\\
&\alpha_1\beta_1+\alpha_3\beta_3 = \tfrac{1}{2}[-n_2-in_4] && , \alpha_1\beta_4-\alpha_3\beta_2 = \tfrac{1}{2}[n_1-l_3] ,\\
&\alpha_2\beta_2+\alpha_4\beta_4 = \tfrac{1}{2}[n_2-in_4] && , \alpha_4\beta_1-\alpha_2\beta_3 = \tfrac{1}{2}[n_1+l_3] .
\end{aligned}$$

These equations are consistent because of the orthogonality relations of $\|l\,m\,n\,q\|$ and put the form of $I_\angle$ into evidence. If $N \neq 0$ we have immediately:

$$\begin{aligned}
&N\alpha_2 = (m_1-iq_1)\,\alpha_3+(-m_3+iq_3)\,\alpha_1 && , N\alpha_4 = -(m_1-iq_1)\,\alpha_1+(-m_3+iq_3)\alpha_3,\\
&N\beta_1 = -(l_2+il_4)\,\alpha_3-(n_2+in_4)\,\alpha_1 && , N\beta_3 = (l_2+il_4)\,\alpha_1-(n_2+in_4)\,\alpha_3,\\
&N\beta_2 = (l_1+n_3)\,\alpha_1-(n_1-l_3)\,\alpha_3 && , N\beta_4 = (l_1+n_3)\,\alpha_3+(n_1-l_3)\,\alpha_1,
\end{aligned}$$

$$\alpha_1^2+\alpha_3^2 = \tfrac{1}{2}N .$$

Supposing the l, m, n, q, to have real values we have

$$\begin{aligned}
&\alpha_1\bar{\beta}_1+\alpha_2\bar{\beta}_2+\alpha_3\bar{\beta}_3+\alpha_4\bar{\beta}_4 = 0,\\
&-\alpha_1\bar{\beta}_3-\alpha_2\bar{\beta}_4+\alpha_3\bar{\beta}_1+\alpha_4\bar{\beta}_2 = 0,\\
&\alpha_1\bar{\alpha}_1+\alpha_2\bar{\alpha}_2+\alpha_3\bar{\alpha}_3+\alpha_4\bar{\alpha}_4 =\\
&\qquad \{[S+(mq)_{24}]\,(\alpha_1\bar{\alpha}_1+\alpha_3\bar{\alpha}_3)+i(mq)_{13}\,[\alpha_1\bar{\alpha}_3-\alpha_3\bar{\alpha}_1]\} : \tfrac{1}{2}N\bar{N},\\
&\beta_1\bar{\beta}_1+\beta_2\bar{\beta}_2+\beta_3\bar{\beta}_3+\beta_4\bar{\beta}_4 =\\
&\qquad = \{[S+(ln)_{13}]\,(\alpha_1\bar{\alpha}_1+\alpha_3\bar{\alpha}_3)+i(ln)_{24}\,[\alpha_1\bar{\alpha}_3-\alpha_3\bar{\alpha}_1]\} : \tfrac{1}{2}N\bar{N},
\end{aligned}$$

and as the minors of $||l\ m\ n\ q||$ are equal to their algebraical minors the last four expressions are equal.

Again

$$-\alpha_1\bar{\alpha}_3-\alpha_2\bar{\alpha}_4+\alpha_3\bar{\alpha}_1+\alpha_4\bar{\alpha}_2=$$
$$=\{i(mq)_{13}(\alpha_1\bar{\alpha}_1+\alpha_3\bar{\alpha}_3)-[S+(mq)_{24}](\alpha_1\bar{\alpha}_3-\alpha_3\bar{\alpha}_1)\}:\tfrac{1}{2}N\bar{N}$$
$$-\beta_1\bar{\beta}_3-\beta_2\bar{\beta}_4+\beta_3\bar{\beta}_1+\beta_4\bar{\beta}_2=$$
$$=\{i(ln)_{24}(\alpha_1\bar{\alpha}_1+\alpha_3\bar{\alpha}_3)-[S+(ln)_{13}](\alpha_1\bar{\alpha}_3-\alpha_3\bar{\alpha}_1)\}:\tfrac{1}{2}N\bar{N}$$

which last expressions are therefore also equal. For real orthogonal transformations of X_4, X_1, X_3, X_2 the matrix Ψ is therefore "almost unitary". Putting

$$\alpha_1=M\cos\varphi \qquad \alpha_3=M\sin\varphi \qquad \varphi=s-it$$

the condition for unitary Ψ becomes

$$\operatorname{tg}\ 2it=\frac{i(mq)_{13}}{S+(mq)_{24}},$$

which equation is soluble if

$$\left|\frac{(mq)_{13}}{S+(mq)_{24}}\right|\leqslant 1.$$

This last condition is satisfied whenever $||l\ m\ n\ q||$ is a real orthogonal matrix as the function

$$F\equiv[1+(mq)_{24}]^2-(mq)_{13}^2$$

for

$$m_1^2+m_2^2+m_3^2+m_4^2-1=0$$
$$q_1^2+q_2^2+q_3^2+q_4^2-1=0$$
$$m_1q_1+m_2q_2+m_3q_3+m_4q_4=0,$$

has all the extrema $\geqq 0$, as is easily obtained e.g. by means of LAGRANGE multipliers. So at the cost of a transformation of X_5, X_6 different from unity we can always make Ψ unitary for real "LORENTZ transformations" of X_1, X_2, X_3, X_4. As in the original DIRAC equations X_5 does not occur, whilst X_6 can only appear in an exponential factor, the transformation of X_5, X_6 does not disturb their LORENTZ invariance.

Conclusion:

1. The generalised DIRAC equations are invariant under general projective G_4-transformations; so are the PAULI-identities between $\varphi'\alpha_i\alpha_k\ldots\ldots\varphi$
2. Restriction to unitary transformation of G_4 causes $\bar{\psi}$ to transform in G_4 contragredient to Ψ and makes it possible to define the quantum mechanical densities $\bar{\psi}\ \alpha_i\ldots\psi$

3. Splitting up the general orthogonal transformation in G_6 into an orthogonal transformation of a sub-G_4 and of a sub-G_2 gives the projective transformation the form

$$\left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -\alpha_3 & -\alpha_4 & \alpha_1 & \alpha_2 \\ -\beta_3 & -\beta_4 & \beta_1 & \beta_2 \end{matrix} \right\|$$

which can be related to two binary transformations $\alpha_1\alpha_2, \beta_1\beta_2$; $\alpha_3\alpha_4, \beta_3\beta_4$.

FORMALISTISCHE BETRACHTUNGEN ÜBER INTUITIONISTISCHE UND VERWANDTE LOGISCHE SYSTEME II

VON

J. RIDDER

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Die affirmativen Kalküle M, Iund K.

§ 5. Das Axiomensystem der Aussagenlogik A_2 wird hier angenommen und fortgesetzt.

Axiom 6^α. $(X \subset \lambda) \subset X'$.

Axiom 6^β. $X' \subset (X \subset \lambda)$.

In diesen Axiomen tritt die einstellige Grundverknüpfung "*non*", angedeutet durch ein Akzent, auf; auszerdem λ, die Andeutung einer falschen Aussage. In Regel E erweitere man die Klasse der Kalkülformeln mit λ, $\mathfrak{R}'$, $\mathfrak{S}'$.

Nach dem Ersetzungsschema E_0 läszt sich das Axiom 6^β auch schreiben: $(X'.X) \subset \lambda$, oder, wegen Satz 6 und Schema 1^β, $X \subset (X' \subset \lambda)$.

Axiom 6^β *ist unabhängig von den Axiomen* 1^α, 2^α, $3°$, 4^α, 6^α *und den Regeln* E, E_0, 1^β, 2^β *und* 4^β; *Axiom* 6^α *ist unabhängig von den Axiomen* 1^α, 2^α, $3°$, 4^α, 6^β *und denselben Regeln.* Die erste Behauptung folgt aus der von HEYTING und JOHANSSON [27]) betrachteten Gruppe:

$\subset$	0	1
0	0	0
1	1	0

$\cdot$	0	1
0	0	1
1	1	1

$+$	0	1
0	0	0
1	0	1

$'$	0	1
	0	0

wenn man dabei ν den Wert 0, λ den Wert 1 zuerteilt (die Kolonnen unter $\subset$, $\cdot$ und $+$ geben die möglichen Werte für das hintere Element dieser zweistelligen Aussagen). Die Axiome, 6^β ausgenommen, erhalten immer den Wert 0, unabhängig von den den lateinischen Buchstaben zuerteilten Zahlenwerten; jede Regel führt von Aussagen mit dem Wert 0 zu Aussagen, die auch immer den Wert 0 haben; dagegen erhält 6^β den Wert 1, falls X den Wert 0 hat. [28])

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[27]) Siehe HEYTING, loc. cit. 2), S. 55; JOHANSSON, loc. cit. 7), S. 124.

[28]) Auch das HEYTINGsche Axiom 4.11 $\{(X \subset Y).(X \subset Y')\} \subset X'$ hat immer den Wert 0; ebenso Axiom 5°. Führt man, in Analogie mit JOHANSSON, loc. cit. S. 129, λ mittels der Definition $\lambda \underset{\text{Df}}{=} (X'.X'')$ ein, so musz λ im obigen Beispiel den gleichen Wert (0) wie ν zugewiesen werden. Nun ist Axiom 5° nicht erfüllt (denn $\lambda \subset 1$ erhält den Wert 1). Das derartig abgeänderte Beispiel zeigt, *dasz Axiom* 5° *unabhängig ist von folgenden Axiomen und Regeln*: 1^α, 2^α, $3°$, 4^α, 6^α, 6^β *und* E, E_0, 1^β, 2^β, 4^β (und von Axiom 4.11). [Auch nicht erfüllt ist das HEYTINGsche Axiom 4.1: $X' \subset (X \subset Y)$].

Die zweite Behauptung folgt mit untenstehender, von JOHANSSON, loc. cit. 7), S. S. 124 betrachteter Gruppe:

$\subset$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	0	0	2	2
3	3	0	3	0	3
4	4	0	0	0	0

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	1	1	1
2	2	1	2	1	2
3	3	1	1	3	3
4	4	1	2	3	4

$+$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	2	4	4
3	0	3	4	3	4
4	0	4	4	4	4

jedoch, in Abweichung von JOHANSSON, mit $\mathfrak{A}' = 1$ ($\mathfrak{A} = 0, 1, 2, 3, 4$), und daneben $\nu = 0$, $\lambda = 4$. Die Axiome 1^α, 2^α, 3°, 4^α und 6^β erhalten immer den Wert 0; dagegen hat Axiom 6^α den Wert 1 für $X = 1$ (und ebenso das Axiom 5°); die Regeln führen von Aussagen mit ausschließlichem Werte 0 zu Aussagen mit derselben Eigenschaft. [29]).

Ohne Benutzung von 6^β läßt sich beweisen:

A. $(X \subset Y) \subset [(Y \subset \lambda) \subset X']$; [30])

ohne Benutzung von Axiom 6^α beweist man:

B. $(X \subset Y) \subset [Y' \subset (X \subset \lambda)]$, [31])

während unter Anwendung von beiden folgt:

C. $(X \subset Y) \subset [Y' \subset X']$. [32]) [33])

A ist keine Folge unseres Axiomensystems bei Ausschließung von Axiom 6^α; B nicht bei Ausschließung von Axiom 6^β. Das folgt unmittelbar aus den beiden vorangehenden Gruppen; in der zweiten Gruppe nehme man dazu $X = 1$, $Y = 1$, in der ersten $X = 0$, $Y = 0$.

Beweisbar (mit 6^α und 6^β) ist das HEYTINGsche Axiom:

4. 11. $\nu \subset [\{[X \subset Y] \cdot [X \subset Y']\} \subset X']$.

Beweis. Nach Satz 12 genügt die Ableitung ohne vorangehendes $\nu \subset$.

(Satz 9) $[\{X \subset (Y \cdot Y')\} \cdot \{(Y \cdot Y') \subset \lambda\} \subset (X \subset \lambda)$;

(E_0, 1^β) $\nu \subset [\{(Y \cdot Y') \subset \lambda\} \subset [\{X \subset (Y \cdot Y')\} \subset (X \subset \lambda)]]$;

(6 , 1^β, E_0) $\nu \subset \{(Y \cdot Y') \subset \lambda\}$;

(S. 11, S. 12) $\{X \subset (Y \cdot Y')\} \subset (X \subset \lambda)$.

(1) (Ax. 6^α, Sch. 1^β) $\{X \subset (Y \cdot Y')\} \subset X'$

(2) (S. 10) $[(X \subset Y) \cdot (X \subset Y')] \subset \{X \subset (Y \cdot Y')\}$.

((1), (2), 1^β) $[(X \subset Y) \cdot (X \subset Y')] \subset X'$.

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[29]) Bei folgender Abänderung dieses Beispiels: $\mathfrak{A}' = (\mathfrak{A} \subset 4)$ ($\mathfrak{A} = 0, 1, 2, 3, 4$), $\nu = 0$, $\lambda = 1$, entsteht ein zweites (komplizierteres) Beispiel für unsere erste Behauptung.

[30]) Bew.: (Satz 9) $[(X \subset Y) \cdot (Y \subset \lambda)] \subset [X \subset \lambda]$; (Ax. 6^α) $[X \subset \lambda] \subset X'$; (Sch. 1^β) $[(X \subset Y) \cdot (Y \subset \lambda)] \subset X'$; (Sch. E_0) $(X \subset Y) \subset [(Y \subset \lambda) \subset X']$.

[31]) Bew.: (Satz 9) $[(X \subset Y) \cdot (Y \subset \lambda)] \subset [X \subset \lambda]$, oder, abgekürzt, $(\mathfrak{D} \cdot \mathfrak{E}) \subset \mathfrak{F}$; (Ax. 6^β) $Y' \subset (Y \subset \lambda)$ oder, abgekürzt, $\mathfrak{G} \subset \mathfrak{E}$; somit $(\mathfrak{D} \cdot \mathfrak{G}) \subset (\mathfrak{D} \cdot \mathfrak{E})$, und $(\mathfrak{D} \cdot \mathfrak{G}) \subset \mathfrak{F}$, also (Sch. E_0) $\mathfrak{D} \subset [\mathfrak{G} \subset \mathfrak{F}]$ oder $(X \subset Y) \subset [Y' \subset (X \subset \lambda)]$.

[32]) Bew.: (Ax. 6^β) $[(X \subset Y) \cdot Y'] \subset [(X \subset Y) \cdot (Y \subset \lambda)]$; (Satz 9) $[(X \subset Y) \cdot (Y \subset \lambda)] \subset [X \subset \lambda]$; (Ax. 6^α) $(X \subset \lambda) \subset X'$; (Sch. 1^β, E_0) $(X \subset Y) \subset [Y' \subset X']$.

[33]) Satz C führt zum Schema $\dfrac{\mathfrak{A} = \mathfrak{B}}{\mathfrak{A}' = \mathfrak{B}'}$.

§ 6. Man kann, umgekehrt, 4. 11 annehmen (und die Kalkülformeln in Regel E entsprechend erweitern), und versuchen 6^{α} und 6^{β} abzuleiten. Dies ist ohnemehr nicht möglich. Denn das mit 4. 11 erweiterte Axiomensystem enthält die Aussage λ nicht; diese musz noch in sinngemäszer Weise eingeführt werden.

Mit 4. 11 läszt sich nach JOHANSSON, loc. cit. 7), S. 129 ableiten:

$$(X' \cdot X'') \subset (Y' \cdot Y'') \subset (X' \cdot X''), \text{ oder } (X' \cdot X'') = (Y' \cdot Y'').$$

Dies führt dann zur Einführung von λ mittels einer evidenten Abänderung von Regel E und des

Axioms θ. $(X' \cdot X'') \subset \lambda$; $\lambda \subset (X' \cdot X'')$.[34] [35]

Nun sind 6^{α} und 6^{β} beweisbar.

Beweis von 6^{α}. (S. 9) $[\{\lambda \subset (X' \cdot X'')\} \cdot (X \subset \lambda)] \subset [X \subset (X' \cdot X'')]$; (Sch. E_0, S. 11, 12, Ax. θ) $(X \subset \lambda) \subset [X \subset (X' \cdot X'')]$; $[X \subset (X' \cdot X'')] \subset$ $\subset$ (Umkehrung v. S. 10) $[(X \subset X') \cdot (X \subset X'')] \subset$ (Ax. 4. 11) X'; (Sch. 1^{β}) $(X \subset \lambda) \subset X'$.

Beweis von 6^{β}. $X \subset X''$ [36]; $(X \cdot X') \subset (X' \cdot X'') \subset$ (Ax. θ) λ; (Sch. 1^{β}) $(X \cdot X') \subset \lambda$.

Der Kalkül M, welchen wir definieren als eine Aussagenlogik A_2, die auszerdem 6^{α} und 6^{β} genügt, ist aequivalent mit einer Logik A_2, in welcher die Axiome 4. 11 *und θ gelten* (Regel E in beiden Kalkülen entsprechend geändert.) Im wesentlichen führen beide Axiomensysteme zum *Minimalkalkül* von JOHANSSON. [37]

Es wird manchmal möglich sein zu einem Theorem des Minimalkalküls zwei Sätze zu finden, deren einer ausschlieszlich unter Anwendung von 6^{α}, und deren anderer nur unter Anwendung von 6^{β} bewiesen werden kann; ein Beispiel liefern C, mit vorangehendem $\nu \subset$, und die zugehörigen Sätze A und B.

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[34]) Die Einführung von λ geschieht bei JOHANSSON, loc. cit., in anderer (und zwar definitorischer) Weise: $\lambda \underset{\text{Df}}{=} (X' \cdot X'')$. — Nimmt man 6^{α} und 6^{β} an, so ist Axiom θ beweisbar, und zwar die erste Hälfte ohne 6^{α}, die zweite Hälfte ohne 6^{β}.

[35]) Die erste, in § 5 betrachtete Gruppe liefert für $X' \cdot X''$ immer den Wert 0, wodurch $(X' \cdot X'') \subset \lambda$ den Wert 1 erhält. Während die zweite Hälfte des Axioms θ eine unmittelbare Folgerung von Axiom 5° ist, ist seine erste Hälfte (ebenso wie Ax. 4. 1 von Fuszn. 28) unabhängig von den Axiomen und Regeln 1^{α}, 2^{α}, 3°, 4^{α}, 5°, 6^{α}; E, E_0, 1^{β}, 2^{β}, 4^{β} und von Ax. 4.11. Vergl. den Beweis von 6^{β}; auch die Fuszn. 34 u. 28. — Die zweite Gruppe von § 5 zeigt, dasz die zweite Hälfte des Axioms θ unabhängig ist von der ersten Hälfte und 1^{α}, 2^{α}, 3°, 4^{α}, 6^{β} und von allen Regeln.

[36]) Siehe JOHANSSON, loc. cit. 7), S. 121 (4.3).

[37]) Nur treten hier auch ‚Sätze' auf, die keine ‚Theoreme' sind, und ‚Theoreme' $\nu \subset \mathfrak{A}$, die λ und (oder) ν in $\mathfrak{A}$ enthalten. — Wir bemerken, dasz aus den Sätzen 4, 14, 15 und dem Schema in Fuszn. 33 folgt, *dasz in jedem Satz ein Ausdruck $\mathfrak{A}' \cdot \mathfrak{A}''$ durch λ, und umgekehrt, ersetzt werden darf. Eine analoge Bemerkung gilt für $\mathfrak{A} \subset \lambda$ und $\mathfrak{A}'$.*

§ 6^{bis}. *Jedes im Kalkül M ableitbare Theorem, welches λ nicht und ν nur am Anfang ($\nu \subset$) enthält, ist auch im Minimalkalkül* (d.h. ohne 3°, 6^{α}, 6^{β}, jedoch bei Zulassung von Axiom 4. 11) *ableitbar; für diese Theoreme führen beide Kalküle genau ebenso weit.*

Beweis. Die Axiome 1^{α}, 2^{α}, 4^{α} und die Regeln E_0, 1^{β}, 2^{β}, 4^{β} von Kalkül M lassen sich abändern wie in Fuszn. 16, 25 angedeutet. Aus einer Ableitung in Kalkül M wird ν entfernt, und die Anwendung von Ax. 3° ersetzt auf die in Fuszn. 16 angegebene Weise.

Es wird weiter genügen zu zeigen, dasz jede Anwendung von Axiom 6^{α} oder Axiom 6^{β} sich durch den Gebrauch von Axiom 4. 11 ersetzen läszt. Dazu ändere man jede im Beweisverfahren auftretende, λ enthaltende Kalkülformel dadurch ab, dasz man λ durch $Y'_m \cdot Y''_m$ ersetzt (m eine genügend hohe natürliche Zahl, damit Y_m nicht unter den im Beweisverfahren auftretenden elementaren Kalkülformeln auftritt). Alle in dieser Weise erhaltenen Sätze sind nun im Minimalkalkül beweisbar. Denn jede Einführung von λ mittels E läszt sich durch eine Einführung von $Y'_m \cdot Y''_m$ mittels E, jede Anwendung von Axiom 6^{α} läszt sich durch Anwendung von

$$(6^{\gamma}) \qquad \{Y \subset (Y'_m \cdot Y''_m)\} \subset Y',$$

jede Anwendung von Axiom 6^{β} durch die von

$$(6^{\delta}) \qquad Y' \subset \{Y \subset (Y'_m \cdot Y''_m)\} \text{ oder } (Y \cdot Y') \subset (Y'_m \cdot Y''_m)$$

ersetzen; und 6^{γ} und 6^{δ} sind im Minimalkalkül ableitbar. [Ableitung von 6^{γ}: $\{Y \subset (Y'_m \cdot Y''_m)\} \subset \{(Y \subset Y'_m) \cdot (Y \subset Y''_m)\}$; (4. 11, E) $\{(Y \subset Y'_m) \cdot (Y \subset Y''_m)\} \subset Y'$; ($1^{\beta}$) $\{Y \subset (Y'_m \cdot Y''_m)\} \subset Y'$. — Ableitung von 6^{δ}: (Johansson, loc. cit. 7), S. 129) $(Y' \cdot Y'') \subset (Y'_m \cdot Y''_m)$; (Joh., loc. cit. S. 121 (4. 3)) $Y \subset Y''$, also $(Y \cdot Y') \subset (Y' \cdot Y'')$; darauf 1^{β}]. [38])

§ 7. Fügt man das Heytingsche Axiom 4. 1: $X' \subset (X \subset Y)$ oder $(X \cdot X') \subset Y$ zum Axiomensystem von Kalkül A_2, so wird die Aussage λ der Bedingung $(X \cdot X') \subset \lambda$, d.i. Axiom 6^{β}, genügen müssen. Auch wenn man auszerdem 4. 11 annimmt, ist die Wahl der λ aufzuerlegenden Bedingungen noch immer teilweise frei. Das zeigt folgende Gruppe, welche Heyting, loc. cit. 3), S. 56, konstruierte zum Beweise der Behauptung, dasz der Satz vom ausgeschlossenen Dritten sich nicht mit den Axiomen und Regeln des intuitionistischen Aussagenkalküls ableiten läszt; als Axiomensystem für diesen Kalkül kann man nehmen das System der Logik A_2 ohne Ax. 3°, doch vermehrt mit 4. 11 und 4. 1 (und mit in allen Formeln der Axiome und Regeln vorangehendem $\nu \subset$).

$\subset$	0	1	2
0	0	0	0
1	1	0	1
2	2	0	0

$\cdot$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$+$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$'$	0	1	2
	1	0	1

$\nu = 0$

[38]) Das voranzusetzende $\nu \subset$ haben wir fortwährend unterdrückt.

Axiom 6^β (und somit auch der erste Teil von Axiom θ) ist erfüllt, unabhängig von dem λ zuerkannten Werte. Setzt man $\lambda = 2$ oder 0, so ist der zweite Teil von Axiom θ, und somit auch Axiom 5°, nicht erfüllt ($X = 0$ liefert für $\lambda \subset (X' \cdot X'')$ den Wert 1); für $\lambda = 1$ ist Axiom 5° dagegen erfüllt.

Wünschen wir die Axiome für λ im nun zu besprechenden Kalkül I so zu wählen, dasz die für den Kalkül M gewählten ihre Gültigkeit behalten, und 4. 1 ableitbar wird, so liegt es auf der Hand Axiom 5° hinzuzufügen, *so dasz der Kalkül I definiert wird als ein Kalkül M, in welchem auch Axiom 5° gilt.* [39])

4. 1 ist nun wirklich ableitbar: (Ax. 5°) $\lambda \subset Y$; (Ax.6^β) $(X \cdot X') \subset \lambda$; (Sch. 1^β) $(X \cdot X') \subset Y$; (Sch. E_0) $X' \subset (X \subset Y)$.

Das principium contradictionis erhält hier, neben $\nu = (X \cdot X')'$ (siehe Heyting, loc. cit. 3), S. 49), auch die Form $\lambda = (X \cdot X')$; in dieser Form ist es gleichwertig mit dem Axiomenpaar 5°, 6^β.

§ 7^{bis}. *Jedes im Kalkül I ableitbare Theorem, welches λ nicht und ν nur am Anfang ($\nu \subset$) enthält, ist auch im intuitionistischen Aussagenkalkül* (d.h. ohne 3°, 5°, 6^α, 6^β, jedoch mit den Axiomen 4. 11 und 4. 1) *ableitbar; für diese Theoreme führen beide Kalküle genau ebenso weit.*

Beweis. Aus Fuszn. 16 und § 6^{bis} geht hervor, dasz es genügt zu zeigen, dasz der mit Axiom 5° korrespondierende Satz

$$(5^\alpha) \qquad (Y'_m \cdot Y''_m) \subset X$$

sich mittels Axiom 4. 1 ableiten läszt. [Ableitung von 5^α: (Ax. 4. 1) $(X \cdot X') \subset Y$, (Regel E) $(Y'_m \cdot Y''_m) \subset X$]. [38])

§ 8. Fügen wir schlieszlich hinzu

Axiom 7^0. $\nu \subset (X + X')$.

Mit dem Axiomenpaar 3°, 7° ist nun gleichwertig $\nu = (X + X')$, d.i. das principium tertii exclusi.

Der Kalkül K sei definiert als ein Kalkül I, in welchem Axiom 7° gilt. Wie wir in den Sätzen 25 und 26 sahen, ist K distributiv; er ist somit ein Kalkül im Sinne unserer Arbeit in Proceed. Akad. Wet. Amsterdam 49 (1946), S. 1158—1160. Dort waren Sätze der Form $\mathfrak{A} \subset \mathfrak{B}$ mit Implikationszeichen in $\mathfrak{A}$ oder (und) $\mathfrak{B}$ nicht möglich. Verabreden wir jedoch eine Implikation $\mathfrak{P} \subset \mathfrak{Q}$ in $\mathfrak{A}$ oder $\mathfrak{B}$ als eine andere Schreibweise von $\mathfrak{P}' + \mathfrak{Q}$ aufzufassen, so werden das hier gegebene Axiom 6^α und Schema E_0 im System der Proceed. ableitbar.

Umgekehrt sind hier $[\mathfrak{A} \subset \mathfrak{B}] \subset (\mathfrak{A}' + \mathfrak{B})$ [40]) und $(\mathfrak{A}' + \mathfrak{B}) \subset [\mathfrak{A} \subset \mathfrak{B}]$

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[39]) λ *und* $\mathfrak{A} \cdot \mathfrak{A}'$ *können nun einander in Sätzen und Theoremen ersetzen.* Vergl. Fuszn. 37.

[40]) $(\mathfrak{A} \subset \mathfrak{B}) \subset$ (Satz C) $(\mathfrak{B}' \subset \mathfrak{A}') \subset [\mathfrak{B}' \subset (\mathfrak{A}' + \mathfrak{B})]$; $(\mathfrak{A} \subset \mathfrak{B}) \subset$ (Sch. E_0) $[\mathfrak{B} \subset (\mathfrak{A}' + \mathfrak{B})]$; also $(\mathfrak{A} \subset \mathfrak{B}) \subset [(\mathfrak{B}' + \mathfrak{B}) \subset (\mathfrak{A}' + \mathfrak{B})]$; (Sch. E_0, Ax. 7°) $(\mathfrak{A} \subset \mathfrak{B}) \subset (\mathfrak{A}' + \mathfrak{B})$.

(ableitbare) Sätze; somit ist in allen Sätzen und Theoremen (vgl. die Fuszn. 37 u. 39) $\mathfrak{A} \subset \mathfrak{B}$ durch $\mathfrak{A}' + \mathfrak{B}$, und umgekehrt, ersetzbar.

Das Axiomensystem für den Kalkül K ist dadurch, wie aus der zit. Arbeit hervorgeht, *völlig aequivalent mit dem Russell-Whiteheadschen Axiomensystem für den klassischen Aussagenkalkül erweitert mit zwei Axiomen:* $\lambda \to X$ *und* $X \to \nu$.

Die negierenden Kalküle M^*, I^* und K^*.

§ 9. Duale Umsetzung (siehe § 4) der Betrachtungen von §§ 5—8, wobei die durch das Akzent angegebene Verneinung als dual gegenüber sich selbst stehend betrachtet werden soll, liefert folgendes.

Der Kalkül A_2^* (§§ 2 und 4) wird hier angenommen und fortgeführt.

In Regel E^* erweitere man die Klasse der Kalkülformeln mit ν, $\mathfrak{R}'$, $\mathfrak{S}'$.

Axiom $6^{\alpha*}$. $X' \subset (\nu \subset X)$.[41])

Axiom $6^{\beta*}$. $(\nu \subset X) \subset X'$.

Nach dem Ersetzungsschema E_0^* läszt sich Axiom $6^{\beta*}$ auch schreiben: $\nu \subset (X + X')$,[42]) oder, wegen Satz 17 und Schema 1^β, $(\nu \subset X') \subset X$.

Axiom $6^{\beta*}$ *ist unabhängig von den Axiomen* 1^α, 2^α, 4^α, $5°$, $6^{\alpha*}$ *und den Regeln* E^*, E_0^*, 1^β, 2^β *und* 4^β; *Axiom* $6^{\alpha*}$ *ist unabhängig von den Axiomen* 1^α, 2^α, 4^α, $5°$, $6^{\beta*}$ *und den Regeln* E^*, E_0^*, 1^β, 2^β *und* 4^β.

Ohne Benutzung von Axiom $6^{\beta*}$ läszt sich beweisen:

A*. $[X' \subset (\nu \subset Y)] \subset (Y \subset X)$;

ohne Benutzung von Axiom $6^{\alpha*}$ beweist man:

B*. $[(\nu \subset X) \subset Y'] \subset (Y \subset X)$,

während bei Anwendung von beiden folgt:

C*. $[X' \subset Y'] \subset (Y \subset X)$,

ein nicht zur intuitionistischen Logik gehöriger Transpositionssatz.

Das mit $6^{\alpha*}$ und $6^{\beta*}$ beweisbare Analogon des Heytingschen Axioms 4. 11 ist:

4. 11*. $[X' \subset \{[Y' \subset X] + [Y \subset X]\}] \subset \lambda$.

§ 10. Man kann, umgekehrt, 4. 11* annehmen (und die Kalkülformeln in Regel E^* entsprechend erweitern), und versuchen $6^{\alpha*}$ und $6^{\beta*}$ abzuleiten. Dies ist nur möglich nach sinngemäszer Einführung von ν.

Mit 4. 11* ist ableitbar:

$$(X' + X'') \subset (Y' + Y'') \subset (X' + X''), \text{ oder } (X' + X'') = (Y' + Y'').$$

Dies leitet zur Einführung von ν mittels einer evidenten Abänderung von Regel E^* und des

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[41]) Interpretation: "Die Negation von $\nu \subset X$ kann nur mit der Negation der Negation von X koexistieren". Oder: "die Verneinung davon, dasz die Verneinung von X nur mit der Verneinung von ν zusammengehen kann, kann nur mit der Negation von X' koexistieren".

[42]) Interpretation: "Die Verneinung von $X + X'$ hat die von ν zur Folge".

Axioms θ^*. $\nu \subset (X' + X'')$; $(X' + X'') \subset \nu$.[43])

Nun sind 6^{α^*} und 6^{β^*} beweisbar.

Der Kalkül M^* (Minimalkalkül in der verneinenden Aussagenlogik), *welchen wir definieren als eine Aussagenlogik A_2^*, die auszerdem 6^{α^*} und 6^{β^*} genügt, ist aequivalent mit einer Logik A_2^*, in welcher die Axiome 4. 11* und θ^* gelten* (Regel E^* in beiden Kalkülen entsprechend abgeändert).[43bis])

Es wird manchmal möglich sein zu einem Theorem dieses Minimalkalküls zwei Sätze zu finden, deren einer ausschlieszlich unter Anwendung von 6^{α^*}, und deren anderer nur unter Anwendung von 6^{β^*} bewiesen werden kann; ein Beispiel liefern C^* (mit nachfolgendem $\subset \lambda$) und die Sätze A^* und B^*.

§ 10[bis]. *Jedes im Kalkül M^* ableitbare (verneinende) Theorem, welches ν nicht enthält und λ nur am Ende* ($\subset \lambda$), *ist, bei Zulassung von Axiom* 4. 11*, *ohne Anwendung der Axiome* 5°, 6^{α^*} *und* 6^{β^*} *ableitbar; für diese Theoreme führt M^* in ursprünglicher und in auf die eben angegebene Weise abgeänderter Form genau ebenso weit.* (Vergl. § 6[bis]).

§ 11. Fügt man das Analogon zum Heytingschen Theorem 4. 1:

4. 1*. $(Y \subset X) \subset X'$ oder $Y \subset (X + X')$

zum Axiomensystem von Kalkül A_2^*, so wird die Aussage ν der Bedingung $\nu \subset (X + X')$, d.i. Axiom 6^{β^*}, genügen müssen. Auch wenn man auszerdem 4. 11* annimmt, ist die Wahl der ν aufzuerlegenden Bedingungen noch immer teilweise frei.

Da wir die im Kalkül M^* für ν angenommenen Axiome im nun zu besprechenden Kalkül I^* beibehalten wollen, während 4. 1* ableitbar werden soll, so liegt es auf der Hand Axiom 3° hinzuzufügen, *so dasz der Kalkül I^* definiert wird als ein Kalkül M^*, in welchem Axiom* 3° *gilt.*[44])

4. 1* ist nun leicht abzuleiten.

Das principium tertii exclusi gilt in I^* in zwei Formulierungen: $\lambda = (X + X')'$ und $\nu = (X + X')$; in der letzten Form ist es gleichwertig mit dem Axiomenpaar 3°, 6^{β^*}. Dagegen ist das principium contradictionis in der Formulierung: $\lambda = (X \cdot X')$ in I^* nicht ableitbar.[45])

§ 11[bis]. *Jedes im Kalkül I^* ableitbare (verneinende) Theorem, welches ν nicht enthält und λ nur am Ende* ($\subset \lambda$), *ist, bei Zulassung der Axiome* 4. 11* *und* 4. 1*, *ohne Anwendung der Axiome* 3°, 5°, 6^{α^*} *und* 6^{β^*} *abzuleiten; für diese Theoreme führt I^* in ursprünglicher und auf die eben angegebene Weise abgeänderter Form ebenso weit.* (Vergl. § 7[bis]).

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[43]) Nimmt man 6^{α^*}und 6^{β^*} an, so ist Axiom θ^* beweisbar.

[43bis]) Regel E in M und Regel E^* in M^* haben den gleichen Wortlaut.

[44]) *ν und $\mathfrak{A} + \mathfrak{A}'$ können nun einander in Sätzen und Theoremen ersetzen.*

[45]) Vergl. § 7.

§ 12. Fügen wir schliesslich hinzu

Axiom 7^*. $(X \cdot X') \subset \lambda$.

Mit dem Axiomenpaar 5°, 7^* ist nun gleichwertig $\lambda = (X \cdot X')$, d.i. das principium contradictionis.

Der verneinende Aussagenkalkül K^* sei definiert als ein Kalkül I^*, in welchem Axiom 7^* gilt. K^* ist distributiv (siehe § 4); er ist somit ein Kalkül im Sinne unserer Arbeit in Proceed. Akad. Wet. Amsterdam 49 (1946), S. 1158—1160. Sätze der Form $\mathfrak{A} \subset \mathfrak{B}$ mit Implikationszeichen in $\mathfrak{A}$ oder (und) $\mathfrak{B}$ waren dort nicht möglich. Verabreden wir eine Implikation $\mathfrak{P} \subset \mathfrak{Q}$ in $\mathfrak{A}$ oder $\mathfrak{B}$ als eine andere Schreibweise von $\mathfrak{P} \cdot \mathfrak{Q}'$ aufzufassen, so werden das hier gegebene Axiom 6^{a^*} und Schema E_0^* im System der Proceed. ableitbar.

Umgekehrt sind hier $[\mathfrak{A} \subset \mathfrak{B}] \subset (\mathfrak{A} \cdot \mathfrak{B}')$ und $(\mathfrak{A} \cdot \mathfrak{B}') \subset [\mathfrak{A} \subset \mathfrak{B}]$ (ableitbare) Sätze; somit ist in allen Sätzen und Theoremen (vgl. die Fuszn. 37 und 39, und § 8) $\mathfrak{A} \subset \mathfrak{B}$ durch $\mathfrak{A} \cdot \mathfrak{B}'$, und umgekehrt, ersetzbar.

Die verneinenden Theoreme des Kalküls K^* fallen zusammen mit den Sätzen $\mathfrak{A} = \lambda$ der Proceedings;[46]) dort waren $\mathfrak{A} = \lambda$ und $\mathfrak{A}' = \nu$ nur andere Formulierungen für dasselbe Resultat; ebenso $\mathfrak{B}' = \lambda$ und $\mathfrak{B} = \nu$.

Das System der affirmativen Theoreme des Kalküls K ist dasselbe wie das der Sätze $\mathfrak{B} = \nu$ der Proceedings.[47])

Somit: Ist $\mathfrak{A} = \lambda$ ein Theorem von Kalkül K^*, so $\mathfrak{A}' = \nu$ ein Theorem von Kalkül K, und, umgekehrt, ein Theorem $\mathfrak{B} = \nu$ von Kalkül K korrespondiert mit dem Theorem $\mathfrak{B}' = \lambda$ in Kalkül K^*.

Die Systeme K und K^* liefern somit nur verschiedene Formalismen für dieselbe (klassische) *Aussagenlogik.*

§ 13. In jedem Satz $\mathfrak{A} \subset \mathfrak{B}$ von K können auszer dem zwischen $\mathfrak{A}$ und $\mathfrak{B}$ stehenden Implikationszeichen noch weitere auftreten; diese deuten wir fortan an durch $\subset_1$. In jedem Satz $\mathfrak{D} \subset \mathfrak{E}$ von K^* können auch $\mathfrak{D}$ und $\mathfrak{E}$ weitere Implikationszeichen enthalten. Diese werden von nun an durch $\subset_2$ angedeutet.

Nun gibt es folgende Dualität zwischen Sätzen und Theoremen von K und K^*:

Ein Satz $\mathfrak{A} \subset \mathfrak{B}$ in K korrespondiert mit einem Satz $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$ in K^*; dabei gehe $\overline{\mathfrak{A}}$ aus $\mathfrak{A}$, $\overline{\mathfrak{B}}$ aus $\mathfrak{B}$ mittels folgender Umsetzungen hervor: α_0) man ändere $+$ in $\cdot$, und umgekehrt; β_0) λ in ν, und umgekehrt; γ_0) $\subset_1$ in $\subset_2$ mit Umsetzung von Antezedens und Sukzedens; ev. vorkommende Akzenten sollen ungeändert bleiben. Ein Theorem $\mathfrak{A} = \nu$ in K korrespondiert mit einem Theorem $\overline{\mathfrak{A}} = \lambda$ in K^*.[48])

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[46]) Jede Implikation $\mathfrak{P} \subset \mathfrak{Q}$ in $\mathfrak{A}$ ist für K^* als $\mathfrak{P} \cdot \mathfrak{Q}'$ zu deuten.

[47]) Für K ist jede Implikation $\mathfrak{P} \subset \mathfrak{Q}$ in $\mathfrak{B}$ als $\mathfrak{P}' + \mathfrak{Q}$ zu deuten.

[48]) Man vergleiche Proc. Kon. Ned. Akad. v. Wet. Amsterdam, **50**, 26, 27 (Satz 42) (1947).

Schlieszlich darf man in K auch $\subset_2$, in K^* $\subset_1$ zulassen; man hat nur $\mathfrak{A} \subset_1 \mathfrak{B}$ als andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$, $\mathfrak{A} \subset_2 \mathfrak{B}$ als andere Schreibweise von $\mathfrak{A} \cdot \mathfrak{B}'$ aufzufassen. Sätze $\mathfrak{A} \subset \mathfrak{B}$ von K sind auch Sätze von K^*, und umgekehrt; denn es sind Sätze des mit K und K^* aequivalenten Kalküls in den Proceed. 49 (1946), 50 (1947).

§ 14. Die Dualität zwischen den affirmativen und verneinenden Aussagenkalkülen, von A_1 und A_1^*, von A_2 und A_2^*, von M und M^*, von I und I^*, und schlieszlich von K und K^* liefern die Möglichkeit aus bekannten Resultaten zu dualen zu gelangen. Wir beschränken uns auf folgende Beispiele.

(Gödel).[49]) Uebersetzt man die Begriffe von K:

$$X',\ X \subset_1 Y,\ X + Y,\ X \cdot Y,$$

so weit sie in der unten folgenden Formel $\mathfrak{A}$ vorkommen, durch folgende Begriffe im Kalkül I:

$$X',\ (X \cdot Y')',\ (X' \cdot Y')',\ X \cdot Y,$$

so ist jedes klassische Theorem $\nu \subset \mathfrak{A}$ ($\mathfrak{A}$ ist eine wahre Aussage), wobei $\mathfrak{A}$ λ und ν nicht enthält, auch ein Theorem des Kalküls I;

und das duale Resultat:

Übersetzt man die Begriffe von K^*:

$$X',\ Y \subset_2 X,\ X \cdot Y,\ X + Y,$$

so weit sie in der unten folgenden Formel $\mathfrak{A}$ vorkommen, durch die folgenden Begriffe im Kalkül I^*:

$$X',\ (Y' + X)',\ (X' + Y')',\ X + Y,$$

so ist jedes klassische Theorem $\mathfrak{A} \subset \lambda$ ($\mathfrak{A}$ ist eine falsche Aussage), wobei $\mathfrak{A}$ λ und ν nicht enthält, auch ein Theorem des Kalküls I^*.

(Glivenko).[50]) Ist $\nu \subset \mathfrak{A}$, wobei $\mathfrak{A}$ λ und ν nicht enhält, ein Theorem von K, so ist $\nu \subset \mathfrak{A}''$ ein Theorem von I; ist $\nu \subset \mathfrak{A}'$ ein klassisches Theorem, mit $\mathfrak{A}$ ohne λ und ν, so ist es auch ein Theorem von I;

und das duale Resultat:

Ist $\mathfrak{A} \subset \lambda$, wobei $\mathfrak{A}$ λ und ν nicht enthält, ein Theorem von K^*, so ist $\mathfrak{A}'' \subset \lambda$ ein Theorem von I^*; ist $\mathfrak{A}' \subset \lambda$ ein klassisches Theorem ($\mathfrak{A}'$ ist falsch), mit $\mathfrak{A}$ ohne λ und ν, so ist es auch ein Theorem von I^*.

(Glivenko).[51]) In I gilt das Theorem

$$\nu \subset \{(X + X')'\}';$$

dual:

In I^* gilt

$$\{(X \cdot X')'\}' \subset \lambda.$$

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[49]) Siehe die Zitate in Fuszn. 6.

[50]) Siehe V. Glivenko, Bull. Cl. Sci. Acad. royale de Belgique (5), **15**, 183 (1929).

[51]) Siehe V. Glivenko, Bull. Cl. Sci. Acad. royale de Belgique (5), **14**, 226 (1928).

Die engeren Prädikatenkalküle.

§ 15. Die Erweiterung der vorangehenden Aussagenkalküle zu engeren Prädikatenkalkülen bietet keine Schwierigkeiten. Die Dualität zwischen den Aussagenkalkülen mit und ohne Asterisk überträgt sich dabei auf die gleichbezeichneten Prädikatenkalküle; All-Aussagen stehen dual gegenüber Existenzaussagen. [52])

P_1 entsteht aus A_1 durch Hinzufügung von:

Axiom 8^α. $(x)\,F(x) \subset F(a)$; und:

Schluszschema 8^β. $\dfrac{\mathfrak{A} \subset F(a) \text{ (in } \mathfrak{A} \text{ kommt } a \text{ nicht vor)}}{\mathfrak{A} \subset (x)\,F(x)}$. [53])

P_2 entsteht aus A_2 durch Hinzufügung von 8^α, 8^β und:

Axiom 9^α. $F(a) \subset (Ex)\,F(x)$; und:

Schluszschema 9^β. $\dfrac{F(a) \subset \mathfrak{A} \text{ (in } \mathfrak{A} \text{ komt } a \text{ nicht vor)}}{(Ex)\,F(x) \subset \mathfrak{A}}$.

MP, IP und KP entstehen aus M bzw. I bzw. K durch Hinzufügung von 8^α, 8^β, $9^\alpha_=$ und 9^β.

§ 15^{bis}. P_1^* entsteht aus A_1^* durch Hinzufügung von 9^α und 9^β, P_2^* aus A_2^* durch Hinzufügung von 9^α, 9^β, 8^α und 8^β; MP^*, IP^* und KP^* entstehen aus M^* bzw. I^* bzw. K^* durch Hinzufügung von 8^α, 8^β, 9^α und 9^β.

KP und KP^* sind nur verschiedene Formalismen für denselben (klassischen) engeren Prädikatenkalkül.

§ 16. (Gödel). [49]) Übersetzt man die Begriffe von KP:

$$X',\ X \subset_1 Y,\ X + Y,\ X \cdot Y,\ (x)\,F(x),\ (Ex)\,F(x),$$

so weit sie in der unten folgenden Formel $\mathfrak{A}$ vorkommen, durch die folgenden Begriffe in IP:

$$X',\ (X \cdot Y')',\ (X' \cdot Y')',\ X \cdot Y,\ (x)\,F(x),\ [(x)\,\{F(x)\}']',$$

so ist jedes klassische Theorem $\nu \subset \mathfrak{A}$, wobei $\mathfrak{A}$ λ und ν nicht enthält, auch ein Theorem von IP;

dual:

Übersetzt man die Begriffe von KP^*:

$$X',\ Y \subset_2 X,\ X \cdot Y,\ X + Y,\ (Ex)\,F(x),\ (x)\,F(x),$$

so weit sie in der unten folgenden Formel $\mathfrak{A}$ vorkommen, durch die folgenden Begriffe in IP^*:

$$X',\ (Y' + X)',\ (X' + Y')',\ X + Y,\ (Ex)\,F(x),\ [(Ex)\,\{F(x)\}']',$$

so ist jedes klassische Theorem $\mathfrak{A} \subset \lambda$, wobei $\mathfrak{A}$ λ und ν nicht enthält, auch ein Theorem von IP^*.

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[52]) Vergl. Proc. Kon. Ned. Akad. v. Wet. Amsterdam, 50, 29 (Satz 42*) (1947).

[53]) Wie sich für P_1 und die weiteren Prädikatenkalküle die Einsetzungsregel E bzw. E^* ändern soll, und wie die Umbenennungsregeln für gebundene Variable lauten sollen, geht unmittelbar aus der Lektüre von Hilbert—Ackermann, Grundzüge der theoretischen Logik, 2e Aufl., 54, 56 u. 57 (1938) hervor.

NEGATIONLESS INTUITIONISTIC MATHEMATICS II

BY

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(Communicated by Prof. L. E. J. BROUWER at the meeting of March 25, 1950)

In 1944 I gave a sketch of some parts of negationless intuitionistic mathematics in these Proceedings; afterwards I started on a more complete and systematic treatment [1]). This note is a sequal to it. As in the meantime, however, many remarks and objections reached me, I preface this note by a concise exposition of my point of view and some explanations to the second note.

In 1947 Prof. L. E. J. BROUWER gave a formulation of the directives of intuitionistic mathematics [2]). It is remarkable that negation does not occur in an explicit way, so one might be inclined to believe negationless mathematics to be a consequence of this formulation. The notion of species, however, is introduced in this way (translated from the Dutch text): "Finally in this construction of mathematics at any stage properties that can be supposed to hold for mathematical conceivabilities already obtained are allowed to be added as new mathematical conceivabilities under the name of species". By this formulation it is possible that there are properties that can be supposed to hold for mathematical conceivabilities already obtained but that are not known to be true. With it negation and null-species are introduced simultaneously but at the cost of evidence. Whatever are the properties that can be supposed? What other criterion could there be than 'to hold for mathematical conceivabilities already obtained'? In the definition of the notion of species the words "can be supposed" should be replaced by "are known". *One should restrict oneself in intuitionistic mathematics to mathematical conceivabilities and properties of those mathematical conceivabilities and one should not make suppositions of which one does not know whether it is possible to fulfil them.* (The well-known turn in mathematics: "Suppose ABC to be rectangular" seems to be a supposition, but mostly means: "Consider a rectangular triangle ABC").

ad § 1. 1. After the introduction of the natural numbers 1, 2, 3 the

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[1]) G. F. C. GRISS, Negatieloze intuïtionistische wiskunde. Versl. Ned. Akad. v. Wetensch., **53**, (1944).

Negationless intuitionistic mathematics. Proc. Kon. Ned. Akad. v. Wetensch., **49**, (1946).

[2]) L. E. J. BROUWER, Richtlijnen der intuïtionistische wiskunde. Proc. Kon. Ned. Akad. v. Wetensch., **50**, (1947).

natural number n' next to the natural number n was introduced by means of induction as follows:

"If, in this way, we have proceeded to E_n $(1, 2, \ldots, n)$, we can again imagine an element n', remaining the same, $n' = n'$, and distinguishable from each element p of E_n $(1, 2, \ldots, n)$, in formula $n' \neq p$, $p \neq n'$. They form the set $E_{n'}$ $(1, 2, \ldots, n')$."

$E_{n'}$ is called the sum of E_n and n', in other words: An element of $E_{n'}$ belongs to E_n or is n'. In this way the disjunction is defined in a particular case. It is evident the disjunction a or b in the usual meaning (the assertion a is true or the assertion b is true), does not occur in negationless mathematics, because there is no question of assertions that are not true. In general our definition of disjunction runs as follows: *a or b is true for all elements of the set V means that the property a holds for a subspecies V' and property b holds for a subspecies V'', V being the sum of V' and V''.*

ad § 1. 2. In accordance with the construction of natural numbers the proofs of properties of those numbers are always given by means of induction, until a system of properties is found, that can serve as a starting point of an axiomatic theory. At the time I used the disjunction in the proofs of the two properties concerning the relations of identity and distinguishability. Now we will show, how it is possible to avoid the use of disjunction in accordance with the remark made ad § 1. 1. For that purpose I formulate the first property: *If b is an element of E_m $(1, 2, \ldots, m)$, then b together with the elements of E_m that are distinguishable from b form E_m.*

Proof: The property holds for E_2. Suppose the proof has proceeded to E_n.1) Consider first an element b of E_n. The elements of $E_{n'}$ that differ from b are n' and those elements of E_n that differ from b. The latter form together with b the set E_n, and E_n together with n' forms $E_{n'}$. 2) Now consider $b = n'$. In this case the elements differing from b form the set E_n, so together with b the set $E_{n'}$. So the property holds for the elements of E_n and for n', so for all elements of $E_{n'}$.

The avoiding of the disjunction has little influence on the proof of the second property.

If for the elements a and b of E_m holds: $a \neq c$ for each $c \neq b$, then $a = b$.

Proof: The property holds for E_2. Suppose the proof has proceeded to E_n. 1) If $b = n'$, then a is distinguishable from each element of E_n, so $a = n'$ and $a = b$. 2) b is element of E_n; choose $c = n'$, then also a is an element of E_n, so $a = b$. The proof has been delivered for all elements of E_n and for n', so for all elements of $E_{n'}$.

Continuation of § 1.

4. *The set A $(1, 2, \ldots)$.*

In consequence of its construction A is the sum of E_n and the species A_n consisting of those elements of A that are distinguishable from

the elements of E_n. Suppose an element of A to be distinguishable from all elements of A_n, then it is contained in E_n; in the case it is distinguishable from all elements of E_n it is an element of A_n. We now again give the proofs of property VI and VII of nr 2.

Proof of VI: A is the sum of E_b and the elements of A that are distinguishable from those of E_b. E_b is the sum of b and the elements of E_b distinguishable from b, so A is the sum of b and the elements of A distinguishable from b.

Proof of VII: a is distinguishable from all elements of A_b thus is an element of E_b. Hence $a = b$.

For the definition and properties of the order-relation in the set A one should consult nr 3. The modifications to be made are only slight ones.

5. *Finite and denumerable infinite species.*

A property of a natural number defines the species of natural numbers having this property. The species may consist of a single element a, e.g. the species of natural numbers identical with a. From the definition it follows that the species contains at least one element, in accordance with the remark made at the beginning. In particular every set is a species.

In the general theory of sets by BROUWER "verschieden" (different) means tacitly 'not the same'. Therefore this part of intuitionistic mathematics bears a paramount negative character.

A species A is said to be in one-to-one correspondance with a species B if to each element of A corresponds a single element of B, while of each element of B it is known to what element of A it corresponds. Then also B is in one-to-one correspondance with A. To two distinguishable elements of A correspond two distinguishable elements of B, and reversely. A and B are said to be of the same cardinal. Two one-to-one correspondances applied one after another give again a one-to-one correspondance.

A species A is mapped uniquely onto a species B, if to each element of A corresponds a single element of B. Then two distinguishable elements of B correspond to distinguishable elements of A.

Property: *If E_p and E_q have the same cardinal, then $p = q$.*

Proof: The property holds for $p = 2$. Suppose the proof has proceeded to n, that means from a one-to-one correspondance of E_n onto E_r it follows $r = n$, while a certain one-to-one correspondance of $E_{n'}$ onto $E_{m'}$ is given. If m' corresponds to n' then E_m and E_n have the same cardinal, so $m = n$, thus $m' = n'$.

If n' corresponds to the element a of E_m and if m' corresponds to b of E_n, then we replace the one-to-one correspondance by another one in which m' corresponds to n' and a to b. Then again E_m and E_n have the same cardinal, thus $m = n$, so $m' = n'$.

We now define: A finite species, of cardinal n, is a species having the same cardinal as E_n. The cardinal n of a finite species E is distinguishable from, less than or greater than the cardinal m of E', according to $n \neq m$, $n < m$, $n > m$.

A denumerable infinite species is a species having the same cardinal as the set of natural numbers.

Property: *If the finite species E has the same cardinal as a proper subspecies of the finite species E', then the cardinal of E is less than that of E'.*

Proof: Suppose E has the cardinal n, then also a proper subspecies E'' of E' has the cardinal n. Now we map the species E' in such a way onto E_m, that E'' is mapped onto E_n. E' contains an element that is distinguishable from each element of E''. This element now is mapped onto an element of E_m, that is distinguishable from each element of E_n.

Likewise the reverse holds: *In the case the cardinal of a finite species E is less than that of a finite species E', E has the same cardinal as a proper subspecies of E'.*

Also: If E is a finite subspecies of the set A of natural numbers, then it is possible to determine a number m, such that E is a subspecies of E_m; so E is a proper subspecies of A.

Each finite species has the same cardinal as a proper subspecies of a denumerable infinite species.

A species containing at least three mutually distinguishable elements is called ordered, if there exists between elements a and b of the species a relation $a < b$ $(b > a)$, satisfying the following conditions:

$$O_1 \quad a \neq b \rightarrow a < b \text{ or } b < a$$
$$O_2 \quad a < b \rightarrow a \neq b$$
$$O_3 \quad a < b \text{ and } b < c \rightarrow a < c.$$

(The second condition replaces the negative condition $a = b$, $a < b$ and $b < a$ exclude one another).

One easily proves by induction:

A finite ordered species contains a first and a final element. Two finite ordered species having the same cardinal number have the same ordinal number.

6. *The fundamental operations.*

We mention only.

Definition of addition: Adding 1 is a transformation that to each number let correspond its successor; adding n' means adding n and then 1, in formula $n + 1 = n'$ and $a + (n + 1) = (a + n) + 1$.

The transformation is unique:

$$a = b \rightarrow a + c = b + c \text{ and } a + c \neq b + c \rightarrow a \neq b.$$

Properties:

$$A_1 \quad a + (b + c) = (a + b) + c$$
$$A_2 \quad a + b = b + a$$
$$A_3 \quad a \neq b \rightarrow a + c \neq b + c.$$

These are the axioms of any commutative group. Characteristic for natural numbers is:

$$N \quad a+b \neq 1$$

and for the order-relation

$$O_4 \quad a > b \rightarrow a+c > b+c.$$

All these properties can easily be proved with induction. Further properties can be deduced from them, without using induction in an explicit way.

Property: $a+b \neq a$.

Proof: $c+b \neq 1 \rightarrow 1+b \neq 1 \rightarrow a+1+b \neq a+1 \rightarrow a+b \neq a$ (N, A_1, A_2, A_3).

Property: $a+c=b+c \rightarrow a=b$.

Proof: Take $d \neq b$, then $d+c \neq b+c \rightarrow d+c \neq a+c \rightarrow d \neq a$. So $a \neq d$ for any $d \neq b$, thus $a=b$ (A_3, VI).

Property: $a+b > a$.

Proof: $b \geqq 1 \rightarrow a+b \geqq a+1 \rightarrow a+b > a$ (nr 3, O_4).

Property: $a+d > b+d \rightarrow a > b$.

Proof: $a+d > b+d \rightarrow a+d \neq b+d \rightarrow a \neq b$.

Take $c < b$, then $c+d < b+d < a+d \rightarrow c+d \neq a+d \rightarrow c \neq a$. $a \neq b$ and $c \neq a$ for each $c < b \rightarrow a > b$ (nr 3).

Definition: Multiplication is a unique transformation defined by

$$a \times 1 = a \text{ and } a(n+1) = an + a,$$

so $a=b \rightarrow ac=bc$ and $ac \neq bc \rightarrow a \neq b$.

Properties:

$$M_1 \quad a(bc) = (ab)c.$$
$$M_2 \quad ab = ba.$$
$$M_3 \quad a(b+c) = ab + ac.$$
$$M_4 \quad a \neq b \rightarrow ac \neq bc.$$
$$O_5 \quad a > b \rightarrow ac > bc.$$

These properties, again, can easily be proved by means of induction. Further

$$ac = bc \rightarrow a = b.$$
$$ac > bc \rightarrow a > b.$$

Property: $z \neq 1 \rightarrow xz \neq 1$.

Proof: $z \neq 1 \rightarrow z > 1 \rightarrow xz > x \rightarrow xz > 1 \rightarrow xz \neq 1$ (nr 3, O_5).

Property: $xy = 1 \rightarrow x = 1$ and $y = 1$.

Proof: Take $z \neq 1 \rightarrow xz \neq 1 \rightarrow xz \neq xy \rightarrow z \neq y$. So $z \neq y$ for any $z \neq 1 \rightarrow y = 1$. Then $x = 1$.

Definition of subtraction. If for three numbers a, b and v holds $v + a = b$, v is called the difference between b and a.

Definition: If $n \neq 1$ we mean by 'n diminishing by 1': determining

the predecessor $'n$ (nr 3); $'n = n - 1$. If $a < b$ and $a \neq 1$ we mean by 'b diminishing by a': diminishing by 1 and after that by $'a$; $b - a = {}'b - {}'a$.

Property: If $b > a$, $b - a$ is the difference between b and a.

Proof: $(b - 1) + 1 = {}'b + 1 = b$. Suppose the proof has proceeded to $'a$. $(b - a) + a = ({}'b - {}'a) + ({}'a + 1) = \{({}'b - {}'a) + {}'a\} + 1 = {}'b + 1 = b$.

Property: There is only one difference between b and a $(b > a)$.

Proof: If $v + a = b$ and $w + a = b$, then $v + a = w + a$, so $v = w$.

We first proved that if $b > a$ at least one value can be found being the difference $b - a$. It need not be told separately that no such value exists in the case $b \leqq a$, for if a difference $b - a$ exists, then $v + a = b$ thus $b > a$ and then $b \neq c$ for each $c \leqq a$.

That there exists at most one value for the difference $b - a$ does not mean that there are no two such values but that one always finds the same value.

Definition of division: If for three numbers a, b and q holds $q\,a = b$, q is called the quotient of b and a.

Property: If a and b have a quotient, there is only one quotient.

Property: If $a \leqq b$ $(a \neq 1)$ two numbers q and $r < a$ can be determined in such a way that $b = qa$ or $b = qa + r$.

Proof: If $b = a$, then $b = 1.\ a$. Let the proof have proceeded to $b = n > a$, then $n = qa$ or $n = qa + r$ $(r < a)$. So $n' = qa + 1$ $(1 < a)$ or $n' = qa + r'$ with $r' < a$ or $r' < a$; in the latter case $n' = qa + a = (q + 1)a$.

On this is founded EUCLIDES' algorithm and the theory of division and separation in primefactors which I only give in outline.

Definition: a and b are called mutually divisible if it is possible to determine d $\neq 1$, p and q such that $a = pd$ and $b = qd$; a and b are called mutually prime if for each $c \neq 1$, p and q holds $a \neq pc$ or $b \neq qc$.

Property: Two numbers are either mutually prime or mutually divisible.

Property: If a and b are mutually prime and c and d are mutually divisible then (a, b) is distinguishable from (c, d).

Definition: a is divisible if it is possible to determine $b \neq 1$ and $c \neq 1$ in such a way that $a = bc$; a is prime if $a \neq bc$ for each $b \neq 1$ and $c \neq 1$.

Property: A number is either prime or divisible.

Property: A divisible numbers can be factorized in a unique way into primefactors.

Property: Each prime number has an immediately succeeding prime number.

§ 2. *The rational number.*

1. *The number* 0.

We add to the set A $(1, 2, \ldots)$ of natural numbers the element 0, distinguishable from all elements of A. How has that to be done?

One may interrupt the enumeration of the elements of A and intersperse the number 0. In the case no operations and orderrelations have been introduced, this will bring forth nothing new. We define however,

$$0 < a, \quad a + 0 = 0 + a = a \quad \text{and} \quad a.0 = 0.a = 0.$$

One might also begin with 0 and add the set of natural numbers. Properly speaking one has to begin all over again and that with $(0, 1, 2, \ldots)$, define the fundamental operations and order-relations, noticing the subset $(1, 2, \ldots)$ is in one-to-one accordance with the set of natural numbers with preservation of fundamental operations and order-relations. The properties with proofs concerning order-relations and operations of natural numbers remain valid with slight modifications. It is evident the number 0 has nothing to do with a cardinalnumber. (no more than the negative number -1 or the fraction $\frac{1}{2}$ leads to a cardinal number).

2. *Entire numbers.*

Introduce in accordance with nr 1 $(0, 1, -1, 2, -2, \ldots)$.

Order relations: $0 < 1$, $-1 < 0$, $-1 < 1$, $n' >$ each element of $(0, 1, \ldots, -n)$ and $-n' <$ each element of $(0, 1, \ldots, n')$.

The further theory of entire numbers does not offer any difficulty.

Negative numbers have as little to do with negation as the number 0 with cardinal numbers.

3. *Rational numbers.*

I will treat the theory of rational numbers more fully as it has to serve as a foundation of the theory of real numbers. If one defines a fraction as a pair of entire numbers the fractions $^2/_3$ and $^3/_2$ ought to be identified. One distinguishes them, however, tacitly by their position. If one defines more precisely by calling an 'ordered' pair of entire numbers a fraction, it is not yet evident how to distinguish $^2/_2$ from 2. One must be able to distinguish in each pair the two numbers, e.g. by indices 1 and 2 or in principle more simply, by writing the numerator in arabic figures and the denominator in roman figures. We choose the manner of writing 5_1 and 4_2, where these symbols are to be considered as one sign.

Definition: We construct a series of natural numbers 1_2, 2_2 etc. and a series of entire numbers 0_1, -1_1, 1_1, etc. where each element of the second series is distinguishable from each element of the first series. Each pair of numbers originating by choosing one number from the first and another one from the second series is called a fraction. The number of the first series is called denumerator, that from the second one numerator.

Two fractions (a_1, b_2) and (c_1, d_2) are called the same if $a_1 = c_1$ and $b_2 = d_2$; they are distinguishable if at least $a_1 \neq c_1$ or $b_2 \neq d_2$. This definition is in accordance with a general definition of identity and distinguishability that will be given later on.

Much more important is the notion equal and different for fractions.

For this purpose we first arrange to calculate with numerators and denumerators by suppressing the indices. We shall denote the relations of identity and distinguishability by $\equiv$ and $\not\equiv$, respectively and the relations of equality and difference by $=$ and $\neq$ and define

$$(a_1\, b_2) = (c_1\, d_2) \text{ if } ad \equiv bc$$
$$(a_1\, b_2) \neq (e_1\, f_2) \text{ if } af \not\equiv be.$$

These relations satisfy the properties I — VII of § 1. 2.

We give the proofs for V and VI.

V. $(a_1\, b_2) = (c_1\, d_2)$ and $(c_1\, d_2) \neq (e_1\, f_2) \to ad \equiv bc$ and $ef \not\equiv ed \to adf \equiv$ $\equiv ebf$ and $cbf \not\equiv ebd \to adf \not\equiv ebd \to af \not\equiv eb \to (a_1\, b_2) \neq (e_1\, f_2)$.

VI. $(a_1\, b_2) \neq (p_1\, q_2)$ for each $(p_1\, q_2) \neq (c_1\, d_2) \to$

$aq \not\equiv bp$ for each p and q for that $pd \not\equiv pc \to$ (for $q \equiv db$)

$adb \not\equiv bp$ for each p for that $pd \not\equiv dbc \to$

$ad \not\equiv p$ for each $p \not\equiv bc \to ad \equiv bc \to (a_1\, b_2) = (c_1\, d_2)$.

We shall always call the relations $\alpha = \beta$ and $\alpha \neq \gamma$ satisfying the conditions I — V relations of equality and difference, if $\alpha \equiv \beta \to \alpha = \beta$ and $\alpha \neq \gamma \to \alpha \not\equiv \gamma$. Sometimes one means by a 'fraction' the species of all mutually equal fractions. In virtue of the developments in § 4 of my first note on negationless mathematics, for 'fractions' an identity-relation and distinguishability-relation holds satisfying conditions I — VII.

NOTE ON LILL'S METHOD OF SOLUTION OF NUMERICAL EQUATIONS

BY

B. MEULENBELD

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of Febr. 25, 1950)

§ 1.

In 1867 LILL [1] gave a graphical method to resolve the real roots of a numerical equation. The main principle in this method is the representation of a polynomial with real coefficients by a right-angled set of sides, called *orthogones*, where the lenghts of the consecutive sides are proportional to the values of the coefficients of the polynomial, and where the directions in which these sides are drawn are dependent on the variations of sign of these coefficients. The value of the polynomial $f(z)$ for each real value of z can be determined with reasonable accuracy.

In the present note I show that this method of representating of a polynomial by an orthogone can be extended. The condition that the consecutive sides are at right angles can be released. Furthermore it is possible to represent in the same figure the polynomials in the variable z: $\frac{f(z)-f(x)}{z-x}$ and $f(x+z)$ for a fixed value of x, and to determine graphically the values of $f'(x)$, $f''(x), \ldots, f^{(n)}(x)$.

§ 2.

We consider a polynomial of degree n

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \qquad (a_0 \neq 0),$$

where a_i $(i = 0, 1, .., n)$ are real numbers.

Now we construct a polygonical set of sides with lengths proportional to the a_i respectively. Each pair of consecutive sides include a fixed angle φ. A rule for the construction of the set of sides may be stated as follows. If two successive coefficients a_i and a_{i+1} have different signs (a variation) the direction of the side represented by a_{i+1} (the a_{i+1}-side) is obtained by a rotation of the a_i-side over an angle φ in positive sense (that is counterclockwise). If a_i and a_{i+1} have same signs, the direction of the a_{i+1}-side is obtained by a rotation of the a_i-side over an angle $180° - \varphi$ in negative sense. In the case of a zero coefficient a_i the direction of the following a_{i+1}-side can be obtained by giving a_i a very small value $+$ or $-$ ε, which tends to zero.

The set of sides found in this way I call a *polygonpart with angle* φ, and the polynomial is said to be represented by such a polygonpart. For $\varphi = \pi/2$ the orthogones of LILL are obtained.

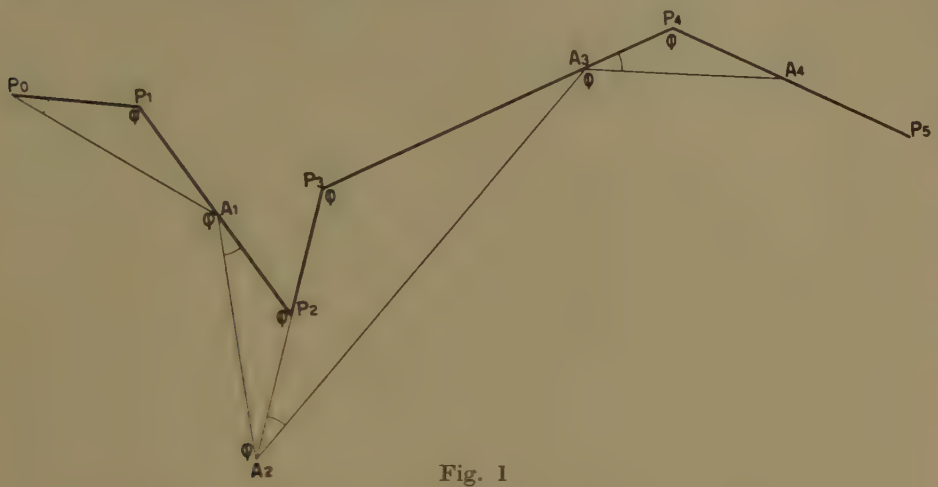

Fig. 1

Let the polygonpart with angle φ $P_0, P_1, \ldots, P_{n+1}$ represent the polynomial:

$$f(z) \equiv a_0 z^n + \ldots + a_n \qquad (a_i \text{ real}, \ a_0 \neq 0).$$

(In fig. 1 is $f(z) \equiv z^4 - 2z^3 - z^2 + 3z - 2$, so $n = 4$).

Without loss of generality we may assume $a_0 > 0$, otherwise we represent the polynomial $-f(z)$.

On the side $P_1 P_2$ we take a point A_1. We put

$$x = -\frac{\overline{P_1 A_1}}{a_0} \quad [1] \tag{1}$$

Now we construct a new polygonpart with angle φ, of which the first side is $P_0 A_1$, and where the angular points lie on the sides or their lengthenings of the first polygonpart. Quite analogously as in the case of the orthogones (see [2]) the following equalities can be deduced from the conformity of the triangles $P_0 P_1 A_1$, $A_1 P_2 A_2$, $A_2 P_3 A_3, \ldots, A_{n-1} P_n A_n$:

$$\frac{\overline{P_0 P_1}}{\overline{P_1 A_1}} = \frac{\overline{A_1 P_2}}{\overline{P_2 A_2}} = \frac{\overline{A_2 P_3}}{\overline{P_3 A_3}} = \ldots = \frac{\overline{A_{n-1} P_n}}{\overline{P_n A_n}},$$

$$\overline{A_1 P_2} = a_1 + \overline{A_1 P_1} = a_1 + a_0 x, \quad [2]$$

$$\overline{A_2 P_3} = a_2 + \overline{A_2 P_2} = a_2 + \overline{A_1 P_2}\, x = a_0 x^2 + a_1 x + a_2,$$

$$\overline{A_3 P_4} = a_3 + \overline{A_3 P_3} = a_3 + \overline{A_2 P_3}\, x = a_0 x^3 + a_1 x^2 + a_2 x + a_3,$$

. . . . . .

$$\overline{A_n P_{n+1}} = a_0 x^n + \ldots + a_{n-1} x + a_n.$$

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[1] We remark that $\overline{AB} = -\overline{BA}$.

[2] We write $\overline{A_1 P_2} = a_1 + a_0 x$ for $\overline{A_1 P_2}$ represents $a_1 + a_0 x$.

Hence the segment $\overline{A_n P_{n+1}}$ represents the value of $f(x)$ with (1). Is $\overline{A_n P_{n+1}} = 0$, then x is a root of the equation $f(z) = 0$. When it is possible to determine A_1 such that the endpoints of the two polygonparts coincide, then a root of $f(z)$ is found. In the case $\varphi = \pi/2$ LILL constructed a simple instrument to find the resolving polygonparts in order to determine the real roots of a given numerical equation.

§ 3.

If we choose on $P_1 P_2$ another point $B_1 \neq A_1$, with $y = -\frac{\overline{P_1 B_1}}{a_0}$ and we construct the corresponding polygonpart $B_1, B_2, \ldots, B_n$, we find a segment $\overline{B_n P_{n+1}}$ representing $f(y)$, so that:

$$\overline{B_n P_{n+1}} = f(y) = a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-1} y + a_0$$
$$= a_0 y^n + (\overline{A_1 P_2} - \overline{A_1 P_1}) y^{n-1} + (\overline{A_2 P_3} - \overline{A_2 P_2}) y^{n-2} + \ldots + (\overline{A_n P_{n+1}} - \overline{A_n P_n})$$
$$= a_0 y^n + (\overline{A_1 P_2} - a_0 x) y^{n-1} + (\overline{A_2 P_3} - \overline{A_1 P_2} x) y^{n-2} + \ldots + \overline{A_n P_{n+1}} - \overline{A_{n-1} P_n} x$$
$$= (y - x) \{a_0 y^{n-1} + \overline{A_1 P_2} y^{n-2} + \overline{A_2 P_3} y^{n-3} + \ldots + \overline{A_{n-1} P_n}\} + \overline{A_n P_{n+1}}.$$

Since $\overline{A_n P_{n+1}} = f(x)$, we have

$$\overline{B_n A_n} = f(y) - f(x) = (y - x) \{a_0 y^{n-1} + \overline{A_1 P_2} y^{n-2} + \overline{A_2 P_3} y^{n-3} + \ldots + \overline{A_{n-1} P_n}\}.$$

From the conformity of the triangles mentioned above if follows:

$$\frac{a_0}{\overline{P_0 A_1}} = \frac{\overline{A_1 P_2}}{\overline{A_1 A_2}} = \frac{\overline{A_2 P_3}}{\overline{A_2 A_3}} = \ldots = \frac{\overline{A_{n-1} P_n}}{\overline{A_{n-1} A_n}}.$$

Hence

$$f(y) - f(x) = (y - x) \left(\overline{P_0 A_1} y^{n-1} + \overline{A_1 A_2} y^{n-2} + \ldots + \overline{A_{n-1} A_n}\right) \frac{a_0}{\overline{P_0 A_1}},$$

and so:

$$\overline{P_0 A_1} y^{n-1} + \overline{A_1 A_2} y^{n-2} + \ldots + \overline{A_{n-1} A_n} = \frac{\overline{P_0 A_1}}{a_0} \frac{f(y) - f(x)}{y - x}.$$

This implies that *the polygonpart* $P_0 A_1 \ldots A_n$ *represents the polynomial* $\frac{f(y) - f(x)}{y - x}$, *multiplied by a factor* $\frac{\overline{P_0 A_1}}{a_0}$.

If a real root x of $f(z) = 0$ is found by a resolving polygonpart, then this polygonpart, as representing the polynomial of lower degree $\frac{f(z)}{z - x}$ can be used to find a next root of $f(z) = 0$.

The representation of the polynomials $f(z)$ and $\frac{f(z) - f(x)}{z - x}$ is very useful to demonstrate several properties of the real roots and of the relations between the roots and the coefficients, also to deduce the discriminants of quadratic, cubic and quartic equations, a.s.o.

§ 4.

Let $P_0, P_1, \ldots, P_{n+1}$ (fig. 2 with $n = 4$) be a polygonpart with angle φ, representing the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n .$$

If $x = -\frac{\overline{P_1 A_1}}{a_0}$, then $f(x) = \overline{A_n P_{n+1}}$, and $P_0, A_1, \ldots, A_n$ represents the polynomial

$$\Theta(z) = \frac{\overline{P_0 A_1}}{a_0} \frac{f(z) - f(x)}{z - x} .$$

Fig. 2

When we construct in this polygonpart a new polygonpart $P_0, B_1, \ldots, B_{n-1}$ with angle φ corresponding with the point B_1 on the side $A_1 A_2$, so that

$$x_1 = -\frac{\overline{A_1 B_1}}{\overline{P_0 A_1}}, \text{ then } \overline{B_{n-1} A_n} = \Theta(x_1) = \frac{\overline{P_0 A_1}}{a_0} \frac{f(x_1) - f(x)}{x_1 - x} ,$$

and $P_0, B_1, \ldots, B_{n-1}$ represents the polynomial

$$\psi(z) = \frac{\overline{P_0 B_1}}{\overline{P_0 A_1}} \frac{\Theta(z) - \Theta(x_1)}{z - x_1} = \frac{\overline{P_0 B_1}}{a_0} \frac{\frac{f(z) - f(x)}{z - x} - \frac{f(x_1) - f(x)}{x_1 - x}}{z - x_1}.$$

If we choose B_1 such that $\angle P_1 P_0 A_1 = \angle A_1 P_0 B_1 = \alpha$, then it follows from the conformity of the triangles $P_1 P_0 A_1$ and $A_1 P_0 B_1$:

$$x = -\frac{\overline{P_1 A_1}}{a_0} = -\frac{\overline{A_1 B_1}}{\overline{P_0 A_1}} = x_1 \quad , \quad \overline{P_0 B_1} = \frac{\overline{P_0 A_1}^2}{a_0} \ ,$$

$$\overline{B_{n-1} A_n} = \lim_{z \to x} \Theta(z) = \frac{\overline{P_0 A_1}}{a_0} f'(x).$$

In this case $P_0, B_1, \ldots, B_{n-1}$ represents the polynomial:

$$\psi(z) = \frac{\overline{P_0 B_1}}{a_0} \frac{\frac{f(z) - f(x)}{z - x} - f'(x)}{z - x} = \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{\frac{f(z) - f(x)}{z - x} - f'(x)}{z - x}.$$

Proceeding with a point C_1 on the side $B_1 B_2$ with $\angle B_1 P_0 C_1 = \alpha$, it is easily seen that

$$C_{n-2} B_{n-1} = \lim_{z \to x} \psi(z) = \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{f''(x)}{2!}.$$

Finally we find sides with lengths

$$\left(\frac{\overline{P_0 A_1}}{a_0}\right)^k \frac{f^{(k)}(x)}{k!} \qquad (k = 0, 1, \ldots, n).$$

Hence for a fixed value of x it is possible to find graphically the values of $f^{(k)}(x)$ $(k = 0, 1, \ldots, n)$. Furthermore it is evident that the angles between each pair of these consecutive sides are $\varphi + \alpha$, so that these sides form a new polygonpart with angle $\varphi + \alpha$ (in fig. 2 $P_5, A_4, B_3, C_2, D_1, P_0$), representing the polynomial:

$$F(z) = f(x) + \left(\frac{\overline{P_0 A_1}}{a_0}\right) \frac{f'(x)}{1!} z + \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{f''(x)}{2!} z^2 + \ldots + \left(\frac{\overline{P_0 A_1}}{a_0}\right)^n \frac{f^{(n)}(x)}{n!} z^n$$

$$= f\left(x + \frac{\overline{P_0 A_1}}{a_0} z\right).$$

From

$$\frac{\overline{P_0 A_1}}{a_0} = \frac{\sqrt{a_0^2 + \overline{P_1 A_1}^2 - 2 a_0 \overline{P_1 A_1} \cos\varphi}}{a_0} = \sqrt{1 + x^2 + 2x \cos\varphi}$$

we have

$$F(z) = f(x + z\sqrt{1 + x^2 + 2x \cos\varphi}),$$

represented by the polygonpart $P_{n+1}, A_n, B_{n-1}, \ldots, P_0$.

For the orthogones of LILL is

$$F(z) = f(x + z\sqrt{1+x^2})'.$$

For small values of x it is possible to choose the angle φ in such way, that the factor $\frac{\overline{P_0 A_1}}{a_0}$ equals the unit. If $|x| < 2$ and $\varphi = bg \cos(-\frac{1}{2}x)$, then $\frac{\overline{P_0 A_1}}{a_0} = 1$, and the values of $\frac{f^{(k)}(x)}{k!}$ $(k = 0, 1, .., n)$ may be determined directly. In this case $F(z) = f(x + z)$, and the polygonpart, representing this polynomial, has the angle $180° - \varphi$.

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REMARQUES SUR LES MÉTRIQUES NON-ARCHIMÉDIENNES. I

PAR

A. F. MONNA

(Communicated by Prof. W. VAN DER WOUDE at the meeting of March 25, 1950)

I. *Espaces métriques.*

§ 1. Définition 1. *Une métrique* $\varrho(x, y)$, *définie sur un espace métrique* E, *est appelée non-archimédienne* [1]) *si pour chaque triple* x, y, z *de points de* E

$$\varrho(x, y) \leqq \max(\varrho(x, z), \varrho(z, y)). \tag{1}$$

Dans chaque espace métrique E ils existent des triples spéciaux tels que (1) est vrai: il suffit de choisir les points x, y, z tels que $\varrho(x, y)$ soit la plus petite des distances $\varrho(x, y)$, $\varrho(x, z)$, $\varrho(z, y)$ (si toutes ces distances sont égales (1) est vrai). Il faut par suite entendre cette définition telle, que la relation (1) doit être vérifiée pour chaque triple ordonné x, y, z de sorte que (1) est vrai pour toutes les permutations de x, y, z.

Pour chaque métrique non-archimédienne on a

$$\varrho(x, y) = \max(\varrho(x, z), \varrho(z, y)) \tag{2}$$

si $\varrho(x, z) \neq \varrho(z, y)$. Supposons en effet qu'on a

$$\varrho(x, z) > \varrho(z, y) \text{ et } \varrho(x, y) < \max(\varrho(x, z), \varrho(z, y)).$$

Il s'ensuit la contradiction

$$\varrho(x, z) \leqq \max(\varrho(x, y), \varrho(y, z)) < \varrho(x, z).$$

Cette propriété exprime que dans un espace à métrique non-archimédienne chaque triangle a deux côtés égaux tandis que la troisième côté n'est pas plus grande.

Théorème 1. *Chaque espace* E *avec la topologie induite par une métrique non-archimédienne est de dimension zéro.*

Démonstration. Soit U le voisinage fermé $\varrho(x, a) \leqq \varepsilon$ de $a \in E$. Si $x_1 \in U$ et $\varrho(y, x_1) \leqq \varepsilon$ on a

$$\varrho(y, a) \leqq \max(\varrho(y, x_1), \varrho(x_1, a)) \leqq \varepsilon$$

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[1]) Dénomination en analogie avec la théorie des valuations non-archimédiennes d'un corps. Pour la relation de l'inégalité (1) à l'axiome d'Archimède voir B. L. v. d. WAERDEN, Moderne Algebra I (Berlin 1937) p. 219, 248.

d'où il suit que U est ouvert. Chaque point de E a donc des voisinages ouvert et fermé arbitrairement petit, de sorte que E est 0-dimensionnel.

Conséquence. En étudiant les espaces topologiques dans lesquels on peut définir une métrique non-archimédienne, on n'a qu'à considérer les espaces de dimension zéro.

Théorème 2. *Si la métrique ϱ est non-archimédienne et si $U(a, \delta)$ et $U(b, \varepsilon)$ désignent les voisinages $\varrho(x, a) \leqq \delta$ et $\varrho(x, b) \leqq \varepsilon$ respectivement de a et de b, alors ou bien l'intersection de $U(a, \delta)$ et $U(b, \varepsilon)$ est vide ou bien un de ces voisinages contient l'autre.*

Démonstration. On peut supposer $\varepsilon \leqq \delta$. Soit z un point commun de $U(a, \delta)$ et $U(b, \varepsilon)$. Alors on a pour tout $p \in U(b, \varepsilon)$

$$\varrho(p, a) \leqq \max\,(\varrho(p, b),\ \varrho(b, z),\ \varrho(z, a)) \leqq \delta$$

donc $U(b, \varepsilon) \subset U(a, \delta)$.

Le théorème 2 admet l'inversion suivante:

Théorème 3. *Soit E un espace métrique. Désignons par U_a les voisinages de $a \in E$, déterminés par une inégalité de la forme $\varrho(x, a) \leqq \varepsilon$. Supposons que pour tout a et b de E on a ou bien* 1) $U_a . U_b = 0$ *ou bien* 2) $U_a \subset U_b$ *ou bien* 3) $U_a \supset U_b$. *Alors la métrique ϱ est non-archimédienne.*

Démonstration. Il faut montrer qu'on a pour tout a, b, c dans E

$$\varrho(a, c) \leqq \max\,(\varrho(a, b),\ \varrho(c, b)).$$

En supposant que cette inégalité n'est pas vraie, ils existent dans E trois points a, b, c tels que

$$\varrho(a, c) > \max\,(\varrho(a, b),\ \varrho(c, b)).$$

Posons $\varrho(a, b) = d$ et supposons (ce qui ne nuit pas la généralité) qu'on a $\varrho(c, b) = d' \leqq d$. Il y a deux cas:

1. $d' < d$. Considérons U_a: $\varrho(x, a) \leqq d$; U_b: $\varrho(y, b) \leqq d'$.

Le point b appartient à U_a et U_b. Puisque

$$\varrho(c, a) > \max\,(\varrho(a, b), \varrho(c, b)) = d,$$

c n'appartient pas à U_a. En vertu de la supposition du théorème il s'ensuit, c appartenant à U_b mais non à U_a, qu'on a $U_b \supset U_a$. Cependant cela est impossible puisque, d étant $> d'$, le point a appartient à U_a mais non à U_b.

2. $d' = d$. Les voisinages U_a: $\varrho(x, a) \leqq d$ et U_c: $\varrho(y, c) \leqq d$ ont le point b en commun. Puisque $\varrho(c, a) > d$, le point c n'appartient pas à U_a et le point a n'appartient pas à U_c, de sorte que ni $U_a \supset U_c$, ni $U_a \subset U_c$, en contradiction avec la supposition du théorème.

La démonstration du théorème 1 subsiste si on remplace la condition que la métrique est non-archimédienne par la condition que chaque point a un voisinage dans lequel la métrique est non-archimédienne. On arrive alors à la définition suivante.

Définition 2. *La métrique $\varrho(x, y)$ s'appelle localement-non-archimédienne si chaque point a un voisinage tel qu'on a* (1) *pour chaque triple x, y, z dans ce voisinage.*

Théorème 4. *Soit ϱ une métrique localement-non-archimédienne dans un espace séparable E. Il existe une métrique équivalente non-archimédienne ϱ'.*

Démonstration. En vertu de la définition 2 à chaque point correspond un voisinage U dans lequel la métrique est non-archimédienne. On peut choisir ces voisinages (qui ne sont pas nécessairement différents pour des points différents) tels qu'ils sont ouverts et fermès et deux à deux disjoints. On le voit comme il suit. L'espace est 0-dimensionnel et, en partant d'un point x, ce point a donc des voisinages ouverts et fermés arbitrairement petits. Choissisons donc comme voisinage U, correspondant à ce point x, un voisinage ouvert et fermé.

Faisons correspondre ce voisinage aussi à chaque point de ce voisinage. L'ensemble $E - U$ est alors ouvert et fermé. En prenant un point de $E - U$, on peut faire correspondre à ce point un voisinage U, ouvert et fermé et tout entier dans $E - U$. Il suffit alors de continuer ce procédé infiniment en faisant parcourir les points un ensemble partout dense dans E afin d'obtenir un système de voisinages U ouverts et fermés et deux à deux disjoints et tel que la métrique est non-archimédienne dans chaque voisinage. On voit de plus qu'il est permis de supposer que la diamètre de chaque voisinage est au plus égal à un nombre donné $\varepsilon > 0$. Soient alors donnés deux points x et y de E et construisons la métrique ϱ' suivante:

$\varrho'(x, y) = \varrho(x, y)$ si x et y appartiennent à un même voisinage U
$\varrho'(x, y) = \varepsilon$ si x et y n'appartiennent pas à un même voisinage U.

Etant donnés les points x, y, z trois cas peuvent de présenter: *a*) x, y, z appartiennent à un même voisinage U; *b*) deux de ces points appartiennent à un même U et le troisième point ne se trouve pas dans un même voisinage avec chacun de ces points; *c*) aucun couple de points appartient a un même U. On voit que dans chaque cas l'inégalité (1) est satisfaite, de sorte que la métrique ϱ' est non-archimédienne. On voit de plus que cette métrique est équivalente à ϱ puisque ϱ et ϱ' coincident localement, ce qui achève la démonstration.

On peut définer des métriques qui sont non-archimédiennes dans un certain sens, qu'on pourrait appeler un sens large (à expliquer dans l'exemple), mais qui ne sont pas localement-non-archimédiennes en aucun point.

Exemple. Considérons dans l'espace euclidien R_2 la métrique habituelle

$$\varrho(x, y) = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{\frac{1}{2}}.$$

Construisons une nouvelle métrique comme il suit:

$$\varrho'(x, y) = \varrho(x, y) \text{ si } \varrho(x, y) \leqq 1$$
$$\varrho'(x, y) = 1 \quad \text{si } \varrho(x, y) > 1.$$

Ces métriques sont équivalentes puisque ils sont localement les mêmes. On vérifie sans peine que ϱ' est une véritable métrique: on montre que l'inégalité triangulaire est satisfaite en considérant tous les cas qui peuvent se présenter. Chaque point possède un voisinage, à savoir le voisinage dont le rayon dans la métrique ϱ vaut $\frac{1}{2}$, dans lequel ϱ' et ϱ sont égaux. ϱ' n'est donc non-archimédienne en aucun point. Cependant chaque triangle dont deux côtés sont $\geqq 1$ dans la métrique ϱ, fournit un triple non-archimédien dans la métrique ϱ', c'est à dire satisfait a (1) dans cette dernière métrique. Tous les triangles „suffisamment larges" sont donc non-archimédiens pour ϱ'.

Définition 3. *Une métrique $\varrho(x, y)$, qui ne satisfait pas à* (1) *pour tout x, y, z, sera appelée archimédienne.*

Théorème 5. *Dans chaque espace, métrisé d'une façon non-archimédienne, il existe une métrique équivalente archimédienne.* [2])

Démonstration. Supposons que l'espace E consiste en au moins trois points et soit ϱ la métrique non-archimédienne donnée. E est de dimension zéro et est donc la somme de trois ensembles non-vides fermés disjoints X, Y, Z:

$$E = X + Y + Z.$$

Introduisons dans E une nouvelle métrique ϱ' comme il suit:

$$\varrho'(x_1, x_2) = \frac{\varrho(x_1, x_2)}{1 + \varrho(x_1, x_2)} \text{ pour } x_1 \in X, x_2 \in X.$$

Analogue $\varrho'(y_1, y_2)$ dans Y et $\varrho'(z_1, z_2)$ dans Z.

$$\varrho'(x, y) = 1 \text{ pour } x \in X,\ y \in Y$$
$$\varrho'(y, z) = 1 \text{ pour } y \in Y,\ z \in Z$$
$$\varrho'(x, z) = 2 \text{ pour } x \in X,\ z \in Z.$$

On voit que la métrique ϱ' est équivalente à ϱ. Puisque pour $x \in X$, $y \in Y$, $z \in Z$ on a

$$\varrho'(x, z) > \max(\varrho'(x, y),\ \varrho'(y, z)),$$

la métrique ϱ' est archimédienne. Cependant cette métrique est localement-non-archimédienne. En effet, pour x_1, x_2, x_3 dans X (analogue pour Y et Z) on a

$$\max(\varrho'(x_1, x_3), \varrho'(x_2, x_3)) \geqq \max\left[\frac{\varrho(x_1, x_3)}{1 + \max(\varrho(x_1, x_3), \varrho(x_2, x_3)}, \frac{\varrho(x_2, x_3)}{1 + \max(\ldots, \ldots)}\right] =$$

$$= \frac{\max(\varrho(x_1, x_3), \varrho(x_2, x_3))}{1 + \max(\ldots, \ldots)} = \frac{1}{1 + \frac{1}{\max(\ldots, \ldots)}} \geqq \frac{1}{1 + \frac{1}{\varrho(x_1, x_2)}} = \varrho'(x_1, x_2).$$

[2]) Ce théorème et la construction de ϱ' m'ont été communiqués par M. J. DE GROOT. J'ai alors remarqué que la métrique ϱ' est encore localement-non-archimédienne de sorte que le théorème ne donne qu'une solution incomplète des problèmes.

Puisque X, Y, Z sont ouverts, on voit donc que ϱ' satisfait à la définition 2.

La métrique ϱ' étant donc restée non-archimédienne dans X, Y, Z on peut poser le problème suivant.

Problème 1*a*. *Soit donné un espace E avec une métrique non-archimédienne. Existe-il une métrique topologiquement équivalente qui n'est pas localement non-archimédienne?*

On voit du théorème suivant que la réponse est affirmative pour les espaces séparables.

Théorème 6. *Chaque espace séparable 0-dimensionnel peut etre métrisé a) de façon non-archimédienne b) tel qu'aucun point n'a un voisinage dans lequel la métrique est non-archimédienne.* [3])

Démonstration. Démontrons d'abord le théorème pour l'ensemble parfait non-dense de Cantor, qui est un espace séparable 0-dimensionnel. Les points de cet espace sont donnés par

$$\sum_{i=1}^{\infty} \frac{\delta_i}{3^i} \qquad (\delta_i = 0 \text{ ou } 2). \tag{3}$$

Posons $1/n$ la distance de deux points si n est la plus petite valeur de l'indice i, à partir duquel diffèrent l'un de l'autre les suites $\{\delta_i\}$, correspondants à ces deux points. Ayant introduite cette métrique, les points

$$\sum_{i=1}^{m} \frac{\delta_i}{3^i} + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^{m+j}} \qquad (\varepsilon_j = 0 \text{ ou } 2) \tag{4}$$

constituent, pour chaque valeur fixée de m, le voisinage du point (3) qui se compose des points à distance $\leqq \frac{1}{m+1}$ à ce point. En faisant parcourir m la suite des nombres naturels, on obtient ainsi un système complet de voisinages du point (3). Cette métrique est non-archimédienne.

En introduisant dans l'ensemble de Cantor la métrique ordinaire des nombres réels (donc $\varrho(x, y) = |x - y|$), on obtient une métrique telle qu'aucun point n'a un voisinage dans lequel cette métrique est non-archimédienne (dans cette métrique l'ensemble de Cantor ne contient pas de triangles non-archimédiens). L'ensemble des points (4) constitue aussi pour cette métrique un voisinage de (3). Le système de ces voisinages est complet. Les métriques ainsi définies sont équivalentes.

L'énoncé général résulte alors de la propriété connue que chaque espace séparable 0-dimensionnel est topologiquement équivalente à un sous-ensemble de l'ensemble parfait non-dense de Cantor (propriété de Sierpinski). Les métriques définies ci-dessus sont transformées dans les métriques désirées.

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[3]) Je dois l'énoncé de ce théorème à M. J. de Groot. Comparer: J. de Groot, A note on 0-dimensional spaces. Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam **50**, 131—135 (1947).

Remarque. L'énoncé du théorème 6 va plus loin qu'une simple réponse au problème 1*a*. Pour une métrique qui n'est pas localement-non-archimédienne il se peut qu'il y a des points qui ont un voisinage dans lequel la métrique est non-archimédienne: on peut seulement dire qu'il y a au moins un point pour lequel il n'existe pas un tel voisinage. Cette remarque donne lieu au problème suivant.

Problème 1*b*. *Soit donné un espace E avec une métrique non-archimédienne. Existe-il une métrique topologiquement équivalente telle qu'aucun point n'a un voisinage dans lequel cette métrique est non-archimédienne?*

La réponse est affirmative pour les espaces séparables, ainsi qu'il résulte du théorème précédent.

C'est une conséquence du théorème 6 que, pour les espaces séparables 0-dimensionnels, il n'y a pas de difference au point de vue topologique entre les métriques archimédiennes et les métriques non-archimédiennes, de sorte que pour l'étude de ces dernières métriques seulement les espaces non-séparables ont d'intéresse.

Dans les problèmes 1*a* et 1*b* on part d'un espace à métrique non-archimédienne. On peut alors poser le problème suivant.

Problème 2. *Chaque espace métrique 0-dimensionnel, peut-il être métrisé topologiquement équivalent non-archimédien?*

Il résulte du théorème 6 que la réponse est affirmative pour les espaces séparables. Le théorème suivant fournit un autre cas simple de la possibilité de métrisation non-archimédienne.

Théorème 7. *Soit E un espace 0-dimensionnel somme d'ensembles fermés disjoints séparables, ayant donc la propriété que chaque point possède un voisinage dans lequel l'espace est séparable(E est localement-séparable). Alors il existe dans E une métrique ϱ' équivalente non-archimédienne.*

Démonstration. D'abord on construit, comme dans la démonstration du théorème 4, un système d'ensembles ouverts et fermés U, deux à deux disjoints et tous séparables. Il résulte du théorème 6 qu'on peut introduire dans chaque U une métrique équivalente non-archimédienne d. On voit de la démonstration de ce théorème qu'on peut arranger, que la diamètre de chaque U est $\leqq \varepsilon$ pour cette métrique d, $\varepsilon > 0$ étant donné. Posons alors

$\varrho'(x, y) = d(x, y)$ si x et y appartiennent à un même U,
$\varrho'(x, y) = \varepsilon$ si x et y n'appartiennent pas à un même U.

Comme au théorème 4 on voit que ϱ' est une métrique non-archimédienne équivalente avec la topologie donnée dans E.

Remarque. Il résulte de ce théorème que ces espaces 0-dimensionnels localement séparables sont métrisables intégralement, c'est à dire sont homéomorphes à un espace métrique. On sait qu'un espace peut être

localement métrisable sans l'être intégralement; voir p. ex. l'espace T_Ω de Urysohn. [4])

§ 2. Dans les espaces compacts métriques la considération des ε-chaînes donne lieu à quelques remarques concernants les métriques non-archimédiennes. [5])

Dans un espace métrique on entend par une ε-chaîne une suite finie de points $x_1, \ldots, x_n$ telle que la distance de chaque couple de points consécutifs est $\leqq \varepsilon$, $\varepsilon > 0$ étant donné. La chaîne joint alors x_1 à x_n. Soit ϱ la métrique donnée dans l'espace E et soient x et y deux points donnés. Désignons par $\triangle_\varrho(x, y)$ la borne inférieure des nombres ε tels qu'il existe une ε-chaîne entre x et y. On a

$$\triangle_\varrho(x, y) \leqq \max(\triangle_\varrho(x, z), \triangle_\varrho(z, y)).$$

Généralement $\triangle_\varrho$ n'est pas une métrique dans E puisque la condition "$\triangle_\varrho(x, y) = 0$ si et seulement si $x = y$" peut être en défaut. Soit x_0 un point fixe dans E et désignons par A l'ensemble des points x tels que $\triangle_\varrho(x, x_0) = 0$. On voit que A est un ensemble fermé. Si $y_0 \in A$ et si B désigne l'ensemble des points y tels que $\triangle_\varrho(y, y_0) = 0$, on a

$$\triangle_\varrho(y, x_0) \leqq \max(\triangle_\varrho(y, y_0), \triangle_\varrho(y_0, x_0)),$$
$$\triangle_\varrho(x, y_0) \leqq \max(\triangle_\varrho(x, x_0), \triangle_\varrho(x_0, y_0)),$$

d'où il suit $A \equiv B$. On arrive donc de cette façon à une partition de E en ensembles fermés disjoints A, chacun avec la propriété que $\triangle_\varrho(x, y) = 0$ pour chaque couple x, y, appartenant à un même A, tandis que $\triangle_\varrho \neq 0$ si x et y n'appartiennent pas à un même A. En considérant les ensembles A comme les éléments d'un nouveau ensemble $P(E)$, $\triangle_\varrho$ est une métrique dans $P(E)$. En effet, soient A et B des éléments de $P(E)$, x_1 et x_2 des points de E appartenants à A, y_1 et y_2 des points de E appartenants à B, on a $\triangle_\varrho(x_1, y_1) = \triangle_\varrho(x_2, y_2)$. $P(E)$ est donc un espace métrique et la métrique est non-archimédienne. Il s'ensuit que l'espace $P(E)$ est 0-dimensionnel.

Si l'on suppose de plus que E est séparable, il résulte d'une propriété que j'ai montrée autrefois: [6])

Soit O un point fixe arbitraire de $P(E)$. Alors il n'existe qu'une infinité dénombrable de points X de $P(E)$ tels que les distances (dans la métrique $\triangle$) de ces points à O sont tous différentes l'une de l'autre.

Supposons ensuite que E soit compact. On sait que les ensembles A définis ci-dessus sont alors les composantes-connexes de E (il résulte de

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[4]) Voir: P. Alexandroff et P. Urysohn, Mémoire sur les espaces topologiques compacts. Kon. Akad. v. Wetensch. Amsterdam 32, 12, 82 (1929).

[5]) Voir: F. Hausdorff, Mengenlehre, p. 158.

[6]) Sur l'approximation de fonctions abstraites. Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam 49, 404—408 (1946).

la compacité que les composantes coincident avec les quasi-composantes de E). $P(E)$ est alors l'espace des composantes-connexes $Q(E)$. Cet espace résulte de E par l'identification de tous les points d'une même composante-connexe en un seul point de $Q(E)$. Nous allons montrer que la topologie induite dans $Q(E)$ par la métrique $\triangle_\varrho(x, y)$ est équivalente avec la topologie T_i telle qu'on la définit ordinairement quand on applique l'opération d'identification.

a) Soit $q_0 \in Q(E)$ et soit Q_0 la composante-connexe correspondant à q_0. Soit $U_\triangle(q_0, \varepsilon)$ le voisinage $\triangle_\varrho(q', q_0) \leqq \varepsilon$ de q_0. Soit $U_{\varrho;\varepsilon}$ le ε-voisinage de Q_0, considéré pour la métrique ϱ. Si Q est une composante-connexe contenant des points qui appartiennent aussi à $U_{\varrho;\varepsilon}$, on a, puisque tout général $\triangle_\varrho \leqq \varrho$. $\triangle_\varrho(x, y) \leqq \varepsilon$ pour tout $x \in Q$, $y \in Q_0$. Si Q parcourt tous les composantes-connexes définies ainsi, les points correspondants q de $Q(E)$ constituent par définition même un voisinage $U_i(q_0)$ de q_0 dans la topologie T_i et on a $\triangle_\varrho(q, q_0) \leqq \varepsilon$. Il en résulte $U_i(q_0) \subset U_\triangle(q_0, \varepsilon)$. Remarquons que, jusqu'ici, on n' pas utilisée la compacité de E.

b) Inversement, soit $U_i(q_0)$ un voisinage donné de q_0, formé par les points q qui correspondent à les composantes-connexes Q contenant des points, appartenants au voisinage $U_{\varrho;\varepsilon}$, défini dans la première partie *a*. Il faut alors montrer que $U_i(q_0)$ contient un voisinage de la métrique-$\triangle$ de q_0. En supposant ceci en défaut, il existe une suite $\delta_1 > \delta_2 > \ldots \to 0$ telle que chaque voisinage $\triangle_\varrho(q, q_0) \leqq \delta_j$ contient des points q qui n'appartiennent pas à $U_i(q_0)$. Cela veut dire que pour chaque valeur de j il existe une composante-connexe Q_j qui n'a aucun point en commun avec $U_{\varrho;\varepsilon}$. Choisissons dans chaque Q_j un point x_j; il existe entre x_j et Q_0 une δ_j-chaîne. E étant compact, la suite $\{x_j\}$ admet au moins un point d'accumulation x. En supposant que $U_{\varrho;\varepsilon}$ soit ouvert, ce qui est permis, ce point x n'appartient pas à $U_{\varrho;\varepsilon}$. De plus, ils existent entre x et Q_0 des chaînes arbitrairement petites d'où résulte que x appartient à Q_0; d'où une contradiction puisque x n'appartient pas à $U_{\varrho;\varepsilon}$.

L'équivalence des deux topologies résulte de *a* et *b*.

La propriété suivante est une conséquence de ce résultat:

L'espace des composantes-connexes d'un espace métrique compact peut etre métrisé d'une façon non-archimédienne.

Cette propriété est d'ailleurs une conséquence immédiate du théorème 6 et d'un théorème de L. E. J. Brouwer exprimant que l'espace des composantes-connexes d'un espace compact séparable est un espace compact séparable 0-dimensionnel.

Supposons ensuite de plus que l'espace E est totalement-discontinu (il s'agit donc maintenant des espaces métriques, compacts, totalement-discontinus. [7]) Les composantes-connexes se réduisent donc aux points mêmes de l'espace. Il est clair que E (métrique-ϱ) coincide avec $Q(E)$

[7]) On sait que ces espaces sont 0-dimensionnels.

(topologie T_i). En vertu du résultat précédent on obtient donc le théorème suivant:

Théorème 8. *Dans chaque espace métrique compact totalement-discontinu la métrique donnée ϱ est équivalente avec la métrique $\triangle_\varrho$.*

Si la métrique donnée elle-même est non-archimédienne on a même

$$\varrho(x, y) = \triangle_\varrho(x, y).$$

En effet, si on avait $\triangle_\varrho(x, y) < \varrho(x, y)$, il existait une suite $x_0 = x$, $x_1, x_2, \ldots, x_n = y$ telle que

$$\varrho(x_i, x_{i+1}) < \varrho(x, y) \quad (i = 0, \ldots, n-1).$$

Puisque

$$\varrho(x, y) \leqq \max_i \varrho(x_i, x_{i+1}),$$

on avait

$$\varrho(x, y) < \varrho(x, y).$$

Puisque $\triangle_\varrho \leqq \varrho$, on a donc $\triangle_\varrho = \varrho$.

Cette démonstration fait voir que cette relation est évidemment vraie dans chaque espace métrique avec une métrique non-archimédienne (donc sans hypothèse de compacité). Cependant, $\triangle_\varrho$ n'est alors pas une métrique dans E mais dans $P(E)$. De plus, on voit qu'une itération de l'opération $\triangle$ — c'est à dire l'application nouvelle de l'opération par des chaînes à la fonction $\triangle(x, y)$ déjà construite — ne donne pas lieu à une nouvelle fonction: $\triangle_\varrho^{(2)}(x, y) = \triangle_\varrho(x, y)$. C'est puisque $\triangle_\varrho(x, y)$ est déjà non-archimédienne.

Le théorème 8 fournit un moyen de construire dans les espaces compacts totalement-discontinus une métrique non-archimédienne à partir d'une métrique archimédienne connue. On peut se demander si l'on peut obtenir toutes les métriques non-archimédiennes dans cette voie. On a donc le problème suivant:

Problème 3. *Soit donnée dans un espace compact totalement-discontinu une métrique non-archimédienne ϱ. Existe-il une métrique ϱ', qui n'est pas localement-non-archimédienne ou même qui est telle qu'aucun point ne contient un voisinage dans lequel ϱ' est non-archimédienne, équivalente à ϱ et telle qu'on ait*

$$\triangle_{\varrho'} = \varrho?$$

Revenons enfin à la démonstration du théorème 6. On voit que pour l'ensemble parfait non-dense de Cantor, qui est compact et totalement-discontinu (puisque 0-dimensionnel) on peut obtenir une métrique non-archimédienne en appliquant l'opération $\triangle$ à la métrique ordinaire et ces métriques sont équivalentes.

§ 3. Dans ce qui précède on a vu qu'une métrique non-archimédienne peut très bien être équivalente à une métrique archimédienne. Ceci

est tout different si l'on considère comme éléments d'un espace E les éléments d'un corps K et si l'on se borne à ne considérer que les topologies qui sont introduites dans E par des valuations de K. Si $|x|$ $(x \in K)$ désigne une valuation de K, la relation

$$\varrho(x, y) = |x - y|$$

définit une métrique dans E, qui est non-archimédienne si la valuation est non-archimédienne et inversement. Le théorème suivant fait voir qu'avec la restriction indiquée une métrique non-archimédienne n'est jamais équivalente à une métrique archimédienne.

Théorème 9. *Soient K un corps métrique, ϱ' et ϱ'' des métriques équivalentes dans K; supposons que ϱ' est non-archimédienne et que ϱ'' est archimédienne. Alors il est impossible que ϱ' et ϱ'' soient tous les deux des valuations de K.*

Démonstration. K contient un corps R isomorphe avec le corps des nombres rationnels; ϱ' et ϱ'' induisent dans R chacune une métrique. Si ϱ résulte d'une valuation de K, alors une valuation est induite dans R. Remarquons alors que chaque valuation non-archimédienne de R est équivalente à une valuation P-adique.[8]) Cependant, une valuation P-adique n'est pas topologiquement équivalente avec la valuation ordinaire archimédienne, d'où ressort le théorème.

Exemples. 1. La valuation P-adique du corps des nombres P-adiques induit une métrique non-archimédienne dans l'espace dont les éléments sont les nombres P-adiques. Cet espace est séparable et 0-dimensionnel et peut donc être métrisé d'une façon archimédienne. Cette dernière métrique ne résulte pas d'une valuation du corps des nombres P-adiques.

2. En introduisant la valuation ordinaire dans le corps des nombres rationnels, ce corps devient un espace archimédien, séparable, 0-dimensionnel, qui peut donc être métrisé d'une façon non-archimédienne. Cette dernière métrique ne résulte pas d'une valuation des nombres rationnels.

Ceci est déjà moins simple si l'on considère des espaces linéaires E sur un corps K muni d'une valuation, même si l'on ne considère que des topologies, induites sur E par une norme dans E (restriction qui est analogue à celle, considérée au début de ce paragraphe pour les corps métriques). Rappelons qu'une norme sur E est une fonction réelle univoque définie sur E, indiquée par $||x||$ $(x \in E)$, satisfaisant aux conditions suivantes 1, 2, 3 et $4a$ ou $4b$:

1. $||\theta|| = 0$,
2. $||x|| > 0$ pour $x \neq \theta$,
3. $||a\,x|| = |a| . ||x||$ pour tout $a \in K$, $x \in E$,
4a. $||x + y|| \leqq ||x|| + ||y||$ (cas archimédien)
4b. $||x + y|| \leqq \max(||x||, ||y||)$ (cas non-archimédien).

[8]) Voir v. d. Waerden, Moderne Algebra I, p. 255.

La relation $\varrho(x, y) = ||x - y||$ définit une métrique dans E. Inversement, avec une métrique ϱ, déduite d'une norme, correspond une valuation de K. Si cette métrique est non-archimédienne, la vulation de K est aussi non-archimédienne. La difficulté pour ces espaces vient puisque l'inverse n'est pas vrai: il se peut que la métrique ϱ est archimédienne, est même telle qu'aucun point ne contient un voisinage dans lequel la métrique est non-archimédienne, tandis que la valuation de K est non-archimédienne. J'ai déjà considéré les espaces ayant cette propriété dans un autre article [9]) et je les ai appelés sémi-non-archimédiens, afin de les distinguer des espaces totalement-non-archimédiens dans lesquels on a l'inégalité $4b$ pour tout $x, y \in E$. Considérons par exemple le cas suivant. Choisissons pour K le corps des nombres P-adiques. Soit $l_P^{(p)}$ (p réel $\geqq 1$) l'espace dont les éléments sont les suites de nombres P-adiques $x_1, x_2, \ldots$, telles que la série

$$\sum_{i=1}^{\infty} |x_i|_P^p$$

converge. En introduisant dans cet espace, qui est linéaire par rapport à K, la norme

$$\|x\| = \left\{ \sum_{i=1}^{\infty} |x_i|_P^p \right\}^{\frac{1}{p}},$$

l'espace devient semi-non-archimédien: l'inégalité $4b$ n'est pas vraie généralement. Par exemple $4b$ est vraie si $|x_i|_P > |y_i|_P$ ($i = 1, 2, \ldots$); on a alors $||x + y|| = ||x||$.

II. *Espaces uniformes.*

On peut introduire une notion „non-archimédien" dans les espaces uniformes, pas nécessairement métriques. Rappelons la définition de ces espaces. [10]) Une structure uniforme est définie dans un espace topologique E si l'on a défini dans l'espace produit $E \times E = E^2$ un ensemble $\mathfrak{F}$ de sous-ensembles $U, V, W, \ldots$, de $E \times E$ avec les propriétés suivantes:

1. l'intersection de tous les ensembles de $\mathfrak{F}$ est la diagonale D (l'ensemble des points (x, x) où $x \in E$).
2. l'intersection de deux ensembles de $\mathfrak{F}$ contient un ensemble de $\mathfrak{F}$.
3. $U \in \mathfrak{F}$ entraîne $U^{-1} \in \mathfrak{F}$.
4. quel que soit $U \in \mathfrak{F}$, il existe un $V \in \mathfrak{F}$ tel que $V^2 \subset U$.

Ici $V\,V = V^2$ désigne l'ensemble des éléments (x, y) (x et $y \in E$) de E^2 ayant la propriété qu'il existe un $z \in E$ tel que $(x, z) \in V$, $(z, y) \in V$. Si $(x, y) \in U$, on a par définition $(y, x) \in U^{-1}$. Etant donné un système $\mathfrak{F}$,

[9]) Over een lineaire P-adische ruimte. Proc. Ned. Akad. v. Wetensch. Amsterdam **46**, 74—82 (1943).

[10]) Voir p. ex. A. Weil, Sur les espaces à structure uniforme et sur la topologie générale. Act. Sci. et ind. Paris **551** (1937).

on peut définir un système de voisinages de chaque $x \in E$ comme il suit: un voisinage $U(x)$ de x se compose des points $y \in E$ tels que $(x, y) \in U$. Un ensemble muni d'une structure uniforme et de la topologie déduite de cette structure de la façon indiquée s'appelle un espace uniforme. Les ensembles de F sont appelés entourages. On peut montrer qu'il est permis de supposer que les entourages sont symétriques, c'est à dire vérifient la relation $\overline{U}^{-1} = U$, et de plus qu'ils sont ouverts. Dans ce qui suit on suppose ceci.

Définition 4. *Une structure uniforme d'un espace uniforme E est appelée non-archimédienne si tous les entourages U, déterminant la structure uniforme, satisfont à la relation*

$$\overset{2}{U} \subset U$$

Dans ce cas on peut donc satisfaire à la condition 4 par $V = U$ [11])

Théorème 10. *La topologie d'un espace métrique, dont la métrique est non-archimédienne, peut se déduire d'une structure uniforme non-archimédienne.*

Démonstration. ϱ étant la métrique donnée, les inégalités $\varrho(x, y) < a$ (a parcouront les nombres réels) définissent des ensembles U_a d'éléments (x, y) qui déterminent une structure uniforme sur l'espace E.¶ La topologie donnée est évidemment équivalente avec la topologie déduite de cette structure. Cette structure est non-archimédienne puisque $\varrho(x, y) < a$ et $\varrho(y, z) < a$ entraînent

$$\varrho(x, z) \leqq \max(\varrho(x, y), \varrho(y, z)) < a,$$

donc

$$\overset{2}{\overline{U}}_a \subset U_a.$$

Cependant une structure uniforme non-archimédienne peut fort bien être compatible avec une métrique archimédienne, même avec une métrique telle qu'aucun point n'a un voisinage dans lequel la métrique est non-archimédienne. En effet, on sait qu'il existe une structure et une seule compatible avec la topologie d'un espace compact. [12]) Si de plus l'espace est métrique et 0-dimensionnel, il peut être métrisé (voir le théorème 6) de façon non-archimédienne et aussi de façon tel qu'aucun point n'a un voisinage dans lequel la métrique est non-archimédienne et ces deux métriques sont équivalentes. Il résulte donc du théorème précédent que la structure unique est non-archimédienne. Les espaces de ce genre méritent le nom non-archimédien: bien qu'ils puissent être métrisés de deux façons, la structure uniforme unique est non-archimédienne.

(To be continued).

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[11]) En vertu de la définition de $\overset{2}{U}$ on a toujours $\overset{2}{U} \supset U$, de sorte que la relation $\overset{2}{U} \subset U$ implique $\overset{2}{U} = U$.

[12]) A. Weil, l.c. p. 24.

NOTE ON THE LOCATION OF ZEROS OF POLYNOMIALS

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GYULA DE SZ. NAGY proved some theorems on the location of zeros of polynomials [1]. These theorems are generalisations of results obtained by G. SZEGÖ [2].

The present note deals with similar problems (Theorems I, II, III and IV). Theorem V is a generalisation of a theorem of the first mentioned author, see [3]. We prove the following

Theorem I.

Let the polynomial

$$f(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$$

have all its zeros in the circle $|z-\alpha| \leqslant R$, *and let the polynomial*

$$g(z) = (z-b_1)(z-b_2)\ldots(z-b_n)$$

have no zero in the circle $|z-\alpha| \leqslant \varrho$, $\varrho < R$.

Then no polynomial

$$h(z) = \lambda f(z) - g(z),\ |\lambda| \leqslant t^n,\ 0 \leqslant t < \frac{\varrho}{R},$$

has a zero in the circle

$$|z-\alpha| \leqslant r = \frac{\varrho - Rt}{1+t}\ (\leqslant \varrho). \tag{1}$$

Proof.

Every point z_0 with $h(z_0) \neq 0$ satisfies

$$\left|\frac{g(z_0)}{f(z_0)}\right| \neq |\lambda|.$$

Now we have for every point z_0 of the region (1)

$$|z_0-a_k| \leqslant |z_0-\alpha| + |a_k-\alpha| \leqslant r+R,$$

and

$$|z_0-b_k| \geqslant |b_k-\alpha| - |z_0-\alpha| > \varrho - r,$$

so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n}\left|\frac{z_0-a_k}{z_0-b_k}\right| < \left(\frac{r+R}{\varrho-r}\right)^n = \frac{1}{t^n} \leqslant \frac{1}{|\lambda|}$$

This completes the proof of Theorem I.

A generalisation of Theorem I is

Theorem II.

Let the polynomial

$$f(z) = (z-a_1)(z-a_2)\dots(z-a_n)$$

have all its zeros in the circle $|z-\alpha| \leqslant \varrho_1$, *and let the polynomial*

$$g(z) = (z-b_1)(z-b_2)\dots(z-b_n)$$

have no zero in the circle

$$|z-\beta| \leqslant \varrho_2,\ \varrho_1 > \varrho_2,\ |\alpha-\beta| < \min(\varrho_1-\varrho_2, \varrho_2).$$

Then no polynomial

$$h(z) = \lambda f(z) - g(z),\ |\lambda|^{1/n} < \frac{\varrho_2 - |\beta-\alpha|}{\varrho_1},$$

has a zero in the region

$$|\lambda|^{1/n} \cdot |z-\alpha| + |z-\beta| \leqslant \varrho_2 - \varrho_1 |\lambda|^{1/n}. \tag{2}$$

Proof. We put $|z_0-\alpha| = r_1$ and $|z_0-\beta| = r_2$, where z_0 is a point of the region (2). From our assumptions it follows that z_0 is an interior point of the circle $|z-\beta| \leqslant \varrho_2$.

Now we have

$$|z_0-a_k| \leqslant |z_0-\alpha| + |a_k-\alpha| \leqslant r_1 + \varrho_1,$$

and

$$|z_0-b_k| \geqslant |b_k-\beta| - |z_0-\beta| > \varrho_2 - r_2,$$

so that

$$\left|\frac{z_0-a_k}{z_0-b_k}\right| < \frac{r_1+\varrho_1}{\varrho_2-r_2},\ \text{and}\ \left|\frac{f(z_0)}{g(z_0)}\right| < \left(\frac{r_1+\varrho_1}{\varrho_2-r_2}\right)^n \leqslant \frac{1}{|\lambda|},$$

and from these last inequalities it follows that z_0 is not a zero of $h(z)$.

Theorem III.

Let G_1 *be a half-plane containing the zeros of the polynomial*

$$f(z) = (z-a_1)(z-a_2)\dots(z-a_n).$$

Let G_2 *be the complementary half-plane containing the zeros of the polynomial*

$$g(z) = (z-b_1)(z-b_2)\dots(z-b_n),$$

and let l *be the boundary line of* G_1 *and* G_2.

Let α *and* β *be two points of* G_1 *with*

$$|\alpha-\beta| > d,\ \text{where}\ d = \max_{k=1,\dots,n} |\alpha-a_k|.$$

Let P *be the parabola with* β *as focus and with* l *as directrix, and let* H

be the hyperbola with foci at α and β, and with the major axis d. Then the polynomial

$$h(z) = f(z) + g(z)$$

has no zero in the region R, the common part of the interior of P, and the interior of H containing α.

Remarks. 1. The set of points from which no tangent can be drawn to a given conic section is said to be the interior of that conic section.

2. If we choose β on the perpendicular from α on l, then the set of points R is not empty.

Proof. We denote by z_0 an arbitrary point of R.

It is obvious, that

$$|z_0 - b_k| > |z_0 - \beta|,$$

and

$$|z_0 - a_k| \leqslant |z_0 - \alpha| + |a_k - \alpha| \leqslant |z_0 - \alpha| + d.$$

Hence

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| < \left(\frac{|z_0 - \alpha| + d}{|z_0 - \beta|}\right)^n < 1, \tag{3}$$

where the last inequality follows from

$$|z_0 - \beta| - |z_0 - \alpha| > d$$

From (3) we see that z_0 is no zero of $h(z)$.

Theorem IV.

Let G_1 be a half-plane containing the zeros of the polynomial

$$f(z) = (z - a_1)(z - a_2) \ldots (z - a_n).$$

Let G_2 be the complementary half-plane containing the points a and α, and let l be the boundary line of G_1 and G_2.

Let P be the parabola with α as focus and l as directrix. Let C be the circle of Apollonius for the points a and α, with ratio t $(0 < t < 1)$, and of which a is an interior point (this means that C is the set of points z with

$$|z - a| = t\,|z - \alpha|).$$

Let $g(z)$ be the polynomial

$$g(z) = (z - a)^n.$$

Then no polynomial

$$h(z) = g(z) + \lambda f(z), \; |\lambda| = t^n, \; 0 < t < 1,$$

has a zero in the region R, which the interiors of P and C have in common.

Proof. We denote by z_0 an arbitrary point of R.

Then

$$|z_0 - a_k| > |z_0 - \alpha|, \text{ and } |z_0 - a| < t\,|z_0 - \alpha|,$$

so that

$$\left|\frac{g(z_0)}{f(z_0)}\right| = \prod_{k=1}^{\infty}\left|\frac{z_0-a}{z_0-a_k}\right| < \left|\frac{z_0-a}{z_0-a}\right|^n < t^n \leqslant |\lambda|.$$

Hence

$$|g(z_0)| < |\lambda f(z_0)|.$$

This means that z_0 is no zero of $h(z)$.

Theorem V.

Let

$$f(z) = (z-a_1)(z-a_2)\ldots(z-a_n),$$

with $$a_k = x_k + i y_k \qquad (k = 1, \ldots, n),$$

$$g(z) = (z-b_1)(z-b_2)\ldots(z-b_n),$$

with $$b_k = x_k + i y_k^*, \; y_k > y_k^* \qquad (k = 1, \ldots, n),$$

$$F(z) = \frac{f(z)}{g(z)} = \frac{(z-a_1)(z-a_2)\ldots(z-a_n)}{(z-b_1)(z-b_2)\ldots(z-b_n)},$$

$$F'(\zeta) = 0, \; F(\zeta) \neq 0, \; \zeta = \xi + i\eta.$$

Then:

(*) *If the intervals*

(4) $$y_k \geqslant y \geqslant y_k^* \qquad (k = 1, 2, \ldots, n)$$

have an interval I in common, η is not an interior point of I.

(**) *ζ is not a point of the region R, which the interiors of the hyperbolas H_k*

$$(x-x_k)^2 - (y-y_k)(y-y_k^*) = 0 \qquad (k = 1, \ldots, n)$$

have in common. ζ is an interior point of one H_k at least, and an exterior point of one the other H_k at least.

(***) *If* $(k = 1, \ldots, n)$

$$2m_k = y_k + y_k^*,$$

then ζ is not a point of the regions R_1 and R_2, the common parts of the regions

$$(x-x_k)(y-m_k) > 0,$$

and

$$(x-x_k)(y-m_k) < 0,$$

respectively.

Proof.

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} = \sum_{k=1}^{n}\left\{\frac{1}{z-a_k} - \frac{1}{z-b_k}\right\} =$$

$$i\sum_{k=1}^{n}\frac{y_k - y_k^*}{\{x-x_k+i(y-y_k)\}\{x-x_k+i(y-y_k^*)\}} =$$

$$i\sum_{k=1}^{n}\frac{y_k - y_k^*}{\{(x-x_k)^2-(y-y_k)(y-y_k^*)\}+i\{2xy-xy_k-xy_k^*-2yx_k+x_ky_k+x_ky_k^*\}}$$

If now $\zeta = \xi + i\eta$, $F'(\zeta) = 0$ and $F(\zeta) \neq 0$, then

$$\frac{F'(\zeta)}{F(\zeta)} = i \sum_{k=1}^{n} \frac{y_k - y_k^*}{A_k + iB_k} = \sum_{k=1}^{n} \frac{B_k(y_k - y_k^*)}{A_k^2 + B_k^2} + i \sum_{k=1}^{n} \frac{A_k(y_k - y_k^*)}{A_k^2 + B_k^2} = 0,$$

$$\sum_{k=1}^{n} \frac{A_k(y_k - y_k^*)}{A_k^2 + B_k^2} = 0, \quad \sum_{k=1}^{n} \frac{B_k(y_k - y_k^*)}{A_k^2 + B_k^2} = 0, \tag{5}$$

with

$$A_k = (\xi - x_k)^2 - (\eta - y_k)(\eta - y_k^*),$$

and

$$B_k = 2\xi\eta - \xi y_k - \xi y_k^* - 2\eta x_k + x_k y_k + x_k y_k^*.$$

If ζ is a point of R, then the inequalities

$$A_k > 0 \qquad (k = 1, \ldots, n) \tag{6}$$

hold. From (6) and our assumption $y_k - y_k^* > 0$ $(k = 1, \ldots, n)$ it follows that the first equality of (5) does not hold.

So (**) is proved.

(*) is an immediate consequence of (**).

Furthermore we have

$$\sum_{k=1}^{n} \frac{B_k(y_k - y_k^*)}{A_k^2 + B_k^2} = \sum_{k=1}^{n} \frac{(y_k - y_k^*)(\xi - x_k)(2\eta - y_k - y_k^*)}{A_k^2 + B_k^2} =$$

$$2 \sum_{k=1}^{n} \frac{(y_k - y_k^*)(\xi - x_k)(\eta - m_k)}{A_k^2 + B_k^2}$$

It can easily be seen that the last expression does not vanish if ζ is a point of R_1, or R_2. This concludes the proof of (***).

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PROJECTIVE GEOMETRIZATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS II: FRENET FORMULAE FOR NORMAL POINTS

BY

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Synopsis. This paper is a continuation of the previous one entitled *Projective geometrization of a system of partial differential equations I.* (These Proceedings, **53**, 318—326) which will be referred to as P I. We shall use also some of the results of affine embedding theory as exposed in the papers *Affine embedding theory*, I, II, III (Proc. Kon. Ned. Akad. v. Wetensch. **52**, no. 5, 7 and 9) which will be referred to as A I, A II, A III.

First two sets of projective tensors K, L will be found and then Frenet formulae for the normal points of a $\mathfrak{P}_m$ in a P_n (cf P I) will be established.

§ 1. *Tensors L and K.*

Theorem (1, 1). *The elimination of the* $P_{a'\ldots}{}^{b}$ *from the equations* (2, 4*a*), *b*) *and* (3, 2) *in* P I *leads to the transformation law for a projective tensor* $L_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}$, $(s = 2, 3, \ldots N)$ *which has the following properties*:

a) for $s = N$ *we have*

$$L_{a_{N+1}a_N\ldots a_1}^{\quad b_N\ldots b_1} \equiv \Phi_{a_{N+1}a_N\ldots a_1}^{\quad b_N\ldots b_1} - \left\{ \Gamma_{a_{N+1}a_N}^{\quad b_N}\, \delta_{a_{N-1}}^{b_{N-1}} \ldots \delta_{a_1}^{b_1} \right\} \tag{1.1a}$$

b) for $s < N$ *we have*

$$L_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1} = \Phi_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1} + \Gamma_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}, \quad s = 2, \ldots, N-1 \tag{1.1b}$$

where $\Gamma_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}$ *is a well-defined function of the* $\Phi_{a_{N+1}\ldots a_r\ldots a_1}^{\quad b_r\ldots b_1}$, $(r = s+1, \ldots, N)$ *as well as of the* $\Gamma_{a_u\ldots a_1}^{\quad b}$, $(u = 2, 3, \ldots, N+2-s)$.

c) $L_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}$ *is homogeneous of degree* $-(N+1-s)$.

$$\dot{L}_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots a_1} = g^{-(N+1-s)}\, L_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1} \tag{1.1c}$$

and moreover

d) it is of the $\left.\begin{matrix} x- \\ y- \end{matrix}\right\}$ *order* $\begin{matrix} N+1\,(0) \\ N+2-s \end{matrix}$.

Proof: The statement *a*) follows at once from the equations (2, 4*a*)

and (3, 2a) in P I. If we eliminate the $P_{a'\ldots}^{\ \ b}$ from (2, 4b) and (3, 2b) in P I we obtain

$$L_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\ \ \ \ \ \ \ b'_s\ldots b'_1} = P_{a'_{N+1}\ldots a'_1}^{a_{N+1}\ldots a_1} P_{b_s\ldots b_1}^{b'_s\ldots b'_1} L_{a_{N+1}\ldots a_s\ldots a_1}^{\ \ \ \ \ \ \ b_s\ldots b_1} + X_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\ \ \ \ \ \ \ b'_s\ldots b'_1}$$

$$(s = 2, \ldots, N-1)$$

where the L's are defined by (1, 1b) (and a similar equation in the parameters $y^{a'}$) and X is a function of the Φ's, $P_{b_1}^{a'}$ and $P_{b'}^{a'_1}$ which does not contain the derivatives of the $P_b^{a'}$, $P_{b'}^{a}$. Following now the same argument as in the proof of the theorem (5, 1) in A I we obtain $X_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\ \ \ \ \ \ \ b'_s\ldots b'_1} = 0$. The remaining statements are easily proved by means of Theorems (2, 2) and (2, 3) in P I.

Note: The formal structure of the projective tensors L's is obviously the same as the one of the affine tensors L's introduced in A II (§ 1).

Theorem (1, 2a). *The elimination of the* $P_{a'\ldots}^{\ \ b}$ *from* (3, 2) *in PI and from*

$$(1.2)\qquad P_{b'_q\ldots b'_1}^{\ \ \ a} = \partial_{b'_q}\left[P_{a'}^{a}\,\Omega_{b'_{q-1}\ldots b'_1}^{\ \ \ \ \ a'} + \Xi_{b'_{q-1}\ldots b'_1}^{\ \ \ \ \ a'} - P_{b'_{q-1}\ldots b'_1}^{b_{q-1}\ldots b_1}\,\Gamma_{b_{q-1}\ldots b_1}^{\ \ \ \ a}\right]^{1)}$$

$$q = 3, 4, \ldots, N$$

leads to a transformation law for a projective tensor $K_{b_q\ldots b_1}^{\ \ \ \ a}$ *which has the following properties*

a) Its definition is

$$(1.3a)\qquad K_{b_q\ldots b_1}^{\ \ \ \ a} \equiv -\partial_{b_q}\Gamma_{b_{q-1}\ldots b_1}^{\ \ \ \ a} + \Gamma_{b_q\ldots b_1}^{\ \ \ \ a} + \Psi_{b_q\ldots b_1}^{\ \ \ \ a},\quad q = 3, 4, \ldots, N$$

where $\Psi_{b_q\ldots b_1}^{\ \ \ \ a}$ *is a well-defined function of the* $\Gamma_{b_r\ldots b_1}^{\ \ \ \ a}$ $r = 2, \ldots, q-1$ *and the first derivatives of the* $\Gamma_{b_u\ldots b_1}^{\ \ \ \ a}$, $u = 2, \ldots, q-2$ *(as well as of* $\Phi_{a_N\ldots a_p\ldots a_1}^{\ \ \ \ \ \ \ b_p\ldots b_1}$, $p = N+3-q, \ldots, N$*) and the first derivatives of* $\Phi_{a_N\ldots a_s\ldots a_1}^{\ \ \ \ \ \ \ b_s\ldots b_1}$ $(s = N+4-q, \ldots, N)$.

b) It is homogeneous of degree $1-q$

$$(1.4a)\qquad \overset{*}{K}_{b_q\ldots b_1}^{\ \ \ \ a} = g^{(1-q)}\,K_{b_q\ldots b_1}^{\ \ \ \ a}$$

and moreover

c) It is of the $\left.\begin{matrix} x- \\ y- \end{matrix}\right\}$ *order* $\begin{matrix} N+2 \\ q \end{matrix}$ (1)

Proof: The elimination of $P_{b'_q\ldots b'_1}^{\ \ \ \ a}$ from (3, 2) in P I and from (1, 2) leads to

$$(1.5)\qquad \left\{\begin{aligned} &P_{b'_q\ldots b'_1}^{b_q\ldots b_1}\left(\Gamma_{b_q\ldots b_1}^{\ \ \ \ a} - \partial_{b_q}\Gamma_{b_{q-1}\ldots b_1}^{\ \ \ \ a}\right) = P_{a'}^{a}\left(\Omega_{b'_q\ldots b'_1}^{\ \ \ \ \ a'} + \Xi_{b'_q\ldots b'_1}^{\ \ \ \ \ a'}\right) - \\ &\quad - \partial_{b'_q}\left[P_{a'}^{a}\left(\Omega_{b'_{q-1}\ldots b'_1}^{\ \ \ \ \ a'} + \Xi_{b'_{q-1}\ldots b'_1}^{\ \ \ \ \ a'}\right)\right] + \Gamma_{b_{q-1}\ldots b_1}^{\ \ \ \ a}\,\partial_{b'_q}P_{b'_{q-1}\ldots b'_1}^{b_{q-1}\ldots b_1} \end{aligned}\right.$$

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[1]) We put here $\Omega_{a'b'}^{\ \ c'} \equiv \Gamma_{a'b'}^{\ \ c'}$ and $\Xi_{a'b'}^{\ \ c'} \equiv 0$.

Because Ξ contains $P^a_{b'}$, $P^{a'}_b$ and no derivatives of these functions, the right hand member in (1, 5) contains $P^a_{b'}$, $P^{a'}_b$ and their first derivatives. The latter may be eliminated by means of (3, 2a) in P I. In doing so we obtain

$$K_{b'_q \ldots b'_1}{}^{a'} = P^{b_q \ldots b_1 a'}_{b'_q \ldots b'_1 a} K_{b_q \ldots b_1}{}^{a} + X_{b'_q \ldots b'_1}{}^{a'}$$

where the K's are defined by (1, 3a) (and a similar equation in the parameters $y^{a'}$) and X contains $P^{a'}_b$ $P^a_{b'}$ but not their derivatives. Using the same argument as previously, we obtain $X_{b'_q \ldots b'_1}{}^{a'} = 0$. The remaining statements follow from (1, 5) and Theorems (3, 2) and (3, 3) in P I.

Theorem (1, 2b). *The elimination of the* $P_{b'_s \ldots b'_1}{}^{a}$, $s = 2, \ldots, N+1$ *from* (2, 4c) *and* (3, 2) *in* P I *and from*

$$P_{b'_{N+1} \ldots b'_1}{}^{a} = \partial_{b'_{N-1}} \left[P^a_{a'} \left(\Omega_{b'_N \ldots b'_1}{}^{a'} + \Xi_{b'_N \ldots b'_1}{}^{a'} \right) - P^{b_N \ldots b_1}_{b'_N \ldots b'_1} \Gamma_{b_N \ldots b_1}{}^{a} \right]$$

leads to a transformation law for a projective tensor $K_{b_{N+1} \ldots b_1}{}^{a}$ *which has the following properties*

a) Its definition is

$$K_{b_{N+1} \ldots b_1}{}^{a} \equiv - \partial_{b_{N+1}} \Gamma_{b_N \ldots b_1}{}^{a} + \Phi_{b_{N+1} \ldots b_1}{}^{a} + \Psi_{b_{N+1} \ldots b_1}{}^{a} \tag{1. 3b}$$

where $\Psi_{b_{N+1} \ldots b_1}{}^{a}$ *is a well-defined function of the* $\Gamma_{b_r \ldots b_1}{}^{a}$ $(r = 2, \ldots, N)$ *and the first derivatives of* $\Gamma_{b_u \ldots b_1}{}^{a}$ $(u = 2, \ldots, N-1)$, *(as well as of* $\Phi_{a_N \ldots a_p \ldots a_1}{}^{b_p \ldots b_1}$, $p = 2, \ldots, N$ *and the first derivatives of* $\Phi_{a_N \ldots a_s \ldots a_1}{}^{b_s \ldots b_1}$, $s = 3, \ldots, N$). [2]

b) It is homogeneous of degree $-N$

$$\dot{K}_{b_{N+1} \ldots b_1}{}^{a} = g^{-N} K_{b_{N+1} \ldots b_1}{}^{a} \tag{1. 4b}$$

and moreover

c) it is of the $\left.\begin{matrix} y- \\ x- \end{matrix}\right\}$ *order* $\begin{matrix} N+2\,(1) \\ N+1 \end{matrix}$

The proof is similar to the previous one.

Note I: The formal structure of the projective tensors K's is obviously the same as the one of the affine tensors K's introduced in A II (§ 1).

Note II: In the next section we introduce the symbols

$$\left\{\!\!\left\{ K_{a_{s+1} \ldots a_r}{}^{b_r} \delta^{b_{r-1} \ldots b_1}_{a_{r-1} \ldots a_1} \right\}\!\!\right\} \quad (s = 3, \ldots, N\,;\; r = 2, \ldots, s-1).$$

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[2] cf. Note I of § 2.

If $\underset{1}{K}_{a_{s+1}}$ and $\underset{2}{K}_{a_s \ldots a_r}^{\quad b_r}$ are two *ideal* factors of $K_{a_{s+1}\ldots a_r}^{\quad b_r}$, then

$$\left\{\left\{ K_{a_{s+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}\right\} \equiv \underset{1}{K}_{a_{s+1}} \left\{ \underset{2}{K}_{a_s \ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}^{3})$$

§ 2. *Frenet formulae for normal points.*

Let $T_{b\ldots}^{a\ldots}$ be a projective tensor. Its covariant derivative will be denoted by $D_c T_{b\ldots}^{a\ldots}$:

$$D_c T_{b\ldots}^{a\ldots} \equiv \partial_c T_{b\ldots}^{a\ldots} + \Gamma_{dc}^{\ a} T_{b\ldots}^{d\ldots} + \ldots - \Gamma_{bc}^{\ d} T_{d\ldots}^{a\ldots} - \ldots . \tag{2.1}$$

Lemma (1, 2). *If $T_{b\ldots}^{a\ldots}$ is homogeneous of degree h then $D_c T_{b\ldots}^{a\ldots}$ is homogeneous of degree $h-1$.*

The proof follows at once from (2, 1) and from Theorem (3, 3) in P I: Because

$$T_{b\ldots}^{a\ldots}(\dot{y}) = \dot{T}_{b\ldots}^{a\ldots} = g^h(y)\, T_{b\ldots}^{a\ldots}(y)$$

we have

$$\begin{aligned}
\dot{D}_c \dot{T}_{b\ldots}^{a\ldots} &= \frac{\partial}{\partial \dot{y}^c} T_{b\ldots}^{a\ldots}(\dot{y}) + \dot{\Gamma}_{dc}^{\ a} \dot{T}_{b\ldots}^{d\ldots} + \ldots - \dot{\Gamma}_{bc}^{\ d} \dot{T}_{d\ldots}^{a\ldots} - \ldots = \\
&= g^{h-1}\left[\partial_c T_{b\ldots}^{a\ldots} + \Gamma_{dc}^{\ a} T_{b\ldots}^{d\ldots} + \ldots - \Gamma_{bc}^{\ d} T_{d\ldots}^{a\ldots} - \ldots\right] = \\
&= g^{h-1} D_c T_{b\ldots}^{a\ldots}.
\end{aligned}$$

Theorem (2, 1). *The points $\mathbf{x}_a$, $\mathbf{n}_{a_s \ldots a_1}$ $(s = 2, \ldots, N)$ satisfy the following equations (Frenet formulae)* [4])

$$\left\{\begin{array}{ll}
a) & D_{a_2}\mathbf{x}_1 = \mathbf{n}_{a_2 a_1} \\
b) & D_{a_s}\mathbf{n}_{a_s a_1} = \mathbf{n}_{a_s a_s a_1} + K_{a_s a_s a_1}^{\quad b}\, \mathbf{x}_b \\
c) & D_{a_{q+1}}\mathbf{n}_{a_q \ldots a_1} = \mathbf{n}_{a_{q+1}\ldots a_1} + \sum_{2}^{q-1}{}^{r}\, \mathbf{n}_{b_r \ldots b_1}\left\{\left\{ K_{a_{q+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}\right\} + K_{a_{q+1}\ldots a_1}^{\quad b}\, \mathbf{x}_b \\
 & \qquad q = 3, \ldots, N-1 \\
d) & D_{a_{N+1}}\mathbf{n}_{a_N \ldots a_1} = L_{a_{N+1} a_N \ldots a_1}^{\quad b_N \ldots b_1}\, \mathbf{n}_{b_N \ldots b_1} \\
 & \qquad + \sum_{2}^{N-1}{}^{r}\, \mathbf{n}_{b_r \ldots b_1}\left[\left\{\left\{ K_{a_{N+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}}^{b_{r-1}} \ldots \delta_{a_1}^{b_1}\right\}\right\} + L_{a_{N+1}\ldots a_r \ldots a_1}^{\quad b_r \ldots b_1}\right] \\
 & \qquad + K_{a_{N+1}\ldots a_1}^{\quad b}\, \mathbf{x}_b .
\end{array}\right. \tag{2.2}$$

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[3]) In A II, III we used the notation $\left\{ K_{a_{s+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}$ instead of $\left\{\left\{ K_{a_{s+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}\right\}$ which may be misleading for anyone who did not closely follow the proof of Theorem (2. 1) in A II and dit not realize that $\left\{ \Gamma_{a_{s+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}$ is symmetric in subscripts while $\left\{ K_{a_{s+1}\ldots a_r}^{\quad b_r}\, \delta_{a_{r-1}\ldots a_1}^{b_r b_{r-1}\ldots b_1}\right\}$ is not.

[4]) The projective normal points $\mathbf{n}_{a\ldots}$ were introduced in P I § 4. If $N=2$ then the equations (2. 2) reduce to (2. 2*a*), and (2. 2*d*) where $\sum_{r=2}^{1} \equiv 0$.

Proof: A similar theorem was proved for the affine case (cf. A II § 2). In spite of the fact, that the proof was based on a special parameter system (which we can not use here) it holds for any parameter system. Because our $\mathfrak{P}_m$ in P_n may be thought of as an $A_{m'}$ in $A_{n'}$ ($m' = m+1$, $n' = n+1$) all theorems which are not based on multiplication with y^a and which hold for $A_{m'}$ in $A_{n'}$ must hold also for $\mathfrak{P}_m$ in P_n. But this is just the case of Frenet formulae. Hence (2, 2) holds also for $\mathfrak{P}_m$ in P_n.

Note I. If we consider (2, 2*b*) as an identity expressed in the $\mathbf{x}_{a_1}$, $\mathbf{x}_{a_2a_1}$ and $\mathbf{x}_{a_3a_2a_1}$ then we get at once for $N > 2$

$$K_{a_3a_2a_1}^{\ \ \ b} = \Gamma_{a_3a_2a_1}^{\ \ \ b} - \partial_{a_3}\Gamma_{a_2a_1}^{\ \ b} + \Gamma_{a_1a_3}^{\ \ c}\Gamma_{ca_2}^{\ \ b} + \Gamma_{a_2a_3}^{\ \ c}\Gamma_{ca_1}^{\ \ b}. \tag{2.3}$$

In a similar manner we may also get the expressions for $K_{a_r\ldots a_1}^{\ \ \ \ b}$ ($r = 4, \ldots, N+1$). The tensor $K_{a_3a_2a_1}^{\ \ \ b}$ is closely related to the first normal tensor A_{jkl}^{i} as defined by Veblen and Thomas [5]) for the symmetric affine case

$$\left\{\begin{array}{l} A_{jkl}^{i} = \partial_l\Gamma_{jk}^{i} - \Lambda_{jkl}^{i} - \Gamma_{jl}^{h}\Gamma_{hk}^{i} - \Gamma_{kl}^{h}\Gamma_{hj}^{i} \\ \qquad \Lambda_{jkl}^{i} \equiv \partial_{(l}\Gamma_{jk)}^{i} - 2\,\Gamma_{(kl}^{h}\Gamma_{j)h}^{\ \ i} \end{array}\right. \tag{2.4}$$

Using the same equation (2, 4) for the definition of the first projective normal tensor A_{jkl}^{i} we obtain from (2, 3) and (2, 4)

$$A_{a_1a_2a_3}^{\ \ \ i} = -K_{a_3a_2a_1}^{\ \ \ i} + K_{(a_3a_2a_1)}. \tag{2.5}$$

Hence we have also in the projective case $A_{(jkl)}^{i} = 0$.

Theorem (2, 2). *The equations* (2, 2) *are g— invariant.*

Proof. The right hand member in (2, 2) for $D_{a_{u+1}}\mathbf{n}_{a_u\ldots a_1}$, ($u = 1, \ldots, N$) is obviously homogeneous of degree $N-u$. On the other hand we have by virtue of Lemma (2, 1)

$$\overset{*}{D}_{a_{u+1}}\overset{*}{\mathbf{n}}_{a_u\ldots a_1} = g^{(N-u)}\,D_{a_u\ldots 1}\mathbf{n}_{a_u\ldots a_1}$$

We shall now use (2, 2) for getting a deeper insight into the role played by the normal points.

§ 3. *Auxiliary results.*

In this section we shall use the osculating spaces $\overset{r}{P}_{m_r}$, $r = 1, \ldots, N$ defined in P I § 1 as well as the tensor

$$Q_b^a = y^c\,\Gamma_{bc}^{a}$$

(cf. P I equation (3, 4) and shall consider only the maximal case without stating it explicitly.

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[5]) Veblen, O. and Thomas, T. Y., "The geometry of paths" (Transaction of the American Mathematical Society, 25, 1923).

Theorem (3, 1). *The space spanned by* $\mathbf{x}$ *and the normal points* $\mathbf{n}_{ab}$ *is at most of dimension* $*m_2$

$$*m_2 \equiv \binom{m+2}{2} - 1 = m_2. \tag{3.1}$$

If and only if

$$Q^a_b = N\,\delta^b_a \tag{3.2}$$

$*m_2$ *reaches its minimal value*

$$*m_2 = m_2 - m_1 \tag{3.3}$$

and the first osculating space $\overset{1}{P}_m$ *and the space spanned by* $\mathbf{x}$ *and* $\mathbf{n}_{ab}$ *have only* $\mathbf{x}$ *in common.*

Proof: First of all we have

$$\begin{cases} y^a\,\mathbf{x}_a = (N+1)\,\mathbf{x} \quad , \quad y^a\,\mathbf{x}_{ab} = N\,\mathbf{x}_a \\ y^a\,\mathbf{n}_{ab} = N\,\mathbf{x}_b - Q^a_b\,\mathbf{x}_a = \left(N\,\delta^a_b - Q^a_b\right)\mathbf{x}_a. \end{cases} \tag{3.4}$$

Let us now introduce a special parameter system so that we have $y^a = \delta^a_o$ at the point P in question. Hence the equations (3, 4) reduce to

$$\mathbf{x}_o = (N+1)\,\mathbf{x}, \quad \mathbf{x}_{ob} = N\,\mathbf{x}_b, \quad \mathbf{n}_{ob} = \left(N\,\delta^a_b - Q^a_b\right)\mathbf{x}_a \text{ at } P \tag{3.5}$$

and the set of points $\mathbf{x}_{ab}$ consists of

$$\mathbf{x}_{oo} = N\,(N+1)\,\mathbf{x}, \quad \mathbf{x}_{o\beta} = \mathbf{x}_{\beta o} = N\,\mathbf{x}_\beta, \quad \mathbf{x}_{a\beta} = \mathbf{x}_{\beta a}, \quad (\alpha, \beta = 1, \ldots, m). \tag{3.6}$$

Because $\mathbf{x}_{(ab)}$ are linearly independent (for our X_m is a maximal one) and span $\overset{2}{P}_{m_2}$, the $\binom{m+1}{2}$ points $\mathbf{x}_{a\beta}$ must be linearly independent and the space spanned by them must not contain points of $\overset{1}{P}_{m_1}$. The same statement obviously holds for the space spanned by the points $\mathbf{n}_{a\beta} = \mathbf{x}_{a\beta} - \Gamma^e_{a\beta}\,\mathbf{x}_e$. Hence the space spanned by $\mathbf{x}, \mathbf{n}_{a\beta}$ has

$$\binom{m+1}{2} = \binom{m+2}{2} - 1 - ([m+1]-1) = m_2 - m_1$$

dimensions and intersects $\overset{1}{P}_{m_1}$ only in $\mathbf{x}$. The spaces spanned by $\mathbf{x}$ and $\mathbf{n}_{a\beta}$, respectively $\mathbf{x}$, $\mathbf{n}_{ab}$ are identical if and only if (3, 2) holds and then $*m_2$ reaches its minimal value given by (3, 3). Assume now that the projective tensor $N\,\delta^a_b - Q^a_b$ is of rank $m+1$. Then the space $\overset{2}{P}_{m_2}$ spanned by $\mathbf{x}$, $\mathbf{x}_a$, $\mathbf{x}_{a\beta}$ is the same as the one spanned by $\mathbf{n}_{oo}$, $\mathbf{n}_{o\beta}$, $\mathbf{n}_{a\beta}$ and consequently the dimension of the latter is given by (3, 1).

Note: If (3, 2) holds then the space spanned by $\mathbf{x}$, $\mathbf{n}_{ab}$ could be thought of as a "first projective normal space" for it is in $\overset{2}{P}_{m_2}$ and intersects $\overset{1}{P}_{m_1}$ only in $\mathbf{x}$. One would expect that a similar statement holds for $\mathbf{n}_{abc}$, namely that the space spanned by $\mathbf{x}$, $\mathbf{n}_{abc}$ is under special

circumstances the second projective normal space (contained in $\overset{3}{P}_{m_3}$ and intersecting $\overset{2}{P}_{m_2}$ only in $\mathbf{x}$). The next theorem shows that the requirement of the existence of two first projective normal spaces (defined in the manner already mentioned) is impossible.

Theorem (3, 2). *If* (3, 2) *holds then* $\overset{2}{P}_{m_2}$ *and the space spanned by* $\mathbf{x}$, $\mathbf{n}_{abc}$ *have more than the point* $\mathbf{x}$ *in common.*

Proof: If (3, 2) holds then

$$y^a D_a \mathbf{n}_{bc} = y^a (\partial_a \mathbf{n}_{bc} - \Gamma_{ba}^{d} \mathbf{n}_{cd} - \Gamma_{ca}^{d} \mathbf{n}_{bd})$$
$$= (N-1)\mathbf{n}_{bc} - 2N\mathbf{n}_{bc} = -(N+1)\mathbf{n}_{bc}.$$

Consequently we have by virtue of (2, 2*b*)

$$y^a \mathbf{n}_{abc} = -(N+1)\mathbf{n}_{bc} - y^a K_{abc}^{d} \mathbf{x}_d. \tag{3. 7}$$

Because (3, 2) holds the space spanned by $\mathbf{x}$, $\mathbf{n}_{bc}$ intersects $\overset{1}{P}_{m_1}$ only in $\mathbf{x}$ (cf. the previous theorem). Hence the right hand member of (3, 7) can not vanish. Consequently there is at least one linear combination of the points $\mathbf{n}_{abc}$ (namely $y^a \mathbf{n}_{abc}$) which is situated in $\overset{2}{P}_{m_2}$ and is different from $\mathbf{x}$.

Note: The theorem already proved shows that the points $\mathbf{n}$ are not appropriate to define "projective normal spaces". If the space spanned by $\mathbf{x}$, $\mathbf{n}_{ab}$ does not overlap $\overset{1}{P}_{m_1}$ (but for the point $\mathbf{x}$) then the space spanned by $\mathbf{x}$ and $\mathbf{n}_{abc}$ overlaps with $\overset{3}{P}_{m_3}$ (not only in $\mathbf{x}$). Hence in order to find "projective normal spaces" we have to use another device. We shall do it in the next paper.

SOME REMARKS ON THE POWER FUNCTION OF WILCOXON's TEST FOR THE PROBLEM OF TWO SAMPLES. I

BY

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(See further Proceedings **53**, *507—520 (1950)).*

1. INTRODUCTION

F. WILCOXON (9) gave a non-parametric solution of the problem of two samples of equal sizes. This solution was generalized and studied in detail by H. B. MANN and D. R. WHITNEY (2).

Let $\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n$ [1]) be $m+n$ independent random variables, the $\mathbf{x}_i$ all having the *continuous* (cumulative) distribution function $F(x)$ and the $\mathbf{y}_j$ all having the *continuous* (cumulative) distribution function $G(x)$.

For any set of values $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ the variables can take [2]) let $U = U(x_1, \ldots, x_m, y_1, \ldots, y_n) = U(E)$ be defined as the number

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[1]) The letter which denotes a variable is printed in bold type when stress is laid upon the random character of the variable, i.e. upon the fact the variable has a distribution function.

[2]) Such a set E is called a "*sample point*"; the set of all sample points E which eventually might be obtained is called the "*sample space*" W.

of pairs of integers (i, j) $(1 \leqq i \leqq m;\ 1 \leqq j \leqq n)$ with $x_i > y_j$, provided $x_i \neq y_j$ for every pair (i, j). Apparently U is zero or a positive integer. Then $\mathbf{U} = U(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n) = U(\mathbf{E})$ is a discrete random variable, defined on the sample space W everywhere with the exception of a set of probability zero consisting of all points E with $x_i = y_j$ for any pair (i, j) $(1 \leqq i \leqq m;\ 1 \leqq j \leqq n)$.

The statistic $U(E)$ was introduced by H. B. MANN and D. R. WHITNEY for testing the hypothesis that $G(x) = F(x)$ holds for all x (shortly: $G = F$) against the alternative that $G(x) < F(x)$ holds for all x (shortly: $G < F$).

The distribution function of $\mathbf{U}$ under the hypothesis $G = F$ has been computed by MANN and WHITNEY for $m \leqq 8$, $n \leqq 8$, and by the Computing Department of the Mathematical Centre at Amsterdam for $m \leqq 10$, $n \leqq 10$. For $m \to \infty$ and $n \to \infty$ the distribution of $(\mathbf{U} - \frac{1}{2}mn)$. $[1/12 \cdot mn\ (m + n + 1)]^{-\frac{1}{2}}$ tends to a normal (0,1)-distribution.

The test, as given by MANN and WHITNEY, consists in rejecting the hypothesis $G = F$ on the level of significance α if and only if $P[\mathbf{U} \leqq U(E) | G = F] \leqq \alpha$, where E is the sample point corresponding with the empirical data.

As for the power function of this test, MANN and WHITNEY remarked that it presents formidable difficulties. They proved however that their test is consistent [3]) with respect to the class of alternatives $G < F$.

2. THE POWER FUNCTION OF A STATISTICAL TEST IN GENERAL

2.1. *Critical region.*

A statistical hypothesis H_0 is tested (against an alternative hypothesis H) by dividing the sample space W into two parts, w and $(W - w)$, and applying the rule that H_0 is rejected if the sample point E falls in w and that H_0 is not rejected if E falls in $(W - w)$. w is called the "*critical region*" of the test. (cf. J. NEYMAN and E. S. PEARSON (4) or J. NEYMAN (3)).

2.2. *The concept of power function.*

The power function of a test, a concept introduced by NEYMAN and PEARSON (5) (cf. also J. NEYMAN (3)), is the probability $\alpha(H) = P[\mathbf{E} \in w | H]$, that the sample point $\mathbf{E}$ falls in the critical region w, calculated under any admissible hypothesis H. H_0 denoting the hypothesis tested (to which there may be an infinity of alternatives), the critical region w is chosen so that:

$$\alpha(H_0) = P[\mathbf{E} \in w \mid H_0] \text{ is equal to } \alpha.$$

---

[3]) A test is called "*consistent*" according to A. WALD and J. WOLFOWITZ (8), if and only if the probability of rejecting the hypothesis tested (here $G = F$) when it is false, tends to 1 as the sample size tends (here: sizes tend) to infinity.

Here α is a given positive number, the so-called level of significance. Apparently α is the probability that H_0 will be rejected (cf. the definition of w in 2. 1), when H_0 is true. Usually α is chosen small, as will be easily understood from I below.

Clearly, if among the many possible critical regions one would exist for which:

$$\text{I.} \quad P[\mathbf{E} \in w \mid H_0] = 0$$

$$\text{II.} \quad P[\mathbf{E} \in w \mid H] = 1 \text{ for all } H \neq H_0,$$

this one would be preferred for testing H_0.

Critical regions for which I and II hold cannot be realized, however, as soon as all sample points have a positive probability under both H_0 and H.

> Remark. w depends, of course, on the number of dimensions N of the sample space W. Making this dependence explicit by writing w_N for w, we see that a test is consistent (cf. the note at the end of 1) if and only if:
>
> $$\lim_{N \to \infty} P[\mathbf{E} \in w_N \mid H] = 1 \quad \text{for all} \quad H \neq H_0.$$
>
> Hence this asymptotic relation corresponding with II defines consistent tests.

2. 3. *The use of the power function.*

The power function is instrumental to judge the "goodness" of a test and to compare several tests.

α) A critical region w with $P[\mathbf{E} \in w \mid H_0] > P[\mathbf{E} \in w \mid H]$ for some alternative H, leads to the hypothesis H_0 being rejected more often under H_0 than under this alternative H. A test based on such a critical region is called "*biased*" (cf. NEYMAN and PEARSON (5) or NEYMAN (3)). An unbiased test is based on a critical region w with $P[\mathbf{E} \in w \mid H_0] < P[\mathbf{E} \in w \mid H]$ for *all* admissible alternatives H. Clearly an unbiased test is in general preferable to a biased one.

β) When two critical regions, w_1 and w_2 with $P[\mathbf{E} \in w_1 \mid H_0] = P[\mathbf{E} \in w_2 \mid H_0]$, both give unbiased tests, w_1 is a closer approximation to the ideal case as sketched in 2. 2 under II, than w_2 if:

$$P[\mathbf{E} \in w_1] \mid H] > P[\mathbf{E} \in w_2 \mid H] \text{ for } \textit{all} \text{ admissible } H \neq H_0.$$

Then the test based on w_1 is called *uniformly more powerful* than the test based on w_2.

> Remark. H specifies the joint distribution function of the N random coordinates of $\mathbf{E}$. So the power function is a function defined on a function space in the most general case, when all sorts of alternative H's are admitted. The class of admissible hypotheses H can be restricted to various degrees. In the most simplified case the only difference between the hypotheses H consists in the value of one parameter. Denote this parameter

by μ and the powerfunction $P[\boldsymbol{E} \in w \mid \mu]$ by $\alpha_w(\mu)$. Let H_0 consist in $\mu = 0$ and let μ be variable in an interval containing $\mu = 0$. If the first and second derivatives of $\alpha_w(\mu)$ for $\mu = 0$ exist and are denoted by $\alpha'_w(0)$ and $\alpha''_w(0)$ respectively, then a necessary condition for unbiasedness with respect to the alternatives $\mu > 0$ is, if $\alpha'_w(0) \neq 0$, that $\alpha'_w(0) > 0$ (or, if $\alpha'_w(0) = 0$, that $\alpha''_w(0) > 0$, etc.), whereas a necessary condition for unbiasedness with respect to the alternatives $\mu \neq 0$ is, that $\alpha'_w(0) = 0$ and, if $\alpha''_w(0) \neq 0$, that $\alpha''_w(0) > 0$ (or, if $\alpha''_w(0) = 0$, that $\alpha^{(3)}_w(0) = 0$ and $\alpha^{(4)}_w(0) > 0$, if these derivatives exist, etc.).

A necessary condition for w_1 being more powerful than w_2 is, with respect to the alternatives $\mu > 0$, that $\alpha'_{w_1}(0) > \alpha'_{w_2}(0)$, if $\alpha'_{w_1}(0) \neq \alpha'_{w_2}(0)$ (or, if $\alpha'_{w_1}(0) = \alpha'_{w_2}(0)$, that $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$, etc.), and with respect to the alternatives $\mu \neq 0$, if $\alpha''_{w_1}(0) \neq \alpha''_{w_2}(0)$, that $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$ (etc.).

3. THE POWER FUNCTION OF WILCOXON'S TEST

3. 1. *General remarks.*

Power functions of WILCOXON's test will be investigated under the following general restrictions on the *distribution functions* $F(x)$ and $G(x)$ (cf. 1):

a) $F(x)$ and $G(x)$ have continuous derivatives, $f(x)$ and $g(x)$, respectively, for all x-values with the exception at most of those bounding the infinite intervals (when present) for which $f(x)$ or $g(x)$ are zero.

b) $G(x) = F(x - \mu)$; $g(x) = f(x - \mu)$.

These restrictions will be assumed valid throughout the rest of the paper unless the contrary is mentioned. According to the restriction *b* the hypotheses H specify μ-values, but F is left unspecified.

The *critical regions* considered are defined by 1, 2 and 3 respectively:

1) $\boldsymbol{U} \leqq U_a$ with the level of significance a
2) $\boldsymbol{U} \geqq mn - U'_a$ with the level of significance a } with $U_a < \frac{1}{2} mn$.
3) $\left| \boldsymbol{U} - \frac{1}{2} mn \right| \geqq \frac{1}{2} mn - U_a$ with the level of significance $2a$ } (for U_a cf 3. 2)

The first region, the only one considered by MANN and WHITNEY, serves to test the hypothesis $\mu = 0$ against $\mu > 0$ and the second region serves to test $\mu = 0$ against $\mu < 0$ (one-sided sets of alternative hypotheses, shortly: "one-sided alternatives"), whereas the third region serves to test $\mu = 0$ against $\mu \neq 0$ ("two-sided alternatives"). Sometimes mathematical difficulties require restrictions on m and n together with restrictions on a and on U_a.

In order to compare the power function of WILCOXON's test with that of STUDENT's test for the difference of two means (under conditions which allow the use of STUDENT's test), in addition to *a*) and *b*) the further restriction on the distribution functions involved:

c) $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

will be imposed in a part of the paper.

3. 2. *The critical region of Wilcoxon's test.*

I. The critical region defined by $\mathbf{U} \leqq U_a$ $(U_a < \frac{1}{2} mn)$ consists of all sample points $\mathbf{E} = (\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ satisfying the inequality $\mathbf{U} \leqq U_a$, where U_a is the maximum of all integers U with $P[\mathbf{U} \leqq U \mid G = F] \leqq \alpha$.

This critical region, w, is the sum of $U_a + 1$ disjoint regions w_k $(0 \leqq k \leqq U_a)$, where w_k is the set of all sample points for which $U = k$. Each region w_k consists of $p'_{m,n}(k)$ disjoint subregions $w_{k,q}$ $(q = 1, \ldots, p'_{m,n}(k))$ with constant $x - y$ — arrangement [4]). For $p'_{m,n}(k)$ MANN and WHITNEY gave the recurrence relation

$$(3.2,1) \quad \begin{cases} p'_{m,n}(k) = p'_{m-1,n}(k-n) + p'_{m,n-1}(k) \\ \left(p'_{i,j}(k) = 0 \text{ if } k < 0;\ p'_{i,0}(k) = p'_{0,i}(k) = \begin{cases} 0, \text{ if } k \neq 0 \\ 1, \text{ if } k = 0 \end{cases}\right) \end{cases}$$

It is seen that $p'_{m,n}(0) = 1 = p'_{m,n}(1)$ if $m \neq 0$, and $n \neq 0$.

Clearly each subregion $w_{k,q}$ is built up out of $m!\,n!$ disjoint sub-subregions, the points of which are characterized by a constant permutation of their m x-coordinates and their n y-coordinates respectively in the constant $x - y$-arrangement corresponding to $w_{k,q}$.

Because of the continuity of the distribution functions of $\mathbf{x}_i$ and $\mathbf{y}_j$ the probability of $\mathbf{E}$ lying in the boundary of one of the above-mentioned regions is zero, so that the boundaries of these regions may be included without changing any probability calculated.

II. The critical region defined by $\mathbf{U} \geqq m\,n - U_a$ $(U_a < \frac{1}{2} m\,n)$ is the sum of $U_a + 1$ disjoint regions w_{mn-k} $(0 \leqq k \leqq U_a)$. The regions w_{mn-k} consist of subregions $w_{(mn-k),q}$ with constant $x - y$-arrangement, which are built up out of sub-subregions of constant permutations, as described for the region $\mathbf{U} \leqq U_a$.

a) By the substitutions $x_i = -x'_{i'}$ $(i, i' = 1, \ldots, m)$; $y_j = -y'_{j'}$ $(j, j' = 1, \ldots, n)$ a one-to-one correspondence is established between the points $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ of the space W and the points $E' = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n)$ of the space W'. Here a pair (i, j) with $x_i \gtrless y_j$ corresponds to a pair (i', j') with $x'_{i'} \lessgtr y'_{j'}$. Hence a point E with $U(E) = m\,n - k$ corresponds to a point E' with $U(E') = k$ $(U(E')$ being defined as the number of pairs (i', j') with $x'_{i'} > y'_{j'})$, and a region w_{mn-k} corresponds to a region w'_k, while the region in W defined by $\mathbf{U} \geqq m\,n - U_a$ corresponds to the region in W' defined by $\mathbf{U} \leqq U_a$.

Remark 1. It is easily seen that $p'_{m,n}(k) = p'_{m,n}(mn - k)$.

Remark 2. The points E of W have m x-coordinates and n y-coordinates.

The points E' of W' have m x'-coordinates and n y'-coordinates.

To make this fact explicit one can write $W_{m,n}$ instead of W and $W'_{m,n}$ instead of W'.

---

[4]) By this term is meant, that the same ordered arrangement of x and y corresponds to every point E of $w_{k,q}$, when the coordinates x_i $(i = 1, \ldots, m)$ and y_j $(j = 1, \ldots, n)$ of E are arranged in order according to increasing magnitude and the suffices of x and y are omitted.

β) A one-to-one correspondence between the points $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ of $W_{m,n}$ and the points $E' = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_m)$ of $W'_{n,m}$ is established by the substitutions $x_i = y'_{j'} + \mu$ $(i, j' = 1, \ldots, m)$ and $y_j = x'_{i'} + \mu$ $(j, i' = 1, \ldots, n)$. Now a pair (i, j) with $x_i \gtrless y_j$ corresponds to a pair (i', j') with $x'_{i'} \lessgtr y'_{j'}$. By the same argument as under α) it is seen that the region of $W_{m,n}$ defined by $U \geqq m\,n - U_a$ corresponds to the region of $W'_{m,n}$ defined by $U \leqq U_a$.

Remark 3. Clearly $p'_{n,m}(k) = p'_{m,n}(m\,n - k)$. Because of remark 1. it is seen that $p'_{n,m}(k) = p'_{m,n}(k)$.

III. The critical region defined by $|U - \frac{1}{2}\,m\,n| \geqq \frac{1}{2}\,m\,n - U_a$ $(U_a < \frac{1}{2}\,m\,n)$ is the sum of the two disjoint regions defined by $U \leqq U_a$ (cf. 3. 2, I) and by $U \geqq m\,n - U_a$ (cf. 3. 2, II), respectively.

3. 3. *General expressions for the power function of Wilcoxon's test.*

Under the restrictions a) and b) of 3.1 imposed on $F(x)$ and $G(x)$, the power function of WILCOXON's test is given by

$$(3.3,1) \qquad \alpha_-(\mu) = \int\limits_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\}$$

for testing $\mu = 0$ against $\mu > 0$,

$$(3.3,2a) \qquad \alpha_-(\mu) = \int\limits_{U \geqq mn - U_a} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\}$$

$$(3.3,2b) \qquad = \int\limits_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(-x_i)\,dx_i\} \prod_{j=1}^{n} \{f(-y_j - \mu)\,dy_j\}$$

$$(3.3,2c) \qquad = \int\limits_{U \leqq U_a} \cdots \int \prod_{i=1}^{n} \{f(x_i)\,dx_i\} \prod_{j=1}^{m} \{f(y_j + \mu)\,dy_j\}$$

for testing $\mu = 0$ against $\mu < 0$,

((3. 3, 2b) follows from (3. 3, 2a) by the substitutions of 3. 2, II, α); (3. 3, 2c) from (3. 3, 2a) by 3. 2, II, β)) and

$$(3.3,3) \qquad \alpha_\pm(\mu) = \alpha_+(\mu) + \alpha_-(\mu)$$

for testing $\mu = 0$ against $\mu \neq 0$.

When $U_a = 0$, one finds, denoting in this case $\alpha_+(\mu)$ by $\alpha^{(0)}_+(\mu)$, that

$$(3.3,4) \qquad \begin{cases} \alpha^{(0)}_+(\mu) = m!\,n! \int\limits_R \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} \\ \text{where } R \text{ is defined by the inequalities:} \\ -\infty < x_1 < x_2 < \ldots < x_m < y_1 < \ldots < y_n < +\infty. \end{cases}$$

From (3. 3, 2c) it follows that

$$(3.3,5a)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = m!\, n! \int\limits_{R} \dots \int \prod_{i=1}^{n} \{f(x_i)\, dx_i\} \prod_{j=1}^{m} \{f(y_j+\mu)\, dy_j\}, \\ \text{with } R \text{ defined by: } -\infty < x_1 < \dots < x_n < y_1 < \dots < y_m < +\infty \end{cases}$$

or, by replacing x_i by $x_i - \mu$ and y_j by $y_j - \mu$, that

$$(3.3,5b)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = m!\, n! \int\limits_{R} \dots \int \prod_{i=1}^{n} \{f(x_i-\mu)\, dx_i\} \prod_{j=1}^{m} \{f(y_j)\, dy_j\} \\ \text{with } R \text{ defined as in (3. 3, 5}a\text{).} \end{cases}$$

Remark. Even without the restriction b) of 3. 1, writing again $g(y)$ for $f(y-\mu)$, the $(m+n)$-fold integral (3. 3, 4) can be reduced to a single one. In fact, writing x for x_m, it is seen that $\alpha_{+}^{(0)}(\mu) =$

$$= m!\, n! \int\limits_{-\infty}^{+\infty} f(x)\, dx \int\limits_{-\infty < x_1 < \dots < x_{m-1} < x} \dots \int \prod_{i=1}^{m-1} \{f(x_i)\, dx_i\} \int\limits_{x < y_1 < \dots < y_n < +\infty} \dots \int \prod_{j=1}^{n} \{g(y_j)\, dy_j\}$$

Because of

$$\int\limits_{a < x_1 < \dots < x_k < b} \dots \int \prod_{i=1}^{k} \{\varphi(x_i)\, dx_i\} = \frac{1}{k!} \left[\int\limits_{a}^{b} \varphi(x)\, dx \right]^k$$

(cf. (7)) one finds:

$$(3.3,6)\quad \begin{cases} \alpha_{+}^{(0)}(\mu) = m \int\limits_{-\infty}^{+\infty} f(x) \cdot F^{m-1}(x) \cdot [1-G(x)]^n\, dx = \\ = \int\limits_{-\infty}^{+\infty} [1-G(x)]^n \cdot d\,[F^m(x)] = -\int\limits_{-\infty}^{+\infty} F^m(x) \cdot d\,[\{1-G(x)\}^n]. \end{cases}$$

In the same way it is found from (3. 3, 5b) that

$$(3.3,7)\quad \begin{cases} \alpha_{-}^{(0)}(\mu) = n \int\limits_{+\infty}^{+\infty} g(x) \cdot G^{n-1}(x) \cdot [1-F(x)]^m\, dx = \\ = \int\limits_{-\infty}^{+\infty} [1-F(x)]^m \cdot d\,[G^n(x)] = -\int\limits_{-\infty}^{+\infty} G^n(x) \cdot d\,[(1-F(x)\}^m] \end{cases}$$

For $\alpha_{+}^{(0)}(\mu)$ one can write instead of (3. 3, 4):

$$(3.3,8)\quad \alpha_{+}^{(0)}(\mu) = m!\, n! \int\limits_{-\infty}^{+\infty} f(x_1)\, dx_1 \int\limits_{x_1}^{+\infty} f(x_2)\, dx_2 \int\limits_{x_2}^{+\infty} \dots \int\limits_{x_m}^{+\infty} f(y_1-\mu)\, dy_1 \int\limits_{y_1}^{+\infty} \dots \int\limits_{y_{n-1}}^{+\infty} f(y_n-\mu)\, dy_n.$$

This expression will be written more shortly by the use of some operators and notations which will be defined in (3. 3, 9, 9a, 9b and 10):

$$(3.3,9)\quad \begin{cases} \int\limits_{y}^{+\infty} \varphi_1(x_1)\, dx_1 \int\limits_{x_1}^{+\infty} \varphi_2(x_2)\, dx_2 \int\limits_{x_2}^{+\infty} \dots \int\limits_{x_{p-1}}^{+\infty} \varphi_p(x_p)\, dx_p = \\ = [(I\varphi_1)(I\varphi_2) \dots (I\varphi_p)](y) = \left[\prod_{\nu=1}^{p} (I\varphi_\nu) \right](y). \end{cases}$$

If $\varphi_\nu(x) = f(x)$ $(\nu = 1, \ldots, k)$, the following abbreviation is used:

$$\left[\prod_{\nu=1}^{p} (I\varphi_\nu)\right](y) = [(If)^p](y). \tag{3.3,9a}$$

If $\varphi_\nu(x) = f(x)$ for all integers $\nu \neq i$ with $1 \leqq \nu \leqq p$ and $\varphi_i(x) = \{f(x)\}^\omega$, then the abbreviation used is:

$$\left[\prod_{\nu=1}^{p} (I\varphi_\nu)\right](y) = [(If)^{i-1}(If^\omega)(If)^{p-i}](y) \tag{3.3,9b}$$

Furthermore, the following notations are used:

$$\begin{cases} f(x-\mu) = f_{-\mu} \;\; ; \;\; f(x+\mu) = f_{+\mu} \;\; ; \;\; f'(x) = f' \;\; ; \\ f'(x-\mu) = f'_{-\mu} \;\; ; \;\; f'(x+\mu) = f'_{+\mu} \text{ and } f''(x) = f''. \end{cases} \tag{3.3,10}$$

With (3. 3, 9, 9a and 10) the expression (3. 3, 8) becomes:

$$a_+^{(0)}(\mu) = m!\,n!\,[(If)^m (If_{-\mu})^n](-\infty) \tag{3.3,11}$$

$$= m!\,n!\,[(If_{+\mu})^m (If)^n](-\infty) \tag{3.3,11a}$$

((3. 3, 11a) follows from (3. 3, 11) by the substitutions

$$x_i = x'_i + \mu \; (1 \leqq i \leqq m) \text{ and } y_j = y'_j + \mu \;\; (1 \leqq j \leqq n).)$$

In a similar way one finds for $a_+(\mu)$, for any (integer) value of U_a which is $\leqq \frac{1}{2} m\,n$:

$$a_-(\mu) = m!\,n! \sum_{U=0}^{U_a} \left[\prod_{h=1}^{r} \{(If)^{m_h} (If_{-\mu})^{n_h}\}\right](-\infty) \tag{3.3,12}$$

$$= m!\,n! \sum_{U=0}^{U_a} \left[\prod_{h=1}^{r} \{(If_{+\mu})^{m_h} (If)^{n_h}\}\right](-\infty) \tag{3.3,12a}$$

$$\begin{cases} \text{Here} \quad m_1 \geqq 0 \text{ and } m_h > 0 \text{ for } h > 0, \\ \qquad\quad n_h > 0 \text{ for } h < r \text{ and } n_r \geqq 0\,; \\ \sum_{h=1}^{r} m_h = m\,, \sum_{h=1}^{r} n_h = n \text{ and } U = \sum_{i=1}^{r} n_i \sum_{j=i+1}^{r} m_j \end{cases} \tag{3.3,12b}$$

$$\begin{cases} \text{The summations (over } U) \text{ in (3. 3, 12) and (3. 3, 12}a) \\ \text{are to be extended over } \textit{all} \text{ combinations of } r, \text{ and of} \\ m_h \text{ and } n_h \;\; (1 \leqq h \leqq r) \text{ which give a value of } U \leqq U_a. \end{cases} \tag{3.3,12c}$$

A similar expression follows for $a_-(\mu)$ by the use of (3. 3, 2c); $a_\pm(\mu)$ then follows from (3. 3, 3).

4. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE ONE-SIDED

4.1. *General theorems.*

4.11. *A theorem on the unbiasedness of the test.*

Theorem 1. *Under the restrictions a) and b) of* 3. 1 *the power function*

$\alpha_+(\mu)$ *given by* (3. 3, 1), *when* α, m *and* n *are constant, is a monotonous non-decreasing function of* μ.

Proof. [5]) By the substitutions $x_i = x'_{i'}$; $y_j = y'_{j'} + \mu$ $(i, i' = 1, \ldots, m;$ $j, j' = 1, \ldots, n)$ a one-to-one correspondence is established between the points $E = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ of the space W and the points $E' = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_n)$ of the space W'. When $U = U(E)$ is defined as in 1. and $U_\mu(E')$, for each point E' with $x'_{i'} \neq y'_{j'} + \mu$ for each pair (i', j'), is defined as the number of pairs (i', j') $(1 \leqq i' \leqq m;\ 1 \leqq j' \leqq n)$ with $x'_{i'} > y'_{j'} + \mu$, then $U(E) = U_\mu(E')$ if E and E' are corresponding points.

Hence

$$(4.1,1) \qquad \alpha_+(\mu) = \int\limits_{U(E) \leqq U_\alpha} \cdots\cdots \int \prod_{i=1}^{m} \{f(x_i)\, dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\, dy_j\}$$

$$(4.1,2) \qquad = \int\limits_{U_\mu(E') \leqq U_\alpha} \cdots\cdots \int \prod_{i'=1}^{m} \{f(x'_{i'})\, dx'_{i'}\} \prod_{j'=1}^{n} \{f(y'_{j'})\, dy'_{j'}\}.$$

Now the integrand of (4. 1, 2) is independent of μ and non-negative for every E'. From the definition of $U_\mu(E')$ follows that, in a fixed point E', $U_\mu(E')$ is a monotonous non-increasing function of μ. Hence the set of all points with $U_\mu(E') \leqq U_\alpha$ cannot decrease when μ increases. So $\alpha_+(\mu)$ is a monotonous non-decreasing function of μ.

Corollary. For all distribution functions F and G satisfying *a*) and *b*) of 3. 1, the critical region $\boldsymbol{U} \leqq U_\alpha$ provides an *unbiased* test of the hypothesis $\mu = 0$ against all alternatives $\mu > 0$ (cf 2. 3, *a*).

Remark. The same holds for the critical region $\boldsymbol{U} \geqq m\,n - U_\alpha$ when the alternatives are given by $\mu < 0$.

4. 12. *On the interchangeability of* m *and* n *in the power function.*

Theorem 2. *Under the restrictions a) and b) of* 3.1 *the power functions* $\alpha_+(\mu)$ *as well as* $\alpha_-(\mu)$ *given by* (3. 3, 1) and by (3. 3, 2) *respectively, when* α *is constant, are identical with those obtained when* m *and* n *are interchanged, provided* $f(x - c)$ *is an even function of* x *for some* c.

Proof. Without loss of generality one may suppose $c = 0$. Now the expression (3. 3, 2*b*) is equal to the expression (3. 3, 2*c*), when $f(x)$ is an even function. The proof of this equality given in 3. 2, II, *a*) and 3. 2, II, β), holds good when $+\mu$ is replaced by $-\mu$. The theorem can also be proved directly from (3. 3, 1) by the substitutions $x'_i = -y'_{j'} + \mu$ $(i, j' = 1, \ldots, m)$ and $y_j = -x'_{i'} + \mu$ $(j, i' = 1, \ldots, n)$.

4. 2. *Calculation of* $\alpha'_+(0)$ *for Wilcoxon's test with specialization to the normal distribution.*

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[5]) This proof is due to Mr J. Hemelrijk.

From the expression (3. 3, 12a) for $a_+(\mu)$, $a'_+(\mu) = \frac{da_+(\mu)}{d\mu}$ is easily calculated.

Defining the operator I^{-1} by

$$I^{-1}\varphi(x) = I^{-1}\varphi = -\varphi'(x) = -\varphi' \tag{4.2,1}$$

it follows from the definition of the operator I in (3. 3, 9) that

$$I^{-1}(I\varphi) = \varphi. \tag{4.2,2}$$

Furthermore defining

$$[\varphi_1(I\varphi_2)](y) = \varphi_1(y)\int_y^{+\infty}\varphi_2(x)\,dx \tag{4.2,3}$$

and

$$[(I\varphi_2)\varphi_1](y) = \int_y^{+\infty}\varphi_2(x)\,\varphi_1(x)\,dx \tag{4.2,4}$$

it is seen that

$$\begin{aligned}&[(If'_{+\mu})(I\varphi)](y) = -[f_{+\mu}(I\varphi)](y) + [If_{+\mu})\varphi](y) = \\ &= -[I^{-1}(If_{-\mu})(I\varphi)](y) + [(If_{+\mu})I^{-1}(I\varphi)](y).\end{aligned} \tag{4.2,5}$$

Hence

$$\begin{aligned}&\left[\frac{\partial}{\partial\mu}\{(If_{-\mu})^m\}\chi\right](y) = \sum_{l=0}^{m-1}[(If_{+\mu})^l(If'_{+\mu})(If_{+\mu})^{m-l-1}\chi](y) = \\ &= -\sum_{l=0}^{m-1}[If_{+\mu})^l I^{-1}(If_{+\mu})^{m-l}\chi](y) + \\ &\qquad + \sum_{l=0}^{m-1}[(If_{+\mu})^{l+1}I^{-1}(If_{+\mu})^{m-l-1}\chi](y) = \\ &= -[I^{-1}(If_{+\mu})^m\chi](y) + [(If_{+\mu})^m I^{-1}\chi](y).\end{aligned} \tag{4.2,6}$$

Now

$$a'_+(\mu) = m!\,n!\sum_{U=0}^{U_\alpha}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\cdot\right.$$
$$\left.\cdot\frac{\partial}{\partial\mu}\{(If_{+\mu})^{m_k}\}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty).$$

Hence, by means of (4. 2, 6):

$$\begin{aligned}a'_+(\mu) = &-m!\,n!\sum_{U=0}^{U_\alpha}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{+\mu})^{m_h}(If)^{n_h}\}I^{-1}(If_{+\mu})^{m_k}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty) \\ &+m!\,n!\sum_{U=0}^{U_\alpha}\sum_{k=1}^{r}\left[\prod_{h=1}^{k-1}\{(If_{-\mu})^{m_h}(If)^{n_h}\}(If_{+\mu})^{m_k}I^{-1}(If)^{n_k}\prod_{h=k+1}^{r}\{(If_{+\mu})^{m_h}(If)^{n_h}\}\right](-\infty).\end{aligned} \tag{4.2,7}$$

If the expression between [ ] begins or ends with I^{-1} for some k, then the corresponding term is to be considered as zero.

From (4. 2, 7) it follows that:

$$\begin{aligned}a'_+(0) = &-m!\,n!\sum_{U=0}^{U_\alpha}\sum_{k=1}^{r}[(If)^{L_k-1}(If^2)(If)^{m_k+n_k-1+M_k}](-\infty) \\ &+m!\,n!\sum_{U=0}^{U_\alpha}\sum_{k=1}^{r}[(If)^{L_k+m_k-1}(If^2)(If)^{n_k-1+M_k}](-\infty).\end{aligned} \tag{4.2,8}$$

Here $L_k = \sum_{h=1}^{k-1} (m_h + n_h)$ and $M_k = \sum_{h=k+1}^{r} (m_h + n_h)$; for m_h, n_h, r and $\sum_U$ cf. (3. 3, 12*b*) and (3. 3, 12*c*). Those terms in the sums of (4. 2, 8) in which the first or the last symbolic power has a negative exponent are to be considered as zero.

The expression (4. 2, 8) for $\alpha'_+(0)$ will be calculated for $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ (cf. *c*) in 3. 1) from the Appendix; cf. (A. 1, 1) and (A. 1, 9). It is seen from A. 1, Remark 2, that the calculations will be restricted to $\tau \leqq 4$, i.e. to $m + n \leqq 5$. Furthermore, only relatively low values of α are considered (cf. 2. 2). If $U_\alpha = 0$ determines an $\alpha > 0{,}15$, then only $U_\alpha = 0$ is considered, otherwise $U_\alpha = 1$, too, will be considered. Cases with $m = 1$ or $n = 1$ are omitted as being trivial. By means of (5. 2, 8), (A. 1, 1) and (A. 1, 9) the following results are obtained:

$$(4.2,9)\quad \left\{ \begin{array}{l} \textit{When } U_\alpha = 0,\ \alpha'_+(0) = m!\, n!\, [(If)^{m-1}\,(If^2)\,(If)^{n-1}]\,(-\infty) = \\ \qquad = \dfrac{m!\, n!\, (m+n-1)\,.\,\Gamma\left(\dfrac{m+n-1}{2}\right)}{2^2\,.\,\pi^{\frac{1}{2}(m+n)}} \cdot V_{1^{m-1},2,1^{n-1}} \end{array} \right.$$

$$(4.2,10)\quad \left\{ \begin{array}{l} \textit{When } U_\alpha = 1,\ \alpha'_+(0) = \\ = +m!\, n![(If)^{m-2}(If^2)\,(If)^n](-\infty) + m!\, n![(If)^m\,(If^2)\,(If)^{n-2}]\,(-\infty) = \\ = \dfrac{m!\, n!\, (m+n-1)\,.\,\Gamma\left(\dfrac{m+n-1}{2}\right)}{2^2\,.\,\pi^{\frac{1}{2}(m+n)}} \cdot \{V_{1^{m-2},2,1^n} + V_{1^m,2,1^{n-2}}\} \end{array} \right.$$

From (4. 2, 9) and (4. 2, 10) the following results are obtained by means of (A. 3, 2) and (A. 4, 4):

m	n	U_α	α	$\alpha'_+(0)$
2	2	0	$\frac{1}{6}$	$\frac{3}{2\pi\sqrt{\pi}} V_{1,2,1} = \frac{1}{\pi\sqrt{\pi}} \arccos\frac{1}{3} = 0{,}22106$
2	3	0	$\frac{1}{10}$	$\frac{12}{\pi^2\sqrt{\pi}} V_{1,2,1\,1} = \frac{3}{2\pi\sqrt{\pi}} \arcsin\frac{\sqrt{3}}{3} = 0{,}16580$
2	3	1	$\frac{1}{5}$	$\frac{12}{\pi^2\sqrt{\pi}} (V_{2,1,1,1} + V_{1,1,2,1}) =$
				$= \frac{3}{2\pi\sqrt{\pi}} \left(\arcsin\frac{\sqrt{18}-\sqrt{3}}{6} + \arcsin\frac{\sqrt{3}}{3}\right) = \frac{1}{2\sqrt{\pi}} = 0{,}28209.$

Remark 1. For the values of α cf. MANN and WHITNEY or 3. 2.

Remark 2. Because of theorem 2 (4. 12) it is not necessary to calculate separately the case $m = 3$, $n = 2$.

4. 3. *A comparison with* $\alpha'_+(0)$ *for Student's test for the difference of two means.*

The alternatives to the hypothesis tested ($\mu = 0$) are $\mu > 0$ (therefore the notation α_+ is used, cf. (3. 3, 1)). Let $\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1 \ldots, \mathbf{y}_n$ be

$m+n=N$ independent random variables, the x_i all having the distribution function $F(x)$ and the y_j all having the distribution function $F(x-\mu)$ with $F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}\xi^2}\,d\xi$. The critical region for testing, according to STUDENT's test, the hypothesis $\mu=0$ against $\mu>0$ at the level of significance α is given by $t \geqq t_\alpha$.

Here t is defined by

$$t=\sqrt{\frac{mn\,(N-2)}{N}}\cdot\frac{\bar{y}-\bar{x}}{\sqrt{ms_x^2+ns_y^2}},$$

where

$$\bar{y}=\frac{1}{n}\sum_{j=1}^{n} y_j;\ \bar{x}=\frac{1}{m}\sum_{i=1}^{m} x_i;\ m\,s_x^2=\sum_{i=1}^{m}(x_i-\bar{x})^2$$

and $n\,s_y^2=\sum_{j=1}^{n}(y_j-\bar{y})^2$, whereas t_α is defined by:

$$\alpha=\frac{1}{\sqrt{(N-2)\,\pi}}\cdot\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}\int_{t_\alpha}^{\infty}\left(1+\frac{x^2}{N-2}\right)^{-\frac{N-1}{2}}dx.$$

t_α can be found from the tables of STUDENT's distribution. For the values of $\alpha=1/6$, $1/10$ and $1/5$ t_α has been tabulated in 3 decimal places only, or not at all. Therefore the quantity $\eta_\alpha=\frac{N-2}{N-2+t_\alpha^2}$ was calculated directly from

$$\alpha=\tfrac{1}{2}\,I_{\eta_\alpha}\left(\frac{N-2}{2},\frac{1}{2}\right), \tag{4.3,1}$$

where $I_x(p,q)$ is the incomplete B-function tabulated by K. PEARSON (6).

The power function of the critical region $t \geqq t_\alpha$ is, analagously to P. L. HSU (1):

$$\alpha_+(\mu)=\begin{cases}\dfrac{e^{-\frac{1}{2}\frac{mn}{N}\mu^2}}{2\sqrt{\pi}\cdot\sqrt{N-2}\cdot\Gamma\left(\frac{N-2}{2}\right)}\cdot\sum_{k=0}^{\infty}\left[\dfrac{2\,mn}{N\,(N-2)}\right]^{\frac{k}{2}}\cdot \\ \cdot\dfrac{\mu^k\cdot\Gamma\left(\frac{k+N-1}{2}\right)}{k!}\int_{t_\alpha^2}^{\infty}x^{\frac{k-1}{2}}\left(1+\dfrac{x}{N-2}\right)^{-\frac{k+N-1}{2}}dx\end{cases} \tag{4.3,2}$$

Hence

$$\alpha_+'(0)=\sqrt{\frac{mn}{2\pi N}}\cdot\left(\frac{N-2}{N-2+t_\alpha^2}\right)^{\frac{N-2}{2}}=\sqrt{\frac{mn}{2\pi N}}\cdot\eta_\alpha^{\frac{N-2}{2}} \tag{4.3,3}$$

From (4. 3, 3) and (4. 3, 1) the following results are obtained:

m	n	α	η_α	$\alpha'_+(0)$	$\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi}$
2	2	1/6	5/9	0,22163	0,00057
2	3	1/10	0,527963	0,16765	0,00185
2	3	1/5	0,758072	0,28845	0,00636

In order to facilitate the comparison with $\alpha'_+(0)$ for WILCOXON's test, a column is added containing the difference of $\alpha'_+(0)$ for the test of STUDENT and for the test of WILCOXON. It is seen that STUDENT's test and WILCOXON's test satisfy a necessary condition that STUDENT's test is more powerful than WILCOXON's test ($\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi} > 0$, cf. the end of the remark in 2. 3), but clearly the difference is very small.

SOME REMARKS ON THE POWER FUNCTION OF WILCOXON's TEST FOR THE PROBLEM OF TWO SAMPLES. III

BY

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5. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE TWO-SIDED

5. 1. *General theorems when alternatives are two-sided.*

5. 11. *Properties of symmetry of the power function.*

Theorem 3. *Under the restrictions a) and b) of* 3. 1 *the power function* $a_{\pm}(\mu)$ *given by* (3. 3, 3), *when a, m and n are constant, is an even function of* μ, *either when* 1) $f(x-c)$ *is an even function of* x *for some* c, *or when* 2) $m = n$ (*or when both conditions are satisfied, of course*).

Proof.

1. From (3. 3, 3, 1 and 2*b*) one obtains:

$$a_{\pm}(\mu) = \int\limits_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(x_i)\, dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\, dy_j\} +$$

$$+ \int\limits_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(-x_i)\, dx_i\} \prod_{j=1}^{n} \{f(-y_j - \mu)\, dy_j\}.$$

Without loss of generality c may be supposed to be equal to zero.

Then $f(x)$ is an even function. Hence:

$$(5.1,1)\quad \begin{cases} \alpha_{\pm}(\mu) = \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} + \\ \qquad + \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j + \mu)\,dy_j\}. \end{cases}$$

By changing $+\mu$ into $-\mu$ the first term of the second member of (5. 1, 1) passes into the second one and vice versa. Hence $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$, q.e.d.

2. From (3. 3, 3, 1 and 2*c*) one obtains:

$$\alpha_{\pm}(\mu) = \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} + \\ + \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{n} \{f(x_i)\,dx_i\} \prod_{j=1}^{m} \{f(y_j + \mu)\,dy_j\}.$$

When $m = n$, it follows that:

$$(5.1,2)\quad \begin{cases} \alpha_{\pm}(\mu) = \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{m} \{f(y_j - \mu)\,dy_j\} + \\ \qquad + \int \dots \int\limits_{U \leqq U_a} \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{m} \{f(y_j + \mu)\,dy_j\}. \end{cases}$$

By changing $+\mu$ into $-\mu$ the first term of the second member of (5. 1, 2) passes into the second one and vice versa. Hence $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$, q.e.d.

Corollary. If the conditions of theorem 3 are satisfied and $\alpha'_{\pm}(0)$ exists, then $\alpha'_{\pm}(0) = 0$.

When $f(x)$ is a non-symmetric function and $m \neq n$, then $\alpha_{\pm}(\mu)$, defined by (3. 3, 3), need not be an even function of μ, as follows from theorem 4.

Theorem 4. *Under the restrictions a) and b) of* 3. 1 *a non-symmetric function* $f(x)$ *and a value of* U_a *can be given, such that* $\alpha'_{\pm}(0)$ *exists and is* $\neq 0$, *if* $m \neq n$.

Proof.

Let $F(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ 1 - e^{-x} & \text{for } x \geqq 0 \end{cases}$, $G(x) = F(x - \mu)$ and $U_a = 0$.

Then by somewhat laborious calculations it can be proved, using (3. 3, 6 and 7), that:

$$(5.1,3)\quad \alpha_{\pm}(\mu) = \begin{cases} \frac{m!\,n!}{(m+n)!}\,e^{n\mu} + (1 - e^{\mu})^n + n e^{-m\mu} \int\limits_0^{e^{\mu}} (1-x)^{n-1} x^m\,dx, & \text{if } \mu < 0. \\ \frac{m!\,n!}{(m+n)!}\,e^{-m\mu} + (1 - e^{-\mu})^m + m e^{n\mu} \int\limits_0^{e^{-\mu}} (1-x)^{m-1} x^n\,dx, & \text{if } \mu > 0. \end{cases}$$

Hence

$$(5.1,4) \qquad a'_{\pm}(0) = \frac{m!\,n!}{(m+n)!}(n-m) \neq 0, \text{ q. e. d.}$$

5.12 *A theorem on possible biasedness of the test when alternatives are two-sided.*

As the critical region given by $|U - \frac{1}{2}mn| \geqq \frac{1}{2}mn - U_a$ cannot possibly provide an unbiased test for the hypothesis $\mu = 0$ against all alternatives $\mu \neq 0$, when $a'_{\pm}(0)$ exists and is $\neq 0$, one should, if possible, take $m = n$ in the applications of the test. In this way one secures the symmetry of the power function. One might hope that unbiasedness then would be secured, too. One might think that as general a theorem would hold good for two-sided alternatives, as theorem 1 and its corollary proved in 4.1, for onesided alternatives. That this is impossible in such a generality, is shown by theorem 5:

Theorem 5. *Under the restrictions a) and b) of* 3.1 *for every m and* $n = m$ *an even function* $f(x)$ *and a value of* U_a *can be given, such that* $a''_{\pm}(0)$ *exists and is* < 0.

Proof. Let $U_a = 0$. By (3.3, 3, 1 and 2*b*) the power function $a_{\pm}(\mu)$, when $f(x)$ is an even function and $m = n$, is equal to (5.1, 2), where now "$U \leqq U_a$" is to replaced by "$U = 0$". By a reduction as described in the remark of 3.3 (under (3.3, 5*b*)) it is found that:

$$(5.1,5) \qquad \begin{cases} a_{\pm}(\mu) = \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1-F(x-\mu))^m]\, dx + \\ + \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1-F(x+\mu))^m]\, dx. \end{cases}$$

Hence, writing $a''_{\pm}(\mu)$ for $\frac{d^2 a_{\pm}(\mu)}{d\mu^2}$,

$$(5.1,6) \qquad a''_{\pm}(0) = 2 \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1-F(x))^m]''\, dx,$$

the primes denoting differentiation with respect to x. Because of $f(x)$ being even, $F(-y) = 1 - F(y)$. Hence:

$$(5.1,7) \qquad \int_{-\infty}^{0} [F^m(x)]' \cdot [(1-F(x))^m]''\, dx = -\int_{0}^{+\infty} [(1-F(y))^m]' \cdot [F^m(y)]''\, dy,$$

as is seen by the substitution $x = -y$. (The primes in the second member of (5.1, 7) denote differentiation with respect to y). By partial integration the second member of (5.1, 7) is seen to be equal to

$$+ \int_{0}^{+\infty} [F^m(y)]' \cdot [(1-F(y))^m]''\, dy,$$

so that (cf. (5.1, 6) and (5.1, 7))

$$(5.1,8) \qquad a''_{\pm}(0) = 4 \int_{0}^{+\infty} [F^m(x)]' \cdot [(1-F(x))^m]''\, dx.$$

Now $[F^m(x)]' \geqq 0$ for every x with $0 \leqq x \leqq \infty$. Hence a sufficient condition for $a''_{\pm}(0)$ being < 0 is that $[(1-F(x))^m_-]'' \leqq 0$ for $0 \leqq x \leqq \infty$, where the equality-sign does not hold in the whole interval $(0, \infty)$.

Such a function $F(x)$, the derivative $f(x)$ of which is an even function, is defined by

$$(5.\,1,\,9) \qquad F(x) = \left\{ \begin{array}{lll} 0 & \text{for} & x \leqq -1 \\ \frac{1}{2}(1-x^2)^{1/m} & \text{for} & -1 \leqq x \leqq 0 \\ 1-\frac{1}{2}(1-x^2)^{1/m} & \text{for} & 0 \leqq x \leqq +1 \\ 1 & \text{for} & x \geqq +1 \end{array} \right\}$$

For $[(1-F(x))^m]'' = 2^{-m}(1-x^2)'' = -2^{1-m}$, if $0 \leqq x \leqq +1$. Hence $a''_{\pm}(0) = -2^{3-m}\left(1-\frac{1}{2^m}\right) < 0$.

Conclusion. The necessary condition for unbiasedness (consisting in $a''_{\pm}(0)$ not being < 0, cf. 2. 3, remark) is not satisfied in general. Hence the critical region defined by $|U - \frac{1}{2}mn| \geqq \frac{1}{2}mn - U_a$ does not provide an unbiased test for the hypothesis $\mu = 0$ against $\mu \neq 0$ without further restrictions being imposed upon F and G.

5. 13. *On the interchangeability of m and n in the power function when alternatives are two-sided.*

Theorem 6. *Under the restrictions a) and b) of* 3.1 *the power function $\alpha_{\pm}(\mu)$, given by* (3. 3, 3), *when α is constant, is identical with the power function $\alpha_{\pm}(\mu)$ obtained when m and n are interchanged, provided $f(x-c)$ is an even function of x for some c.*

Proof. (3. 3, 3) and theorem 2.

5. 2. *Calculation of $a''_{\pm}(0)$ for Wilcoxon's test with specialization to the normal distribution.*

Throughout 5. 2 $f(x)$ is assumed to be an even function. After (5. 2, 8*b*) $f(x)$ is taken to be equal to $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$. From (3. 3, 3, 1 and 2*b*) it follows that ($f(x)$ being an even function): $\alpha_{\pm}(\mu) = \alpha_+(\mu) + \alpha_-(\mu) =$

$$= \int_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} \{f(y_j - \mu)\,dy_j\} +$$

$$+ \int_{U \leqq U_a} \cdots \int \prod_{i=1}^{m} \{f(x_i)\,dx_i\} \prod_{j=1}^{n} f(y_j + \mu)\,dy_j\}.$$

Hence $\alpha_{\pm}(\mu) = \alpha_+(\mu) + \alpha_+(-\mu)$, so that $\alpha_{\pm}(0) = 2\,\alpha_+(0) = 2\,\alpha$
$a'_{\pm}(0) = a'_+(0) - a'_+(0) = 0$ and $a''_{\pm}(0) = 2\,a''_+(0)$.

Now $a''_+(0)$ is calculated, starting from (4. 2, 7):
The operator I^{-2} is defined by

$$(5.\,2,\,1) \qquad I^{-2}\varphi(x) = I^{-1}I^{-1}\varphi = -I^{-1}\varphi' = +\varphi''(x) = +\varphi''.$$

From the definitions in 4. 2 it then follows that

$$(5.2,2)\quad \begin{cases} \left[\frac{\partial}{\partial\mu}\{I^{-1}(If_{+\mu})^m\}\chi\right](y) = \left[I^{-1}\frac{\partial}{\partial\mu}\{(If_{+\mu})^m\}\chi\right](y) = (by\ (4.2,6)) = \\ = -[I^{-2}(If_{+\mu})^m\chi](y) + [I^{-1}(If_{+\mu})^m I^{-1}\chi](y). \end{cases}$$

Furthermore from (5. 2, 1) and (4. 2, 6):

$$(5.2,3)\quad \left[\frac{\partial}{\partial\mu}(If_{+\mu})^m I^{-1}\chi\right](y) = -[I^{-1}(If_{+\mu})^m I^{-1}\chi](y) + [(If_{+\mu})^m I^{-2}\chi](y).$$

From (4. 2, 7), by means of (4. 2, 6), (5. 2, 2) and (5. 2, 3), one obtains: $a''_{\pm}(0) = 2\,a''_{+}(0) =$

$$(5.2,4) = \begin{cases} 4\,m!\,n! \sum_{U=0}^{U_a} \sum_{z=2}^{2r-1} \sum_{w=1}^{z-1} [(-1)^{w+z}(If)^{P_w} I^{-1}(If)^{Q_{w,z}} I^{-1}(If)^{R_z}](-\infty) \\ + 2\,m!\,n! \sum_{U=0}^{U_a} \sum_{w=1}^{2r-1} [(If)^{P_w} I^{-2}(If)^{S_w}](-\infty). \end{cases}$$

Here:

$$P_w = \begin{cases} \sum_{h=1}^{\frac{1}{2}w} (m_h + n_h), & \text{if } w \text{ is even.} \\ \sum_{h=1}^{\frac{1}{2}(w-1)} (m_h + n_h) + m_{\frac{1}{2}(w+1)}, & \text{if } w \text{ is odd.} \end{cases}$$

$$R_z = \sum_{h=1}^{r} (m_h + n_h) - P_z = \begin{cases} \sum_{h=\frac{1}{2}z+1}^{r} (m_h + n_h), & \text{if } z \text{ is even.} \\ n_{\frac{1}{2}(z+1)} + \sum_{h=\frac{1}{2}(z+3)}^{r} (m_h + n_h), & \text{if } z \text{ is odd.} \end{cases}$$

$$Q_{w,z} = \sum_{h=1}^{r} (m_h + n_h) - P_w - R_z.$$

$$S_w = \sum_{h=1}^{r} (m_h + n_h) - P_w.$$

If in (5. 2, 4) the expression between [ ] begins or ends with I^{-1} or I^{-2} for some w or z, then the corresponding term is to be considered as zero.

Remark 1. The summation variables w and z should not be confounded, of course, with the w of 2. 1 (critical region) and the z (coordinates) of the Appendix.

As to the meaning of m_h, n_h and r, cf. (3. 3, 12b).

The summations over U in (5. 2, 4) are to be extended over *all* combinations of r, and of m_h and n_h $(1 \leqq h \leqq r)$ which give a value of $U \leqq U_a$. If $U = 0$, then $r = 1$ and the sum over z in (5. 2, 4) is empty. Hence in the first term of (5. 2, 4) $\sum_{U=0}^{U_a}$ can be replaced by $\sum_{U=1}^{U_a}$.

Remark 2. Introducing the notations:

$$d_{2h-1} = m_h,\ d_{2h} = n_h, \text{ and}$$

$$f_{\mu,j} = \begin{cases} f \quad (= f(x)), & \text{if } j \text{ is even,} \\ f_{+\mu}\,(= f(x+\mu)), & \text{if } j \text{ is odd,} \end{cases}$$

$\alpha''_{\pm}(\mu)$ can be written as:

$$(5.\,2,\,5)\quad \begin{cases} 4\,m!\,n! \sum\limits_{U=1}^{U_a} \sum\limits_{z=2}^{2r-1} \sum\limits_{w=1}^{z-1} \left[\prod\limits_{j=1}^{w} \{(If_{\mu,j})^{d_j}\}\, I^{-1} \prod\limits_{j=w+1}^{z} \{(If_{\mu,j})^{d_j}\}\, I^{-1} \prod\limits_{j=z+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \\ + 2\,m!\,n! \sum\limits_{U=0}^{U_r} \sum\limits_{w=1}^{2r-1} \left[\prod\limits_{j=1}^{w} \{(If_{\mu,j})^{d_j}\}\, I^{-2} \prod\limits_{j=w+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \text{ (end of Remark 2.)} \end{cases}$$

In the applications of (5. 2, 4 or 5) the following reductions are to be used:

$$(5.\,2,\,6)\quad \begin{cases} [(If)^K\, I^{-1}\, (If)^L\, I^{-1}\, (If)^M]\,(-\infty) = \\ = \begin{cases} [(If)^{K-1}\, (If^2)\, (If)^{L-2}\, (If^2)\, (If)^{M-1}]\,(-\infty), & \text{if } L \geqq 2, \\ [(If)^{K-1}\, (If^3)\, (If)^{M-1}]\,(-\infty), & \text{if } L = 1 \end{cases} \end{cases}$$

and: $[(If)^K\, I^{-2}\, (If)^L]\,(-\infty) =$

$$(5.\,2,\,7)\quad \begin{cases} = [(If)^K\, I^{-1}\, f(If)^{L-1}]\,(-\infty) = \\ = -[(If)^K\, f'(If)^{L-1}]\,(-\infty) + [(If)^K\, ff\,(If)^{L-2}]\,(-\infty) = \\ = -[(If)^{K-2}\, (If)\, (Iff')\, (If)^{L-1}]\,(-\infty) + [(If)^{K-1}\, (If^3)\, (If)^{L-2}]\,(-\infty) = \\ = +\tfrac{1}{2}[(If)^{K-2}\, (If^3)\, (If)^{L-1}]\,(-\infty) + \tfrac{1}{2}[(If)^{K-1}\, (If^3)\, (If)^{L-2}]\,(-\infty). \end{cases}$$

By means of (5. 2, 6) and (5. 2, 7) it follows from (5. 2, 4) that $\alpha''_{\pm}(0) =$

$$(5.\,2,\,8)\quad \begin{cases} 4\,m!\,n! \sum\limits_{U=1}^{U_a} \sum\limits_{z=2}^{2r-1} \sum\limits_{w=1}^{z-1} [(-1)^{w+z}\, (If)^{P_w - 1}\, (If^2)\, (If)^{Q_{w,z}-2}\, (If^2)\, (If)^{R_z-1}]\,(-\infty) \\ + m!\,n! \sum\limits_{U=0}^{U_a} \sum\limits_{w=1}^{2r-1} [(If)^{P_w-2}\, (If^3)\, (If)^{S_w-1}]\,(-\infty) \\ + m!\,n! \sum\limits_{U=0}^{U_a} \sum\limits_{w=1}^{2r-1} [(If)^{P_w-1}\, (If^3)\, (If)^{S_w-2}]\,(-\infty). \end{cases}$$

For P_w, $Q_{w,z}$, R_z and S_w and for the summations over U see the indications under (5. 2, 4).

If $Q_{w,z} = 1$ for some w or z, then the corresponding term in the first sum of the second member of (5. 2, 8) is to be replaced by:

$$(5.\,2,\,8\alpha)\quad 4\,m!\,n!\,[(-1)^{w+z}\, (If)^{P_w-1}\, (If^3)\, (If)^{R_z-1}]\,(-\infty).$$

In (5. 2, 6), (5. 2, 7) and (5. 2, 8) those terms in which the first or the last symbolic power has a negative exponent, are to be considered as zero.

The expression (5. 2, 8) for $a''_{\pm}(0)$ will be calculated for $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-1/2\, x^2}$ (cf. *c*) in 3. 1) by means of the Appendix; cf. (A. 1, 1) and (A. 1, 9). It is seen from A. 1, Remark 2, that the calculations will be restricted to $\tau \leqq 4$, i.e. to $m + n \leqq 6$. Furthermore, only relatively low values of 2α are considered (cf. 2. 2; $2\alpha = \alpha_{\pm}(0)$, see 5. 2, some lines before (5. 2, 1)). If $U_\alpha = 0$ determines a value of $2\alpha > 0{,}15$, then only $U_\alpha = 0$ is considered, otherwise $U_\alpha = 1$, too, will be considered. Cases with $m = 1$ or $n = 1$ are omitted as being trivial. By means of (5. 2, 8), (A. 1, 1) and A. 1, 9) the following results are obtained:

$$(5.2,9)\qquad \left\{\begin{array}{l} \textit{When } U_\alpha = 0,\ a''_{\pm}(0) = \begin{cases} + m!\, n!\, [(If)^{m-2}\, (If^3)\, (If)^{n-1}]\, (-\infty) + \\ + m!\, n!\, [(If)^{m-1}\, (If^3)\, (If)^{n-2}]\, (-\infty) = \end{cases} \\ = \dfrac{m!\, n!\, (m+n-2)\, .\, \Gamma\left(\dfrac{m+n-2}{2}\right)}{2^2\, .\, \pi^{\frac{m+n}{2}}\, .\, \sqrt{3}}\, [V_{1^{m-2},3,1^{n-1}} + V_{1^{m-1},3,1^{n-2}}]. \end{array}\right.$$

$$(5.2,10)\qquad \left\{\begin{array}{l} \textit{When } U_\alpha = 1,\ a''_{\pm}(0) = m!\, n!\, \{+ 4\, [(If)^{m-2}\, (If^2)^2\, (If)^{n-2}]\, (-\infty)\} + \\ + m!\, n!\, \{+ [(If)^{m-3}\, (If^3)\, (If)^n]\, (-\infty) - [(If)^{m-2}\, (If^3)\, (If)^{n-1}]\, (-\infty)\} + \\ + m!\, n!\, \{- [(If)^{m-1}\, (If^3)\, (If)^{n-2}]\, (-\infty) + [(If)^m\, (If^3)\, (If)^{n-3}]\, (-\infty)\} = \\ = \dfrac{m!\, n!\, (m+n-2)\, .\, \Gamma\left(\dfrac{m+n-2}{2}\right)}{2^2\, .\, \pi^{\frac{m+n}{2}}}\, [2 \cdot V_{1^{m-2},2,2,1^{n-2}} + \dfrac{1}{\sqrt{3}} \cdot V_{1^{m-3},3,1^n} - \\ - \dfrac{1}{\sqrt{3}} \cdot V_{1^{m-2},3,1^{n-1}} - \dfrac{1}{\sqrt{3}} \cdot V_{1^{m-1},3,1^{n-2}} + \dfrac{1}{\sqrt{3}} \cdot V_{1^m,3,1^{n-3}}]. \end{array}\right.$$

Those terms in the last members of (5. 2, 9) and (5. 2, 10) in which a symbolic power has a negative exponent, are to be considered as zero.

From (5. 2, 9) and (5. 2, 10) the following results are obtained by means of (A. 2, 1), (A. 3, 2) and (A. 4, 4):

m	n	U_α	2α	$a''_{\pm}(0)$
2	2	0	$\frac{1}{3}$	$\frac{2}{\pi^2\sqrt{3}}\, [V_{3,1} + V_{1,3}] = \frac{2}{\pi\sqrt{3}} = 0{,}36755$
2	3	0	$\frac{1}{5}$	$\frac{3\sqrt{3}}{2\pi^2}\, [V_{3,1,1} + V_{1,3,1}] = \frac{\sqrt{3}}{\pi^2}\left[\arccos\frac{\sqrt{6}}{4} + \arccos\frac{1}{4}\right]$ $= \frac{\sqrt{3}}{\pi^2}\left[\pi - \arccos\frac{\sqrt{6}}{4}\right] = 0{,}39132$
2	4	0	$\frac{2}{15}$	$\frac{16\sqrt{3}}{\pi^3}\, [V_{3,1,1,1} + V_{1,3,1,1}] =$ $= \frac{2\sqrt{3}}{\pi^2}\left[\arcsin\frac{\sqrt{30} - \sqrt{6}}{8} + \arcsin\frac{\sqrt{6}}{4}\right] =$ $= \frac{2\sqrt{3}}{\pi^2}\arccos\frac{1}{2} = \frac{2\sqrt{3}}{3\pi} = 0{,}36755$

m	n	U_α	2α	$\alpha''_\pm(0)$
2	4	1	$\frac{4}{15}$	$\frac{6}{\pi^2}\left[2\,V_{2,2,1,1}-\frac{1}{\sqrt{3}}\,V_{3,1,1,1}\right]=$
				$=\frac{6}{\pi^2}\left[2\arcsin\frac{\sqrt{20}-\sqrt{2}}{6}-\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{30}-\sqrt{6}}{8}\right]$
				$=0{,}51398.$
3	3	0	$\frac{1}{10}$	$\frac{24\sqrt{3}}{\pi^3}\,V_{1,3,1,1}=\frac{3\sqrt{3}}{\pi^2}\arcsin\frac{\sqrt{6}}{4}=0{,}34698.$
3	3	1	$\frac{1}{5}$	$\frac{72}{\pi^2}\left[V_{1,2,2,1}+\frac{1}{\sqrt{3}}\,V_{3,1,1,1}-\frac{1}{\sqrt{3}}\,V_{1,3,1,1}\right]=$
				$=\frac{9}{\pi^2}\left[\arcsin\frac{2}{3}+\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{30}-\sqrt{6}}{8}-\right.$
				$\left.-\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{6}}{4}\right]=0{,}52282.$

Remark 3. For the values of α cf. MANN and WHITNEY (2) or 3. 2.

Remark 4. Because of theorem 6 (5. 13) it is not necessary to calculate separately the case $\begin{cases} m=3,\ n=2 \\ m=4,\ n=2 \end{cases}$.

5. 3. *A comparison with $\alpha''_\pm(0)$ for Student's test for the difference of two means.*

The alternatives to the hypothesis tested ($\mu=0$) are $\mu\neq 0$ (therefore the notation $\alpha_\pm$ is used, cf. (3. 3, 3)). Under the same assumptions about $x_1,\ldots,x_m, y_1,\ldots,y_n$ as in 4. 3, the critical region for testing, according to STUDENT's test, the hypothesis $\mu=0$ against $\mu\neq 0$ at the level of significance 2α is given by $|t|\geqq t_\alpha$. Here t and t_α are defined as in 4. 3. As t_α for the α-values needed has been tabulated in 3 decimal places only or not at all, the quantity $\eta_\alpha=\frac{N-2}{N-2+t_\alpha^2}$ $(N=m+n)$ was calculated directly from

$$\alpha=\tfrac{1}{2}\,I_{\eta_\alpha}\left(\frac{N-2}{2},\tfrac{1}{2}\right), \tag{5. 3, 1}$$

where $I_x(p,q)$ is the incomplete B-function tabulated by K. PEARSON (6).

The power function of the critical region $|t|\geqq t_\alpha$ is (cf. P. L. HSU (1)):

$$\alpha_\pm(\mu)=\begin{cases} \dfrac{e^{-\frac{1}{2}\frac{mn}{N}\mu^2}}{\sqrt{N-2}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot\sum\limits_{k=0}^{\infty}\left[\dfrac{mn}{2N(N-2)}\right]^k\cdot \\ \cdot\dfrac{\mu^{2k}\,.\,\Gamma\left(\frac{2k+N-1}{2}\right)}{k!\,\Gamma(k+\frac{1}{2})}\displaystyle\int_{t_\alpha^2}^{\infty}x^{k-\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-k-\frac{N-1}{2}}dx. \end{cases} \tag{5. 3, 2}$$

Hence $\alpha''_\pm(0)=$

$$=\frac{mn\,.\,\Gamma\left(\frac{N-1}{2}\right)}{N\sqrt{N-2}\,.\,\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot$$

$$\cdot\left[\frac{N-1}{N-2}\int_{t_\alpha^2}^{\infty}x^{\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-\frac{N+1}{2}}dx-\int_{t_\alpha^2}^{\infty}x^{-\frac{1}{2}}\left(1+\frac{x}{N-2}\right)^{-\frac{N-1}{2}}dx\right].$$

By the substitution $1+\frac{x}{N-2}=y^{-1}$ it easily follows that:

$$(5.3,3)\qquad \begin{cases} \alpha''_{\pm}(0)=\dfrac{mn\,.\,\Gamma\left(\frac{N-1}{2}\right)}{N\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\left[(N-1)\int\limits_0^{\eta_a} y^{\frac{N-4}{2}}(1-y)^{\frac{1}{2}}\,dy-\int\limits_0^{\eta_a} y^{\frac{N-4}{2}}(1-y)^{-\frac{1}{2}}\,dy\right] \\ \qquad\qquad =\dfrac{mn}{N}\cdot\dfrac{2\,\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\pi}\,.\,\Gamma\left(\frac{N-2}{2}\right)}\cdot\eta_a^{\frac{N-2}{2}}\cdot(1-\eta_a)^{\frac{1}{2}}. \end{cases}$$

Here $N=m+n$.

In the cases, for which $\alpha''_{\pm}(0)$ was calculated for WILCOXON's test (cf. 5. 2, at the end) the following results are obtained for STUDENT's test from (5. 3, 3) and (5. 3, 1):

m	n	$2\,\alpha$	η_a	$\alpha''_{\pm}(0)$	$\alpha''_{\pm}(0)_{St}-\alpha''_{\pm}(0)_{Wi}$
2	2	1/3	5/9	0,37037	0,00282
2	3	1/5	0.527963	0,40270	0,01138
2	4	2/15	0,530988	0,38618	0,01863
2	4	4/15	0,706294	0,54070	0,02672
3	3	1/10	0,468123	0,35959	0,01261
3	3	1/5	0,629850	0,54306	0,02024

In order to facilitate the comparison with $\alpha''_{\pm}(0)$ for WILCOXON's test, a column is added containing the difference of $\alpha''_{\pm}(0)$ for the test of STUDENT and for the test of WILCOXON. This comparison shows that under the conditions which allow the use of STUDENT's test, [6]) a necessary condition for STUDENT's test being more powerful than WILCOXON's one is satisfied (cf. the remark in 2. 3), but in the cases investigated the difference is very small.

Remark. The following provisional result for large m and n was obtained for the power function of WILCOXON's test when alternatives are two-sided:

$$\alpha''_{\pm}(0)_{Wi}\approx\frac{\sqrt{2}}{\sqrt{\pi}}\cdot e^{-\frac{1}{2}\zeta_a^2}\cdot\zeta_a\cdot\frac{6\,mn}{2\pi\,(m+n)},$$

---

[6]) When $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ and $g(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2}$ and a critical region $|t|\geqq t_a$ is defined as in the beginning of 5.3, then $\alpha''_{\pm}(\mu)$ depends on σ. Denoting $\alpha_{\pm}(0)$ by $\beta(\sigma^2)$ to make this dependence explicit, it follows from HSU (1) that, when $m=2$, $n=3$, and $2\,\alpha=1/5$, : $\beta'(1)=0{,}0447$ for STUDENT's test (the prime denotes differentiation with respect to σ^2), whereas calculations showed that the corresponding quantity for WILCOXON's test is equal to 0,0357. So, in this example at least, WILCOXON's test is less sensitive to the invalidity of the assumption $\sigma=1$ than STUDENT's test.

except for a relative error of the order $1/m$ and $1/n$. Here ζ_a is defined by:

$$a = \frac{1}{\sqrt{2\pi}} \int_{\zeta_a}^{\infty} e^{-\frac{1}{2}x^2}\, dx,$$

where $2\,a$ is the level of significance of the test.

For STUDENT's test it is obtained from (5. 3, 3) that:

$$a'_{\pm}(0)_{St} \approx \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{-\frac{1}{2}\zeta_a^2} \cdot \zeta_a \cdot \frac{mn}{m+n},$$

(except for a relative error of the order $1/m$ and $1/n$). Hence $a'_{\pm}(0)_{Wi}/a'_{\pm}(0)_{St} \approx 3/\pi$, so that the difference in power between the two tests is not great indeed. And then, WILCOXON's test is more general in that $a\ (H_0)$ has the same value whatever $F(x)$, provided $F(x)$ is continuous.

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6. *Summary.*

Some properties of the critical region and the power function of WILCOXON's non-parametric solution of the problem of two samples are studied. Under the conditions which allow the use of STUDENT's test the difference in power between the two tests is investigated, as well for one-sided as for two-sided sets of alternative hypotheses, when the sum of the sample-sizes is $\leqq 5$ and $\leqq 6$ respectively. In these cases the difference is rather small. Indications are that for large sample sizes, too, the difference in power is not great. In an appendix the relation of the power function of WILCOXON's test with the volume of a spherical simplex is exposed, which shows the limitation of the sample sizes (to 5 and 6 resp.) to be relevant. In an introduction the concepts of critical region and of power function in general are exposed.

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A. APPENDIX

CALCULATION OF A CERTAIN MULTIPLE INTEGRAL

A. 1. *The relation of the integral with a spherical simplex.*

Let

$$J = \left[\prod_{i=1}^{\tau} (If^{k_i})\right](-\infty) = \int_G \cdots \int \prod_{i=1}^{\tau} \{f^{k_i}(x_i)\, dx_i\} \tag{A. 1, 1}$$

where

$$k_i > 0,\ \sum_{i=1}^{\tau} k_i = N,\ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and the region G is defined by $-\infty < x_1 < x_2 < \ldots < x_\tau < +\infty$.

Then

$$J = \frac{1}{(2\pi)^{\frac{1}{2}N}} \int_G \cdots \int e^{-\frac{1}{2}\sum_{i=1}^{\tau} k_i x_i^2} \prod_{i=1}^{\tau} dx_i \tag{A. 1, 1a}$$

Be $\cos\theta_j$ denoted with c_j and $\sin\theta_j$ with s_j $(j = 1, \ldots, \tau - 1)$. Make the substitutions:

$$x_i = \frac{1}{\sqrt{k_i}} \cdot R \cdot s_{\tau-i} \cdot \prod_{j=\tau-i+1}^{\tau-1} c_j \qquad (i = 1, \ldots, \tau) \tag{A. 1, 2}$$

with $s_0 = 1$; $0 \leqq \theta_1 \leqq 2\pi$; $0 \leqq \theta_j \leqq \pi$ $(j = 2, \ldots, \tau - 1)$.

Then

$$\frac{\partial(x_1, x_2, \ldots, x_{\tau-1}, x_\tau)}{\partial(R, \theta_1, \ldots, \theta_{\tau-2}, \theta_{\tau-1})} = \frac{1}{\sqrt{\prod_{i=1}^{\tau} k_i}} R^{\tau-1} \cdot \prod_{j=1}^{\tau-1} c_j^{j-1}.$$

Hence

$$J = \frac{1}{(2\pi)^{\frac{1}{2}N} \cdot \sqrt{\prod_{i=1}^{\tau} k_i}} \int_0^{\infty} e^{-\frac{1}{2}R^2} \cdot R^{\tau-1}\, dR \cdot \mathcal{L} = \frac{\Gamma\left(\frac{\tau}{2}\right)}{2^{\frac{1}{2}(N-\tau-2)}\, \pi^{\frac{1}{2}N} \sqrt{\prod_{i=1}^{\tau} k_i}} \cdot \mathcal{L} \tag{A. 1, 3}$$

Here

$$\text{(A. 1, 4)} \qquad \mathcal{L} = \int_{G'} \dots \int \prod_{j=1}^{\tau-1} \{\cos^{j-1}\theta_j \, d\theta_j\},$$

where the region G' is defined by:

$$-\frac{1}{\sqrt{k_1}} < \frac{1}{\sqrt{k_1}} s_{\tau-1} < \frac{1}{\sqrt{k_2}} s_{\tau-2} c_{\tau-1} < \dots < \frac{1}{\sqrt{k_\tau}} \prod_{j=1}^{\tau-1} c_j < +\frac{1}{\sqrt{k_\tau}}.$$

Now the (τ-dimensional) volume of that part of the τ-dimensional [7]) unit sphere, the orthogonal Cartesian coordinates $z_i (i = 1, \dots, \tau)$ of which satisfy the inequalities (defining the region G'') $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \dots < \frac{z_\tau}{\sqrt{k_\tau}}$ is given by

$$\text{(A. 1, 5)} \qquad \int_{G''} \dots \int \prod_{i=1}^{\tau} dz_i = V_{k_1, k_2, \dots, k_\tau} \quad \text{(say)}$$

(cf. Remark 1). Bij the substitutions $z_i = -z_i' \, (i = 1, \dots, \tau)$ it is easily seen that

$$\text{(A. 1, 6)} \qquad V_{k_1, k_2, \dots, k_{\tau-1}, k_\tau} = V_{k_\tau, k_{\tau-1}, \dots, k_2, k_1}.$$

Moreover by the substitutions:

$$\text{(A. 1, 7)} \qquad z_i = R \cdot s_{\tau-i} \cdot \prod_{j=\tau-i+1}^{\tau-1} c_j \qquad (i = 1, \dots, \tau)$$

with $s_0 = 1$; $0 \leqq \theta_1 \leqq 2\pi$; $0 \leqq \theta_j \leqq \pi \, (j = 2, \dots, \tau - 1)$ it is seen that

$$\text{(A. 1, 8)} \qquad V_{k_1, \dots, k_\tau} = \int_0^1 R^{\tau-1} \, dR \cdot \mathcal{L} = (1/n) \cdot \mathcal{L}.$$

Thus by (A. 1, 3) and (A. 1, 8):

$$\text{(A. 1, 9)} \qquad J = \frac{\tau \cdot \Gamma\left(\frac{\tau}{2}\right)}{2^{\frac{1}{2}(N-\tau+2)} \pi^{\frac{1}{2}N} \sqrt{\prod_{i=1}^{\tau} k_i}} \cdot V_{k_1, \dots, k_\tau}.$$

Remark 1. If in $V_{k_1, \dots, k_\tau}$ $k_{i+1} = k_{i+2} = \dots = k_{i+h} = k$, the sequence of suffices $k_{i+1}, \dots, k_{i+h}$ will be denoted shortly with the symbolic power k^h.

Remark 2. $V_{k_1, k_2, \dots, k_\tau}$ will be calculated below for $\tau = 2, 3, 4$.

A. 2. *The case* $\tau = 2$.

V_{k_1, k_2} is the area of that part of the unit circle the coordinates z_1, z_2 of which satisfy $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}}$. Clearly this is just half of the total area. So

$$\text{(A. 2, 1)} \qquad V_{k_1, k_2} = \pi/2 \text{ for every } k_1, k_2.$$

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[7]) τ here is the number of dimensions of the underlying space.

A. 3. *The case* $\tau = 3$.

V_{k_1, k_2, k_3} is the volume of that part of the 3-dimensional unit sphere, the coordinates z_1, z_2, z_3 of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}}.$$

This region is bounded by the surface of the sphere and by two planes, the equations of which are given by

(A. 3, 1) $$\frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} = 0 \quad \text{and} \quad \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} = 0.$$

The volume V_{k_1, k_2, k_3} is built up out of points which make both the first members of the equations (A. 3, 1) negative.

The corresponding angle between the two planes is given by $\pi - \varphi$, where φ is the angle between the positive normals.

$$\varphi = \arccos \frac{-\frac{1}{\sqrt{k_2}} \cdot \frac{1}{\sqrt{k_2}}}{\sqrt{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\left(\frac{1}{k_2} + \frac{1}{k_3}\right)}} = \pi - \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

So

(A. 3, 2) $$V_{k_1, k_2, k_3} = \frac{\pi - \varphi}{2\pi} \cdot \frac{4}{3}\pi = \frac{2}{3} \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

A. 4. *The case* $\tau = 4$.

V_{k_1, k_2, k_3, k_4} is the volume of that part of the 4-dimensional unit sphere, the coordinates z_1, z_2, z_3, z_4 of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}} < \frac{z_4}{\sqrt{k_4}}.$$

This region is bounded by the "surface" of the sphere and by three hyperplanes, the equations of which are given by

(A. 4, 1)
$$\begin{cases} \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} = 0 & (I) \\ \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} = 0 & (II) \\ \frac{z_3}{\sqrt{k_3}} - \frac{z_4}{\sqrt{k_4}} = 0 & (III) \end{cases}$$

The hyperplane $P \equiv \sum_{i=1}^{4} z_i \sqrt{k_i} = 0$ is perpendicular to the line

$$\frac{z_1}{\sqrt{k_1}} = \frac{z_2}{\sqrt{k_2}} = \frac{z_3}{\sqrt{k_3}} = \frac{z_4}{\sqrt{k_4}},$$

which is contained in each of the three bounding hyperplanes. The points, P has in common with V_{k_1, k_2, k_3, k_4}, fill up that part of the

3-dimensional unit sphere in the hyperplane P with $(0, 0, 0, 0)$ as a centre, which is common to the point sets for which

$$P = 0;\ \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} < 0;\ \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} < 0;\ \frac{z_3}{\sqrt{k_3}} - \frac{z_4}{\sqrt{k_4}} < 0.$$

Be v the volume of this part of the 3-dimensional unit sphere in the hyperplane P. Then:

$$V_{k_1,k_2,k_3,k_4} = v \cdot \int_{-1}^{+1} (1-q^2)^{3/2}\, dq = \frac{3\pi}{8}\, v. \tag{A. 4, 2}$$

As to v, the following equality holds good:
$v/\frac{4}{3}\pi = \Phi/4\pi$, where Φ is the spherical excess of the (spherical) triangle defined by the three hyperplanes mentioned above; $\frac{4}{3}\pi$ is the volume and 4π the surface of the 3-dimensional unit sphere. So

$$v = \tfrac{1}{8}\Phi = \tfrac{1}{8}\left[(\mathrm{I}, \mathrm{II}) + (\mathrm{II}, \mathrm{III}) + (\mathrm{III}, \mathrm{I}) - \pi\right] \tag{A. 4, 3}$$

where (I, II) is the angle between the hyperplanes I and II corresponding to the inequalities defining v and the same holds for (II, III) and (III, I). (I, II) is the supplement of the angle between the positive normals on I and II respectively; etc. So $(\mathrm{III}, \mathrm{I}) = \pi/2$, whereas

$$(\mathrm{I}, \mathrm{II}) = \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1+k_2)(k_2+k_3)}};\ (\mathrm{II}, \mathrm{III}) = \arccos \frac{\sqrt{k_2 k_4}}{\sqrt{(k_3+k_3)(k_3+k_4)}}.$$

Hence

$$\Phi = \arcsin \frac{\sqrt{k_2 k_3}\left[\sqrt{(k_1+k_2+k_3)(k_2+k_3+k_4)} - \sqrt{k_1 k_4}\right]}{(k_2+k_3)\sqrt{(k_1+k_2)(k_3+k_4)}}$$

and, by (A. 4, 3) and (A. 4, 2) it follows that

$$V_{k_1,k_2,k_3,k_4} = \frac{\pi}{8} \frac{\sqrt{k_2 k_3}\left[\sqrt{(k_1+k_2+k_3)(k_2+k_3+k_4)} - \sqrt{k_1 k_4}\right]}{(k_2+k_3)\sqrt{(k_1+k_2)(k_3+k_4)}}. \tag{A. 4, 4}$$

A RANK-INVARIANT METHOD OF LINEAR AND POLYNOMIAL REGRESSION ANALYSIS
II [1])

BY

H. THEIL

(Communicated by Prof. D. VAN DANTZIG at the meeting of March 25, 1950)

2. CONFIDENCE REGIONS FOR THE PARAMETERS OF LINEAR REGRESSION EQUATIONS IN THREE AND MORE VARIABLES.

The probability set.

2.0. The probability set Γ underlying the probability statements of this section is the $n(\nu+2)$-dimensional Cartesian space $R_{n(\nu+2)}$ with coordinates

$$u_{11},\ldots,u_{1n},\ldots,u_{\nu 1},\ldots,u_{\nu n},\ v_1,\ldots,v_n,\ w_1,\ldots,w_n.$$

Every random variable will be supposed to be defined on this probability set.

In this first place we consider $n(\nu+2)$ random variables $\mathbf{u}_{\lambda i}, \mathbf{v}_i, \mathbf{w}_i$ $(\lambda = 1, \ldots, \nu;\ i = 1, \ldots, n)$. Furthermore we consider $(n+1)\nu+1$ parameters $\alpha_0, \alpha_\lambda, \xi_{\lambda i}$ $(i = 1, \ldots n;\ \lambda = 1, \ldots, \nu)$ and put:

$$\left.\begin{aligned}
&(5)\quad \theta_i = \alpha_0 + \sum_{\lambda=1}^{\nu} \alpha_\lambda \xi_{\lambda i}\\
&(6)\quad \eta_i = \theta_i + \mathbf{w}_i\\
&(7)\quad \mathbf{x}_{\lambda i} = \xi_{\lambda i} + \mathbf{u}_{\lambda i}\\
&(8)\quad \mathbf{y}_i = \eta_i + \mathbf{v}_i
\end{aligned}\right\}\ \begin{matrix} i = 1,\ldots,n \\ \lambda = 1,\ldots,\nu. \end{matrix}$$

So the variables $\mathbf{x}_{\lambda i}$ and $\mathbf{y}_i$ have a simultaneous distribution on Γ, and are therefore random variables.

We call $\xi_{\lambda i}$ the parameter values of the variable ξ_λ. The equation (5) is the multiple regression equation. The random variables $\mathbf{w}_i$ are called "the true deviations from linearity", while the random variables $\mathbf{u}_{\lambda i}$ and $\mathbf{v}_i$ are called "the errors of observation" of the values $\xi_{\lambda i}$ and η_i respectively.

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[1]) This paper is the second of a series of papers, the first of which appeared in these Proceedings, **53**, 386—392 (1950).

Putting

$$\mathbf{z}_i = -\sum_{\lambda=1}^{\nu} \alpha_\lambda \mathbf{u}_{\lambda i} + \mathbf{v}_i + \mathbf{w}_i$$

we have

$$\mathbf{y}_i = \alpha_0 + \sum_{\lambda=1}^{\nu} \alpha_\lambda \mathbf{x}_{\lambda i} + \mathbf{z}_i,$$

the random variables $\mathbf{z}_i$ being called "the apparent deviations from linearity".

Confidence regions for $\alpha_0, \alpha_1, \ldots, \alpha_\nu$.

2. 1. In order to give confidence regions for the $(\nu+1)$ parameters α_0, α_λ $(\lambda = 1, \ldots, \nu)$ we impose the following *conditions*:

Condition I: The n $(\nu+2)$-uples $(\mathbf{u}_{1i}, \ldots, \mathbf{u}_{\nu i}, \mathbf{v}_i, \mathbf{w}_i)$ are stochastically independent.

Condition II: 1. Each of the errors $\mathbf{u}_{\lambda i}$ vanishes outside a finite interval $|\mathbf{u}_{\lambda i}| \leqq g_{\lambda i}$.

2. For each $i \neq j$ we have $|\xi_{\lambda i} - \xi_{\lambda j}| > g_{\lambda i} + g_{\lambda j}$.

Furthermore we impose *for the incomplete method* to be mentioned:

Condition III:

$$P[\mathbf{z}_i < \mathbf{z}_j] = P[\mathbf{z}_i > \mathbf{z}_j] = \tfrac{1}{2} \text{ for } i \neq j$$

and *for the complete method*:

Condition IIIa: Each $\mathbf{z}_i$ has the same continuous distribution function.

2. 2. Secondly we *define* the following quantities:

$$\mathbf{G}^{(\lambda')}(i) = \mathbf{y}_i - \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^{\nu} \alpha_\lambda \mathbf{x}_{\lambda i} =$$

$$= \alpha_0 + \alpha_{\lambda'} \mathbf{x}_{\lambda' i} + \mathbf{z}_i \qquad (\lambda' = 1, \ldots, \nu;\ i = 1, \ldots, n).$$

Furthermore, after arranging the n observed points $(y_i, x_{1i}, \ldots, x_{\nu i})$ according to increasing values of $x_{\lambda'}$ (which, by condition II, is identical with the arrangement according to increasing values of $\xi_{\lambda'}$):

$$x_{\lambda' 1} < x_{\lambda' 2} < \ldots < x_{\lambda' n}$$

we define the quantities

$$\mathbf{K}^{(\lambda')}(ij) = \frac{\mathbf{G}^{(\lambda')}(i) - \mathbf{G}^{(\lambda')}(j)}{\mathbf{x}_{\lambda' i} - \mathbf{x}_{\lambda' j}} =$$

$$= \frac{\mathbf{y}_i - \mathbf{y}_j}{\mathbf{x}_{\lambda' i} - \mathbf{x}_{\lambda' j}} - \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^{\nu} \alpha_\lambda \frac{\mathbf{x}_{\lambda i} - \mathbf{x}_{\lambda j}}{\mathbf{x}_{\lambda' i} - \mathbf{x}_{\lambda' j}} =$$

$$= \alpha_{\lambda'} + \frac{\mathbf{z}_i - \mathbf{z}_j}{\mathbf{x}_{\lambda' i} - \mathbf{x}_{\lambda'}} \qquad (i = 1, \ldots, n-1;\ j = i+1, \ldots, n).$$

For any set of values $\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu$ we arrange the quantities $K^{(\lambda')}(i\,j)$ according to increasing magnitude; we define $K_i^{(\lambda')}$ as the quantity with rank i in this arrangement:

$$K_1^{(\lambda')} < K_2^{(\lambda')} < \ldots < K_{\binom{n}{2}}^{(\lambda')}$$

Finally we define the intervals $\boldsymbol{I}_{\lambda'}(\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu)$ as the intervals

$$\left(K_q^{(\lambda')},\ K_{\binom{n}{2}-q-1}^{(\lambda')}\right)$$

with $2q \leqq \binom{n}{2}$; $\boldsymbol{A}_{\lambda'}$ as the union of

$$\boldsymbol{I}_{\lambda'}(\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu) \text{ for all } \alpha_\lambda\ (\lambda = 1, \ldots, \nu;\ \lambda \neq \lambda');$$

and $\boldsymbol{A}$ as the union of all $\boldsymbol{A}_{\lambda'}$ $(\lambda' = 1, \ldots, \nu)$.

2. 3. We have the following theorem concerning the *complete method* for three and more variables:

Theorem 4: Under conditions I, II and III*a* the region $\boldsymbol{A}$ is a confidence region for the parameters $\alpha_1, \ldots, \alpha_\nu$, the level of significance being $\leqq 2\nu . P[q-1\,|\,n]$ [2]).

Proof: If the set of assumed parameters values $\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu$ is the "true" set, it follows from the analysis in section 1.3., that $\boldsymbol{I}_{\lambda'}(\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu)$ is a confidence interval for $\alpha_{\lambda'}$ to the level of significance $2\,P[q-1\,|\,n]$. Hence it follows that if $(\alpha_1, \ldots, \alpha_\nu)$ represents the "true" point in the $\alpha_1, \ldots, \alpha_\nu$-space, we have

$$P[(\alpha_1, \ldots, \alpha_\nu) \in \boldsymbol{A}_{\lambda'}] = 1 - 2\,P[q-1\,|\,n], \qquad (\lambda' = 1, \ldots, \nu),$$

which proves the theorem.

2. 4. If condition III (but not necessarily III*a*) is fulfilled, the method mentioned above can be replaced by the following one. We replace the quantities

$$\boldsymbol{K}^{(\lambda')}(i\,j) \qquad (\lambda' = 1, \ldots, \nu;\ i = 1, \ldots, n-1;\ j = i+1, \ldots, n)$$

by

$$\boldsymbol{K}^{(\lambda')}(i,\ n_1 + i) \qquad (\lambda' = 1, \ldots, \nu;\ i = 1, \ldots, n_1).$$ [3])

The intervals $\boldsymbol{I}'_{\lambda'}(\alpha_1, \ldots, \alpha_{\lambda'-1}, \alpha_{\lambda'+1}, \ldots, \alpha_\nu)$ are now defined as the intervals bounded by the values of $\boldsymbol{K}^{(\lambda')}(i,\ n_1 + i)$ with rank r_1 and $(n_1 - r_1 + 1)$ respectively, if they are arranged in ascending order; whereas the definitions of $\boldsymbol{A}'_{\lambda'}$ as the union of all $\boldsymbol{I}'_{\lambda'}$ and of $\boldsymbol{A}'$ as the union of all $\boldsymbol{A}'_{\lambda'}$

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[2]) For the definition of $P[q-1\,|\,n]$ the reader is referred to section 1.3. (part I of this paper).

[3]) $n_1 = \frac{1}{2}n$. Cf. section 1.2.

remain unchanged. The following theorem of the *incomplete method* for three and more variables will now be obvious from the analysis of section 1. 1.:

Theorem 5. Under conditions I, II and III the region $\boldsymbol{A}'$ is a confidence region for the parameters $\alpha_1, \ldots, \alpha_\nu$, the level of significance being $\leqq 2\nu.\ I_{\frac{1}{2}}(r_1, n_1 - r_1 + 1)$.

2. 5. A confidence region for the parameters $\alpha_0, \alpha_1, \ldots, \alpha_\nu$ can be constructed, if the median of $\boldsymbol{z}_i$ is known, e.g. if the following condition is fulfilled:

Condition IV: The median of each $\boldsymbol{z}_i$ is zero.

The method for the construction of this confidence region is analogous to the one given in section 1. 2.

An illustration for the special case $\nu = 2$.

2. 6. The form of the region $\boldsymbol{A}_\lambda$ or $\boldsymbol{A}'_\lambda$ will now be indicated for the case of three variables:

$$\boldsymbol{y}_i = \alpha_0 + \alpha_1 \boldsymbol{x}_{1i} + \alpha_2 \boldsymbol{x}_{2i} + \boldsymbol{z}_i.$$

Using the incomplete method we find n_1 functions of α_2:

$$\boldsymbol{K}^{(1)}(i, n_1 + i) = \frac{\boldsymbol{y}_i - \boldsymbol{y}_{n_1+i}}{\boldsymbol{x}_{1i} - \boldsymbol{x}_{1,n_1+i}} - \alpha_2 \frac{\boldsymbol{x}_{2i} - \boldsymbol{x}_{2,n_1+i}}{\boldsymbol{x}_{1i} - \boldsymbol{x}_{1,n_1+i}},$$

which are estimates of α_1, given α_2. They are represented by straight lines in the α_1, α_2-plane. For any value of α_2 we can arrange these quantities in ascending order. As long as (under continuous variation of α_2) the numbers i_1 and i_2 for which the statistics $K^{(1)}(i_1, n_1 + i_1)$ and $K^{(1)}(i_2, n_1 + i_2)$

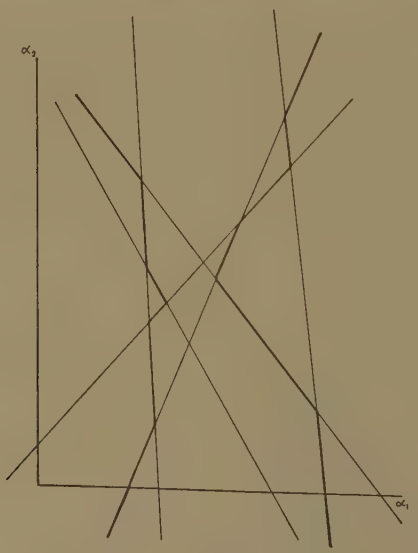

Fig. 1. $n_1 = 6,\ r_1 = 2$.

have the r_1-th and $(n_1 - r_1 + 1)$-th rank according to increasing order (with r_1 as defined in section 2. 4.) remain constant, the extreme points of the confidence intervals vary along straight lines. If, when passing some value a_2^* of a_2 either i_1 or i_2 changes, the corresponding straight line passes into another one, intersecting the first one in a point with $a_2 = a_2^*$.

So a diagram can be constructed, in which the n_1 straight lines are drawn in the a_1, a_2-plane. This gives the stochastic region $\mathbf{A}_1'$ depending on the given observations and bounded to the left and to the right by broken lines.

According to Theorem 5 it contains the true point (a_1, a_2) with the probability

$$1 - 2\, I_{\frac{1}{2}}(r_1, n_1 - r_1 + 1).$$

The region $\mathbf{A}_2'$, bounded above and below, can be constructed in a similar way; then the observed points must be arranged in ascending order of $\boldsymbol{x}_2$.

Publication of the Statistical Department of the "Mathematisch Centrum", Amsterdam.

THE OVARIAN OCCLUSION APPARATUS IN THE VIVIPAROUS CYPRINODONTS LEBISTES RETICULATUS PETERS AND XIPHOPHORUS HELLERI HECKEL

BY

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(Communicated by Prof. CHR. P. RAVEN at the meeting of March 25, 1950)

In order to retain the developing embryos in the uterus a mechanical occlusion apparatus, the cervix uteri, is present in all, except the lower mammals. A detailed description of this caudal part of the uterus has been given e.g. in 1927 by STIEVE for the human female and in 1932 by VAN DEN BROEK for several mammals.

As to the mechanical function of the cervix there exists no common agreement. STIEVE (1927) supposed that its solidity is due partly to the large quantity of tough mucus, produced by the cervical glands and partly to the cavernous tissue, which forms the principal part of it. During parturition the blood is driven out of the cavernous spaces by the pressure of the fluid, enclosed by the foetal membranes, with the result that the cervix channel is dilated.

According to GOERTTLER (1929) the dilation of the cervix uteri is caused by contraction of a complicated system of muscle fibres in the cervix wall.

The observation of STIEVE (1927) that the tissue of the human cervix uteri closely resembles that of a corpus cavernosum, has been confirmed by DE SNOO (1939, 1946). Moreover, this author was of opinion that the pressure of the fluid bladder, formed by the foetal membranes, is not able to open the cervix lumen, as he observed that during labour the cervix uteri is neither elongated nor strained. Therefore he considers the cervix to be a passive occlusion apparatus, which is capable of holding the foetus within the uterus during pregnancy and which towards the end of the gestation period becomes softer by a reversible colloidal chemical process. By means of stretching experiments with strips of cervix tissue DE SNOO tried to obtain confirmatory evidence. This special cervix tissue, which consists of cells intermediate between connective tissue cells and muscle fibres, was found by him in the human female and in a number of mammals, viz. the monkeys *Macacus cynomolgus* and *Chrysothrix sciureus*, and in the bitch, cat, mare, cow, ewe, hedgehog

and Guinea pig. In Marsupials it was absent, a fact ascribed to the short duration of the gestation period in this mammalian group.

According to DE SNOO the cervix tissue is therefore a specific structure of the higher mammals. He was unable to find it in some ovo-viviparous lizards and snakes. As he did not investigate viviparous fishes, it was decided to examine the viviparous Cyprinodonts *Lebistes reticulatus* and *Xiphophorus helleri* in this respect.

The ovary of *Lebistes* lies in the body cavity, surrounded by the liver, a double loop of the intestine, the muscular body wall and the air bladder. Originally there are two ovaries, but these fuse early in development, and from this moment possess a common oviduct.

As in other Teleosts the wall of the ovary passes over into the wall of the oviduct. Here a number of lobes was found (fig. 1). They are

Fig. 1. Occlusion apparatus of an adult female of *Lebistes reticulatus*. The longitudinal walls, consisting of occlusion tissue, are covered by ovarian epithelium. In the ovary a number of oocytes.

covered with epithelium cells similar to those lining the ovarian cavity; under the epithelium a stroma with connective tissue cells and a longitudinal layer of peculiar tissue occur (fig. 2). Perpendicular to the latter

a radial layer of similar tissue is found. Both layers consist of cells strongly resembling those of the occlusion tissue of the cervix of the mammalian uterus [cf. figs. 125, 126, 128 and 136 of DE SNOO (1939)].

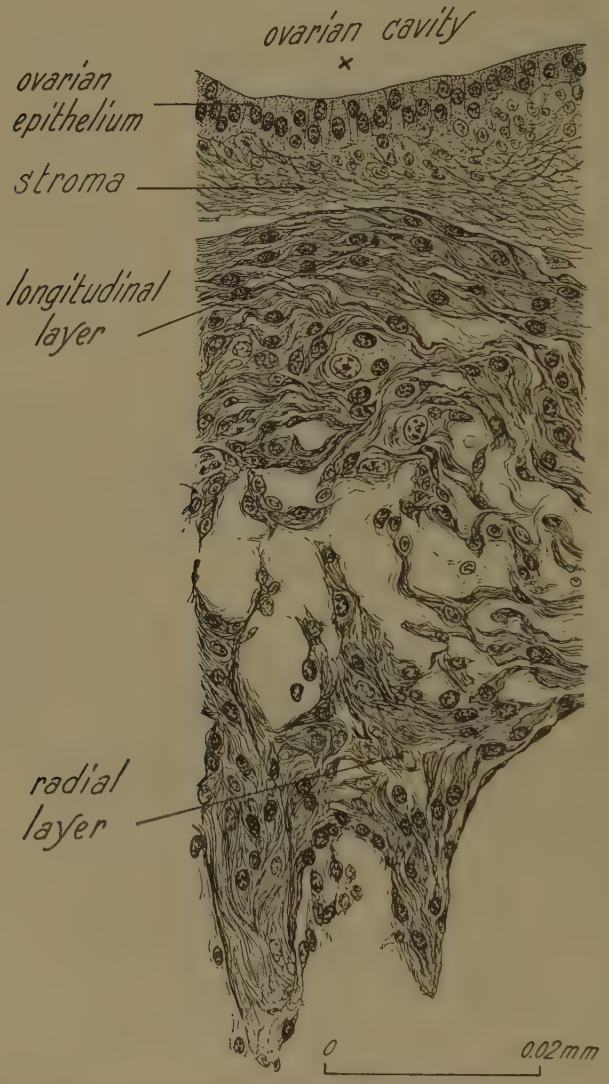

Fig. 2. Part of the occlusion apparatus of *Lebistes reticulatus*, highly magnified. The following details are visible: ovarian epithelium, stroma, longitudinal and radial layers of occlusion tissue.

These lobes are already present in the ovary of the juvenile female of *Xiphophorus* (fig. 3). They develop by an invagination of the ovarian epithelium, a process which proceeds backwards as well as forwards.

In the adult female *Xiphophorus* the occlusion apparatus is much more clearly differentiated. The lobes are more numerous and the spaces

between them larger. Secondary lobes may also develop, while the subepithelial connective tissue layer has increased in volume.

In some respect the morphology of the occlusion apparatus resembles that of the muscular wall of certain hollow organs which are capable of maintaining a distinct tonus without much loss of energy.

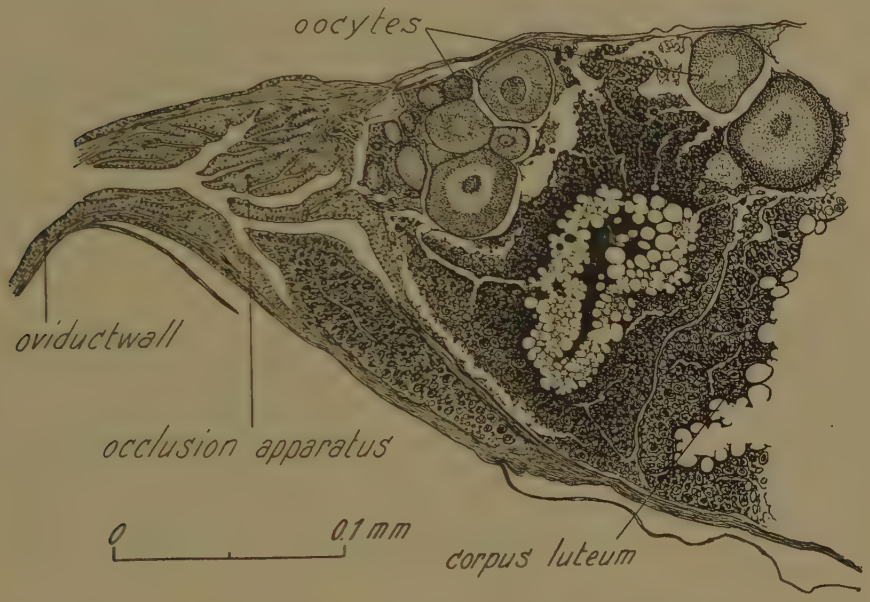

Fig. 3. Occlusion apparatus of a juvenile *Xiphophorus helleri*. Simple system of lobes, developed by an invagination process from the ovary wall. Several oocytes of different sizes are visible as well as corpora lutea.

As an ovarian occlusion apparatus occurs in *Lebistes* as well as in *Xiphophorus*, it is likely that this apparatus is generally found in the viviparous Cyprinodonts. In these fishes in which the number of young may even be very large [up to 97 in *Lebistes* (SCHMIDT, 1920), or more than 200 in *Xiphophorus*] the presence of an occlusion apparatus, which can resist a high internal pressure, is very appropriate.

As to the expulsion of the young, DE SNOO (1939) distinguished an expulsion without and with peristalsis. The first mechanism is present in the primates, whereas the second is to be found e.g. in Ungulates.

According to STOLK (1950) in the Cyprinodonts the expulsion of the young must take place with the help of peristalsis of the oviduct wall. This opinion is based on the fact that in *Lebistes* the oviduct wall possesses a well developed musculature, whereas the wall of the ovary is devoid of a muscle layer. Moreover, the young are born without any detectable pressure of the ventral body wall: during parturition the ventral muscles remain quiet. Consequently an intra-abdominal pressure cannot play any significant role during parturition. Finally the fact that parturition

generally takes place in transverse position is in favour of a peristaltic mechanism.

It therefore appears that in the Cyprinodonts the embryos are kept within the ovary with the help of the occlusion apparatus, present at the transition of the ovary into the oviduct. It is probable that at the end of the gestation period the tissue constituting this apparatus relaxes. This is possible due, first to a reversible colloidal chemical process, and second to the high pressure of the ovarian fluid in which a great number of fully grown embryos is present. After the occlusion apparatus has been opened, the embryos are transported through the oviduct with the help of the peristalsis of its wall.

It is likely that further research will also reveal an occlusion apparatus in other groups of fishes as well as in those of ovo-viviparous snakes and lizards, in which a gestation period is found.

Summary.

In the viviparous Cyprinodonts *Lebistes reticulatus* and *Xiphophorus helleri* an occlusion apparatus is present, where the ovary passes into the oviduct. The tissue composing it resembles that of the cervix uteri in the human female, described by DE SNOO (1939, 1946). After the occlusion apparatus has been opened the young pass through the oviduct, presumably with the help of the peristalsis of its wall.

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ELEKTRONEN-MIKROSKOPISCHE STRUKTURUNTERSUCHUNG AN SPERMIEN. V.

VON

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I. *Einleitung.*

Unsere bisherigen elektronen-mikroskopischen Untersuchungen über den Aufbau des Spermiums (BRETSCHNEIDER, 1947—1949) haben sich ausnahmslos auf diejenigen des Stiers, Pferdes und Hundes bezogen. Es hing dies zusammen mit dem Problem und der Anwendung der künstlichen Besamung bei diesen Haussäugern hierzulande. Dabei, stellte sich u.a. das Bedürfnis einer besseren Einsicht in den Bau des normalen und abnormalen Spermiums für die morphologische Beurteilung des Spermas heraus. Es ergab sich dabei, dasz unsere bisherigen licht optischen Kenntnisse nur durch die Anwendung des Elektronenmikroskopes erweitert werden konnten. Für die Vertiefung unserer Einsichten war es geboten, vergleichsweise auch die Spermien anderer Tiere heran zu ziehen, wobei sich heraus stellte, dasz an diesen anderen Objekten manche Strukturen oder deren Zustandekommen die Analyse merklich förderten. Die hierbei erzielten Ergebnisse wollen wir nun in einigen Einzelbeiträgen mitteilen.

II. *Präparation.*

Die Spermata wurden vor der Fixation drei mal gespült in physiologischer Kochsalzlösung und abzentrifugiert und zur letzten Lösung die Fixationsflüssigkeit hinzu gefügt. Für die Quellungsversuche wurde die letzte Spülung mit einer hypotonischen Flüssigkeit ausgeführt. Zur Darstellung der Spiralen mit Eisen wurde das frische unverdünnte Bullensperma mit einer 1—5 % wässerigen Lösung von Eisenzucker (Saccharas ferricus med.) versetzt und im Brutschrank bei 38° C bewahrt. Es wird von uns nun untersucht ob diese Methode auch für andere Objekte, und zwar bei intravitaler Applikation, elektronenmikroskopisch aus zu werten ist.

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[1]) 44. Mitteilung der "Werkgemeenschap voor Endrocrinologie".

Die Spermien von *Loligo vulgaris* wurden durch Zerzupfen der Spermatophoren in Seewasser gewonnen und in Formol fixiert. Die Spermien der Drohne entnahmen wir den Vasa deferentia und fixierten sie in ½ % Überosmiumsäure. Das Froschsperma wurde der Cloaca entnommen und mit ½ % Überosmiumsäure fixiert. Die Spermien von *Triton taeniatus* wurden durch Zerzupfen der Spermatophoren in Leitungswasser gewonnen und mit Mercurochrom fixiert. Die Beschattung der Präparate erfolgte mit Gold, die Aufnahmen mit dem Delfter Elektronenmikroskop bei Spannungen von 80—100 kV.

III. *Die Galea capitis und deren Homologe.*

Aus Untersuchungen G. B. Teunissens's (5) ging hervor, dasz eine bestimmte als "Acrosomkappe" bezeichnete Missbildung der Galea bei Bullen zur Sterilität führen kann und dasz diese Abnormalität erblich ist. Durch diese Feststellung wurde unsere Aufmerksamkeit auf die Wichtigkeit der Kopfkappe für die Befruchtung gelenkt. Obwohl aus den vielen Untersuchungen über die Spermiohistogenese immer wieder hervorging, dasz die Anlage der Galea aus dem bläschenförrmigen Acroblast und dem in ihm eingeschlossenen dichteren Acrosom stattfindet, war das weitere Schicksal dieser beiden Anteile nicht überzeugend zu verfolgen und im ausgereiften Spermium nicht mehr nachweisbar. Es war darum angezeigt die Galea elektronen-mikroskopisch zu untersuchen, da es bei den geringen Abmessungen dieser Strukturen möglich war, dasz die beiden Teile wohl auch im reifen Spermium vorhanden sind, licht-optisch aber nicht sichtbar werden. Tatsächlich hat die elektronen mikroskopische Analyse diese Frage einigermassen geklärt.

a. Die Galea capitis beim Cavia.

Wie wir früher mitteilten (3) besitzen die beschriebenen Säuger (Pferd, Rind und Hund) eine distinkte Kopfkappe, die sich über mehr als ein Drittel des Kopfes als eine echte Kappe stülpt. Sie kann bei Anwendung hypotonischer Lösungen oder durch bakterielle Lysis als Ganzes vom Kopf abgestreift werden. Bekanntlich besitzt das Spermium vom *Cavia* einen grossen platten Kopf und eine üppig entwickelte Kopfkappe. In dieser lässt sich auch am ausgereiften ejakulierten Spermium eine deutliche Trennung in zwei Schichten erkennen, die sich durch ihre Massendicke elektronen optisch unterscheiden und die vorallem deutlich werden, wenn man die Spermien in hypotonischer Kochsalzlösung spült. Dadurch quillt die Kappensubstanz vorallem in der Längsrichtung der Kopfachse und hebt sich vom Kopfprofil ab. Abb. 1. An dem annäherend scheibenförmigen Kopf bemerkt man eine massendichtere apicale Hälfte, die durch die Summation von Kopf- und Kappensubstanz zustande kommt, und zeigt, dass der distale Rand der Kopfkappe auch nach der Quellung mit der Kopfoberfläche fester verbunden bleibt. Innerhalb der Kopfkappe unterscheiden wir eine dünne daher lichtere Aussenkappe die bis

zum distalen Galearand verläuft und eine dichtere, dunklere Innenkappe. Am rechten Rande und im Zentrum finden sich zwei schmale, gekrümmte Substanzanhäufungen, ausserdem granuläre and flockenartige Gebilde, die scheinbar durch die Hypotonie zu quellen beginnen. An einer stärkeren elektronen-optischen Vergrösserung, Abb. 3, fällt eine gleichmässige granuläre Struktur auf, die wahrscheinlich bereits den Elementarteilchen der Kappensubstanz zuzuschreiben ist.

Die schon am Stier gemachte Beobachtung der quellenden Wirkung hypotonischer Lösungen auf die Galeasubstanz wandten wir auch hier noch weiter an. Verdünnt und wäscht man frisches Sperma mit Leitungswasser (Abb. 2), dann quillt die Galeasubstanz der Aussenschicht stark an, dehnt sich aus und hebt sich vom Kopf ab. Gleichzeitig fliesst im Zentrum die an sich weniger formbeständige Substanz der inneren Schicht als ein Koagulat aus der Kopfkappe hervor. Man sieht dies am rechten Rand der Kappe in der Abb. 2. Es scheint sich also sowohl morphologisch als auch physiko-chemisch um zwei getrennte Bestandteile zu handeln, welche vielleicht auch zwei verschiedenen Funktionen dienen. Dies verwundert um so weniger, als gerade beim *Cavia*, dem klassischen Untersuchungsobjekt der Spermiumanalyse, bereits licht-optisch festgestellt werden konnte, dass Innen- und Aussenkappe wahrscheinlich von zwei verschiedenen Strukturen: dem Acrosom (Innenschicht) und dem Acroblast (Aussenschicht) abstammen.

b) Die „Acrosomkappe" beim Bullen [2]).

Obwohl beim Bullen ein doppelter Aufbau der Galea der Zartheit wegen auch elektronen-mikroskopisch nicht fest zu stellen war, findet dies in einem besonderen Falle doch statt. Teunissen (1945) stellte eine pathologische Ausbildung der Kappe fest, bei der diese als ein kompakter Tropfen am Kopfpol fixiert bleibt. Man nahm an, dass sich die Galea in diesem Falle nicht völlig ausdifferenzierte sondern auf einem cytogenetisch jungem Stadium stecken bleibt. Die elektronen-mikroskopische Analyse bestätigt inzwischen diese Annahme. Man beobachtet nl. in vielen Spermien noch, dass diese pathologische Galea aus den ursprünglichen zwei Teilen Acroblast und eingeschlossenem Acrosom besteht, (Abb. 4). Man sieht in den elektronen-optischen Aufnahmen deutlich die zentrale elektronen-optisch dichtere Anschwellung (Acrosom) umgeben von der weniger dichten Substanz der Aussenkappe, welche sich am Kopfpol ausbreitet, (Abb. 4, links unten). Bei anderen Spermien wieder sieht man, wie sich die Galeanlage wie eine Beule vom Kopf abhebt oder seitlich abfliesst (Abb. 5). Auch kann das Acrosomgranulum fehlen oder in verschiedene kleinere Granula zerteilt sein. Kennzeichnend für diese

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[2]) Die Abbildungen stammen von Semen eines in England befindlichen friesischen Stieres. Wir danken Herrn Lord Rotschild (Cambridge) für die bereitwillige Überlassung dieser für ihm in Delft gemachten Aufnahmen.

Abnormalität ist übereinstimmend, dass eine Ausdifferenzierung unterbleibt und die Substanz der Galea Anlage am vorderen Kopfpol angehäuft bleibt.

c) Die Galea capitis beim Hahn.

Im Gegensatz zu der normal sehr dünnen Kopfkappe beim Bullen fanden wir beim Hahn eine kompakte Galea. Sie befindet sich als ein zapfenförmiger Fortsatz am vorderen Kopfpol und lässt eine deutliche Grenze gegenüber dem eigentlichen Kopf erkennen, da die Galeabasis etwas breiter ist als der Kopfpol (Fig. 6).

Dass die Galea beim Hahn nicht bloss wie beim Bullen als eine dünne Kappe den Kopf bekleidet, sondern aus zwei kompakteren Teilen besteht, ergibt sich aus der folgenden Beobachtung. An zahlreichen Spermien eines Ejakulates war der äussere Teil der Kappe abgeworfen oder im Begriffe abgestreift zu werden. (Abb. 6 bis 10). Dadurch wurde ein 0′7 μ bis 1′0 μ langer und 85 mμ bis 135 mμ breiter stilettförmiger Forzatz sichtbar, der am Kopfpol als innerer Kappenteil mit dem Kopf verbunden blieb (Abb. 10). Dieser stilettförmige Innenteil ist nicht etwa die eigentliche Kopfspitze, denn oft trennte sich auch diese Spitze vom Kopfapex ab, wodurch der Kopf mit einer knopfförmigen Verdickung endigt. Wir betrachten diesen Forsatz für den inneren Teil der Kopfkappe. Aus der folgenden Beobachtung erhellt, dass sich Innen- und Aussenkappe gegenüber hypotonischen Einflüssen ganz verschieden verhalten. Behandelt man nl. die Spermien mit hypotonischen Kochsalzlösungen (von 0′4—0′2 %) dann quillt lediglich die Aussenkappe stark an. Wir sehen den Beginn der Quellung in der Abbildung 9. Während die Innenkappe noch als stilettförmiges Gebilde zu sehen ist, bläht sich die Aussenkappe durch starke Wasseraufnahme auf. In der Abb. 8 hat sich der vordere Teil der Kappe noch stärker verflüssigt, während der distale Teil noch als eine elektronenoptisch dichtere Hülse zu sehen ist. Die Kappe hat sich überdies vom Kopf abgehoben und liegt neben ihm. Lässt man die hypotonische Lösung stundenlang einwirken, dann verquillt die äussere Kappe zu einer grossen Gallertblase (Abb. 11), während der innere Teil seine Form behält und nur etwas dicker wird. Wie beim Cavia, lässt sich also auch beim Hahn auf Grund der Quellungsversuche der Schluss ziehen, dass die Kopfkappe aus zwei verschiedenen Differenzierungen besteht.

d) Die Galea capitis der Drohne.

Zu dem Aufbau der Galea aus zwei deutlich zu unterscheidenden Teilen gesellt sich bei der Biene noch eine merkwürdige Form (Abb. 12). Der 0′5 μ breite und 5 μ lange walzenförmige Kopf trägt eine nahezu ebenso lange (4′2 μ) Galea, welche eine komplizierte Differenzierung aufweist. Sie besteht aus einem zylinderförmigen Aussenteil in welchem axial ein 160 mμ dicker Stab- der Innenteil der Kopfkappe- verläuft. Sein zugespitztes Vorderende ragt ungefähr 0′7 μ aus dem Zylinder hervor

und ist in diesem Falle also länger als der Aussenteil der Galea. Dieser Achsenstab wird basal breiter und heftet sich mit einer ringartigen Verdickung am proximalen Kopfpol fest. An der freien Spitze befindet sich ein kleiner elektronenoptisch dichterer Knopf, während eine 50 mμ dicke Spiralfibrille in einigen Windungen um das freie Ende herum zu laufen scheint. Der Zylinderförmige Aussenteil der Galea besteht aus zwei Teilen: einer Art Manchette welche das vordere erste Drittel bildet und dem anschliessenden Hauptstück. Die Grenze zwischen Manchette und Hauptstück wird durch einen elektronen-optisch dichteren Ring markiert. An der Oberfläche des gesammten Zylinders scheint eine 50 mμ dicke Spiralfibrille zu verlaufen, welche vorallem in der Manchette sichtbar ist. Da der Kopf an sich schon licht optisch relativ sehr schmal ist, war der komplizierte Bau dieser Kopfkappe bisher noch gänzlich unbekannt.

e) Die Galea capitis des Tintenfisches Loligo vulgaris.

Im Gegensatz zu dieser kompliziert gebauten Kopfkappe der Drohne steht diejenige des Tintenfisches welche einen primitiven Typus vergegenwärtigt. Sie sitzt ebenfalls dem proximalen Pol des Kopfes an und besteht wahrscheinlich aus einer sehr wasserreichen Substanz, da sie — wie aus dem sehr schmalen Schatten der Abb. 13 zu entnehmen ist — zu einer nur dünnen Lamelle eintrocknet. Das stumpfe Vorderende dieser Kopfkappe ist elektronen-optisch weniger dicht und scheint kraterförmig ein zu sinken. Bei vielen Spermien beobachteten wir in dieser Vertiefung ein elektronen-optisch sehr dichtes Granulum, das vielleicht als das Homologon des Innenteils der Kopfkappe an zu sehen ist. (Abb. 14). Die übrige Kappensubstanz ist homogen, zeigt aber vorallem an dem freien Vorderrande eine Längsstreifung die wahrscheinlich von der Längsorientierung der Bauelemente herrührt.

IV. *Zur Genese der Spiralen des Verbindungsstückes.*

Während bei den Vertebraten immer ein gut ausgebildetes Verbindungsstück zu beobachten ist, wird dies bei Invertebraten schwieriger. So zeigt das Bienenspermium überhaupt kein verbindendes Stück zwischen Kopf und Schwanz. (Abb. 12). Beim Tintenfisch (Abb. 13—14), sitzt ein Homologon des Verbindingsstückes als kegelartiges Gebilde dem distalen Kopfteil seitlich an, während der Schwanz unmittelbar an dem Kopf zu inserieren scheint. Im Verbindungsstück nimmt man ein grösseres elektronen-optisch dichteres Granulum wahr.

Wir haben seinerzeit (1949) beim Spermium des Bullen beschrieben, dass sich im Verbindungsstück eine Doppelspirale befindet. Seit den Untersuchungen Benda's (1898), Meves (1899) u.a. leitet man diese Spiralen von den Mitochondrien ab. In den ausdifferenzierten Spermien wurde bisher auch elektronen-mikroskopisch keine Andeutung gefunden, die darauf hindeutet, dass diese Spiralen aus ursprünglich granulären

Mitochondrien bestünden. Es gelang uns nun in Spermata, welche durch provozierte Ejakulation gewonnen wurden, neben den normalen auch noch nicht völlig ausdifferenzierte Spermien zu untersuchen, bei welchen die Genese der Spiralen zu verfolgen war.

Bei Fröschen (*Rana esculenta*) welche im Jänner zwecks Schwangerschaftbestimmung mit Harn schwangerer Frauen eingespritzt waren, erfolgt Ejakulation. Dabei zeigt sich, dass neben ausdifferenzierten Spermien auch noch zahlreiche Metaspermien ausgeschieden werden. Sie zeichneten sich aus durch den Besitz ihres Zytoplasmas, in welchem zahlreiche (ungefähr 110) granuläre 2 μ bis 3 μ grosse Mitochondrien liegen, (Abb. 16). An der Basis des bananenförmigen Kopfes befindet sich in ausdifferenzierten Spermien beim Frosch nur ein kleines kegelförmiges Verbindungsstück, (siehe rechtes Spermium der Abb. 16) an dem der dünne Schwanz inseriert. In dem Metaspermium dieser Abbildung links befindet es sich noch in der Entwicklung indem es aus einer Anhäufung von Mitochondrien entsteht. Der grösste Teil des Plasmas mit seinen Mitochondrien fliesst dem Schwanze entlang ab. In der Abb. 15 ist dieser Vorgang nahezu beendet und erkennen wir noch den Aufbau des etwa traubenförmigen Verbindungsstückes aus Mitochondrien. Das Restplasma mit seinen überzähligen Mitochondrien ist im Begriffe sich aufzulösen und gleitet dem Schwanze entlang. An manchen Mitochondrien lässt sich ein dichteres Zentrum und eine lichtere Schale erkennen. Im Schwanz lassen sich mit Sicherheit nur 7 Subfibrillen feststellen, aus dem Grad der photographischen Schwärzung ist aber zu vermuten, dass zwei Subfibrillen überdeckt sind, woraus man auf 9 Fibrillen schliessen könnte. Eine Spiralfibrille vermisst man aber. Im Gegensatz zum Frosch, in dessen kurzem Verbindungsstück keine deutliche spiralige Anordnung der Mitochondrien zu sehen ist, finden wir beim Hahn eine gut ausgebildete Spirale. Ein für die künstliche Besamung provoziertes Ejakulat besass viele noch nicht ausdifferenzierte Spermien. Sie wurden mit einer hypotonischen 0,4 % NaCl-Lösung gespült, wobei die äussere Hülle des Verbindungsstückes und Schwanzes platzte und dadurch die Analyse des inneren Aufbaues erleichtert wurde. Im Gegensatz zu den bereits ausdifferenzierten Spermien deren Verbindungsstück trotz der Hypotonie intakt blieben (Abb. 22), quillt die Spirale der nichtausdifferenzierten Spermien durch den Wasserzutritt. Wir sehen dann — je nach dem Grade der Entwicklung — dass die Spirale in distinkte 350—500 mμ grosse granuläre Einzelmitochondrien auseinanderfällt. (Abb. 17 und 20). In extremen Fällen der hypotonischen Wirkung und wahrscheinlich an den noch am wenigsten ausdifferenzierten Spermien krümmt sich der Kopf sichelförmig und zieht sich der, seiner Aussenhülle beraubte, Schwanz stark zusammen, wodurch die Mitochondrien der Spirale als einzelne Granula sichtbar werden (Abb. 17). Sie liegen dann in den zahlreichen Schlingen der Subfibrillen lose eingebettet und sind dann für die Feststellung ihrer Zahl zugänglich. Wir stellten zwischen 18 und 21

fest. Die normale Lokalisation der Mitochondrien im Verbindungsstück zeigt die Abbildung 20, in welcher die Mitochondrien als 20 lose Granula um die fibrilläre Achse herum liegen. Dahingegen findet in der Abbildung 21 bereits eine Anordnung der Mitochondrien in einer Spirallinie um den Achsenstrang herum statt. In einem weiter fortgeschrittenem Stadium (Abb. 18) verschmelzen die aneinander gereihten Mitochondrien zu einer Spirale, an welcher deutliche Abgrenzungen der Granula nicht mehr gut zu sehen sind. Ihr Volumen nimmt überdies ab, wodurch im ausdifferenzierten Spermium (Abb. 22) das gesammte Verbindungsstück schmäler wird als der Kopf. Eine Spiralstruktur ist dann nur noch andeutungsweise vorhanden. Das Verbindungsstück wird auch bei diesem Spermium distal durch eine ringförmige Struktur — dem Ring von JENSEN — abgeschlossen. (Abb. 19).

V. *Der Einfluss der Schwellung auf den Spermienschwanz.*

Unter der hypotonischen Einwirkung verändert sich der Schwanz so weitgehend, dass man etwas mehr von seinem Aufbau zu sehen bekommt. Durch den Druck des eingedrungenen Wassers tritt Schwellung ein und wird die zarte Schwanzmembrane gesprengt. Dadurch werden die 9 Subfibrillen aus ihrem Plasmamantel befreit und kontrahieren sich stark, wie dies bereits die Abb. 11 erkennen liess. Da die Schwellung wie licht-optisch festgestellt wurde plötzlich beim Hinzufügen der hypotonischen Kochsalzlösung eintritt, erfolgt dieser Vorgang noch im lebenden Zustand des Spermiums. In den meisten Fällen verknäueln sich die Subfibrillen mit dem plasmolysierenden Schwanzplasma und ziehen sich kopfwärts zusammen. Auch der Kopf krümmt sich sichelförmig und nimmt in seiner Krümmung das Fibrillenknäuel und das Schwanzplasma auf, (Abb. 17). In verschiedenen Aufnahmen (Abb. 7, 10, 11, 17, 19) ist die grobe netzförmige Struktur des Schwanzplasmas als Maschenwerk zu sehen. Die ca. 50 mμ dicke Spiralfibrille des Schwanzes wird infolge der hypotonischen Einwirkung gesprengt und ist nur noch als Fragmente an den Subfibrillen zu erkennen (Abb. 20) links. Hinsichtlich der chemischen Zusammensetzung der Spirale im Verbindungsstück lässt sich elektronen-optisch ebenfalls durch die Quellungsversuche mit hypotonischen Lösungen etwas näheres aussagen. Man bemerkt dann bei den Säugerspermien, dass die Lipoidfraktion und zwar hauptsächlich das Lecithin reichlich Wasser aufnimmt und durch die Membran des Verbindungsstückes hindurchdringt (Abb. 1 und 3). Dabei behält dieses gequollene Lecithin seine topographische Beziehung zur Spirale, denn es folgt auch ausserhalb der Membrane noch im Grossen und Ganzen den Spiralwindungen. Aus der elektronen-optischen Massenverteilung dieser Myelinfigur mit dem helleren Zentrum, kann man den Schluss ziehen, dass es sich ursprünglich um eine lipoide Oberflächenschicht der Spirale handelte. Diese Beobachtung stimmt überein mit der allgemeinen Annahme, dass die Mitochondrien gekennzeichnet sind durch eine solche Lipoidhülle.

VI. *Die isolierte und gestreckte Doppelspirale beim Bullen.*

In einem besonderen Falle liess sich die Doppelspirale eines Bullenspermiums isoliert darstellen. Bei einem wahrscheinlich bereits längere Zeit, totem Spermium lösten sich nach der proteolytischen Einwirkung von Chloramin-T (Bretschneider 1947—1949) die bereits mazerierten Subfibrillen gänzlich auf, während die lipoidreichen Spiralen übrig blieben. Diese wurden beim Zentrifugieren des Spermas stark in die Länge gezogen, wobei sich die Windungen streckten. Beim eintrocknen legten sie sich in abgeplattete Schlingen übereinander und ergaben nach der Beschattung das in der Abb. 23 wiedergegebene Bild. Wir konnten diese Spiralen über ihren gesammten Verlauf verfolgen, aber nirgends etwaige Grenzen ursprünglicher Einzelmitochondrien entdecken. In der rechten unteren Ecke sieht man noch einen Teil eines normalen Spermiums mit den dicht stehenden Windungen der Doppelspirale.

VII. *Zur Färbung der Spirale des Verbindungsstückes.*

Eine ergänzende Beobachtung zu unseren früheren Mitteilungen über die Spirale des Verbindungsstückes bezieht sich auf die Eigenschaft der Spiralen des Bullenspermiums postmortal Eisenionen stark zu speichern. Versetzt man normales Sperma welches regelmässig eine gewisse Zahl bereits toter Spermien besitzt mit einer Eisenzuckerlösung und wäscht nachher gut aus, dann tritt elektronen-mikroskopisch die Spirale deutlich hervor, Abb. 24—26.

Durch diese selektive Anhäufung des schweren Eisenions entsteht ein kontrastreiches Bild. Ausserdem quellen in der Eisenzuckerlösung die Spiralen etwas und lockern sich dadurch auf. Soweit der Verlauf der Spiralen nicht durch das Zentrifugieren gestört ist, erkennt man aus dem parallelen Verlauf die doppelten Spiralwindungen. Sie endigen kurz vor dem Ring von Jensen, dem man in der Abb. 24 ebenfalls noch sieht. Die Schwanzstrukturen und die vordere Kopfhälfte nehmen keine Eisenionen auf, Abb. 26, dahingegen speichert die becherförmige distale Kopfhülle das Eisen merklich. Spermien welche zur Zeit der Eiseneinwirkung noch lebten, nehmen auch in ihren Spiralen kein Eisen auf. Die Gegenprobe wurde gemacht indem Sperma mit überwiegend lebenden Spermien durch Hitze abgetötet und danach mit Eisenzucker versetzt wurde. Dann adsorbieren die Spiralen aller Spermien das Eisenion. Wir vermuten dass der Lipoidanteil der Spiralen für die Speicherung des Eisenions verantwortlich ist und zwar deshalb nur an bereits toten Spermien, weil deren äussere Hülle zerstört oder zumindest für so grosse Moleküle permeabel wird.

VIII. *Der undulierende Geisselsaum des Salamanderspermiums.*

Bekanntlich besitzt das Spermium von *Triton taeniatus* einen langen undulierenden Geisselsaum, mit welchem sich das Spermium trypanosomenähnlich fortbewegt. Abb. 28. Der Saum besteht aus einem äusserst

dünnen Plasmafilm der seitlich vom Schwanz absteht und nach aussen zu begrenzt wird durch einen sogenannten „Randfaden". Da dieser viel länger ist als der Spermiumschwanz, faltet sich der Geisselsaum krausenartig auf. Bei starken elektronen-mikroskopischen Bergrösserungen zeigt sich dass der einfache Randfaden nicht aus einer Fibrille besteht, sondern aus zumindest 9 im Durchmesser 160 mμ dicken Subfibrillen, Ab. 27. Einige dieser randständigen Subfibrillen verkleben miteinander zu einem dickeren Strang, die anderen liegen nebeneinander in einer Ebene auf der undulierenden Membrane.

Wie aus dem nur geringen Unterschied der photographischen Schwärzung zwischen Objektträgeruntergrund und Geisselsaum zu ersehen ist, muss diese Membrane äusserst dünn sein. Ueber eine Eigenstruktur ist deshalb elektronen-optisch nichts auszusagen, weil die gerichtete Eigenstruktur des Cellulose-filmes vom Objektträger vorherrscht. In der elektronen-optisch schwächeren Aufnahme, Abb. 28, ist im linken Spermium der Uebergang vom Verbindungsstück in den Hauptteil des Schwanzes zu sehen, rechts das Verbindungsstück allein. In ihm fällt wieder ein Spiralkörper auf dessen Windungen 100 mμ im Durchmesser messen und anscheinend rechtsgewunden verlaufen; dies im Gegensatz zu den Säugerspermien, die linksgewundene Spiralen besitzen. Am Uebergang vom Verbindungsstück in den Schwanz (linkes Spermium) sehen wir als elektronen-optisch dichtere Struktur den zentralen Achsenstrang und das lichtere umgebende Schwanzplasma mit der Aussenmembrane. An beiden inseriert die undulierende Saumgeissel.

IX. *Einige theoretische Bemerkungen.*

Die Galea capitis erweist sich als ein obligater Bestandteil des Spermiums sowohl der Vertebraten als der Invertebraten. Ihre konstante topographische Beziehung zum Kopfpol unterstützt die Meinung, dass sie mit der Bahnung des Weges zur Eizelle etwas zu schaffen hat. Man denkt sich die Kopfkappe als Träger der Hyaluronidase, doch sprechen neuere physiologische Untersuchungen mehr für ein zweifaches Fermentsystem das an der Aufschliessung der Hyaluronsäure und der Eihüllen betätigt ist, (siehe die Zusammenfassung von Bielig und Medem, 1949). Vielleicht steht die Doppelnatur der Kopfkappe aus Innen- und Aussenkappe damit in Beziehung. Morphologisch lässt sich dies vorläufig nicht ermitteln. Aus den hier beschriebenen Kopfkappen beim Cavia, Hahn und vorallem der Drohne erhellt, dass diese beiden Teile der Kopfkappe nicht etwa einen einfachen Sekrettropfen darstellen, sondern einen recht komplizierten Bau aufweisen und selbst bestimmter Stützfibrillen (Drohne) nicht entbehren. Aus der Beobachtung mit hypotonischen Lösungen in welchen vorallem die äussere Kappe stark quillt, kann man schliessen, dass sie aus einer Substanz besteht, welche unter normalen isotonischen und pH Milieuverhältnissen wasserarm sein muss. Sie erinnert dadurch an jene biologischen Feinstrukturen die sich ebenfalls

aus wasserarmen konzentrierten Muzinen zusammensetzen und erst im Wasser stark schwellen (Eihüllen, Trichocysten, Spermatophoren, Cuticulae etz.). Da die Kopfkappe intracellulär entsteht, also unter den richtenden Kräften des Cytoplasmas und nicht als ein Sekretprodukt anzusehen ist, wird die spezifische Eigenform begreiflich. Es zeigt sich dass Störungen in der Differenzierung (Acrosomkappe beim Stier) zum Nichtfunktionieren des Spermiums im Befruchtungsprozess führen (Sterilität). Zur Funktion gehört scheinbar nicht allein das Vorhandensein des Materials, sondern auch seine bestimmte Anordnung als Form.

Die Feststellung, dass eine Vielheit loser unregelmässig im Plasma verteilter Granula (Mitochondrien) in einem bestimmten Zeitpunkt der Entwicklung sich planmässig ausrichten und durch Verschmelzung eine neue Zellstruktur bilden, ist eine interessante morphogenetische Erscheinung. Man kann sich die Feinstruktur eines kugeligen Mitochondriums konzentrisch oder radiär vorstellen. Dahingegen weist sowohl die glatt verlaufende ausdifferenzierte Spirale, die selbst nach erheblichen Zug — siehe das Beispiel der zentrifugierten- und isolierten Spirale — keine Abgrenzung oder Brüche erkennen lässt als auch die beobachtete Doppelbrechung darauf hin, dass seine Feinstruktur längsorientiert sein muss. Es muss demnach im Augenblicke der Verschmelzung der Mitochondrien eine Umorientierung der Feinstruktur aus der konzentrischen oder radiären in die Längsrichtung stattfinden. Gresson (1941) stellte an der Maus licht mikroskopisch fest, dass die Spirale der in das Eiplasma eingedrungenen Spermien wieder in granuläre Mitochondrien zerfällt, dieser Vorgang also selbst reversibel ist. Welche poläre Kräfte hier wirksam sind, ist zur Zeit völlig unbekannt. Bemerkenswert ist fernerhin, dass sich nur eine bestimmte in engen Grenzen schwankende Zahl von Mitochondrien zur Spirale verbinden, der Rest aber abgestossen wird. Morphologisch ist weder der Grösse noch der Form nach ein Unterschied zwischen beiden zu sehen.

Bemerkenswert ist die Feststellung am Geisselsaum des Tritonspermiums, dass dieser nicht einheitlich ist, sondern aus einer Anzahl Subfibrillen besteht, deren Dicke ca 160 Å beträgt. Auch lassen diese Subfibrillen die Andeutung eines periodischen Aufbaues erkennen. Obwohl wir bei diesen Vergrösserungen noch nicht den makromolekularen Aufbau zur Anschauung bringen, lässt diese Zusammensetzung aus dünnen Einzelfibrillen erkennen, dass sich das Plasma weit kleinerer Elementarteile bedient, als wir bisher licht-optisch darstellen konnten.

Eine weitere interessante Erscheinung ist die immer wieder auftretende Tendenz der Feinstrukturen der lebenden Substanz zur spiraligen Anordnung. Im Verbindungsstück ordnen sich die einzelnen Mitochondrien zu einer Spiralform zusammen und verschmelzen miteinander. Die Hülle des Schwanzes besitzt spiralige Fibrillen, und zwar in einer Anordnung wie wir sie auch in der Geissel von Flagellaten (Brown, 1945) oder der Cilie von Ciliaten wieder finden (Bretschneider, 1949/50). Aber auch

die Kopfkappe des Bienenspermiums gebraucht die Spiralfibrills als stützende Struktur. In den meisten Fällen (mit Ausnahme des Triton spermiums) handelt es sich um linksgewundene Spiralen. Es lohnt sich dieses Spiralproblem weiter zu verfolgen, da es möglich ist, dass diese Linkswindung viel mehr verbreitet ist und vielleicht letzten Endes zusammenhängt mit einem asymmetrischen Bau des Eiweissmoleküls.

X. *Zusammenfassung.*

In Anbetracht der funktionellen Wichtigkeit, welche man heute der Galea capitis im Befruchtungsprozess zuschreibt, untersuchten wir in erster Linie diese Struktur des Spermiums, da sich heraus stellte, dass eine gewisse pathologisch veränderte und zwar nicht ausdifferenzierte Kopfkappe beim Bullen zur Sterilität führt.

1. Bei den bisher untersuchten Vertebraten und Invertebraten liessen sich vergleichend-cytologisch zwei Bautypen unterscheiden. Bei der einen Form umhüllt die Galea capitis einen Teil des Spermakopfes kappenförmig (Pferd, Rind, Hund, Cavia), bei der anderen ist die Galea als eine frei hervorragende Fortsetzung des Kopfes zu betrachten (pathologische Kopfkappe beim Bullen, Hahn, Drohne, Tintenfisch).

2. Im Allgemeinen liessen sich nun elektronen-mikroskopisch auch am reifen Spermium zwei Teile an der Galea feststellen: ein Innenteil der genetisch vom Acrosom herstammt und ein Aussenteil, der sich vom Acroblastbläschen ableiten lässt. Sowohl morphologisch- als Teile mit einer bestimmten Eigenform- als auch ihrem chemisch-physikalischen Verhalten bei der Quellung nach sind diese beiden Teile als zwei verschiedene Strukturen zu betrachten.

3. Dem Grade ihrer Differenzierung nach ist die Galea des Tintenfisches *Loligo* am einfachsten gebaut, bei der Drohne am kompliziertesten. Die Galea erweist sich bei der Drohne als ein aus verschiedenen formbeständigen Teilen zusammengesetztes Gebilde, bei dem Tintenfisch als eine offenbar sehr wasserreiche etwas konisch geformte Protuberanz am vorderen Pol.

4. Die pathologisch bei manchen Bullen vorkommende sogenannte „Acrosomkappe" — einer in der Entwicklung gehemmten Galea — gleicht in ihrem Bau dem primitiven Typus des Cephalopodenspermiums, bei welchem der Innenteil in Form eines Granulums in der Substanz des Aussenteiles eingebettet ist.

5. Durch die Untersuchung von Spermien welche noch nicht völlig ausdifferenziert waren, konnte beim Frosch und Hahn die Genese der Spirale im Verbindungsstück aus ursprünglich distinkten granulären Mitochondrien elektronen-mikroskopisch nachgewiesen werden. Es zeigte sich dabei, dass nach der Anordnung der einzelnen Mitochondrien in einer Spirallinie diese schliesslich so völlig mit einander verschmelzen, dass sie einen einheitlichen Spiralfaden bilden.

6. Die Aufnahme einer isolierten Doppelspirale des Bullenspermiums

wird beschrieben und festgestellt, dass es sich auch hier um einen einheitlichen Spiralfaden handelt.

7. Es wird gezeigt, dass die Spiralen toter Spermien nach Behandlung mit Eisenzucker selektiv Eisenionen adsorbieren und auf diese Weise elektronen-optisch sehr kontrastreich abgebildet werden können.

8. Schliesslich wird der undulierende Saum des Triton-spermiums beschrieben, dessen sog. Randfaden aus ca. 9 Stück 160 mμ dicken Subfibrillen besteht und einen sehr dünnen Plasmasaum begrenzt.

9. Hinsichtlich der elektronen-mikroskopischen Technik wird gezeigt:

a) dass die Anwendung der durch Hypotonie erzeugten Quellung eines der Mitteln ist, um elektronen-optisch zu dicke Objekte der Elektronen-Mikroskopie zugänglich zu machen, (Quellung der Kopfkappe und Spirale, Sprengung der Aussenmembrane im Verbindungsstück und Schwanz, verbunden mit dem Sichtbarmachen von Mitochondrien Subfibrillen und Plasma).

b) dass durch Anwendung einer Eisenverbindung (Saccharas ferricus) das Eisenion als spezifisches Färbemittel zu einer guten elektronenoptischen Kontrastwirkung führt, (Spiral- und Kopfbecherfärbung).

Die Untersuchungen wurden durch die Unterstützung der "Niederländischen Organisation für angewandte Landwirtschaftliche Untersuchungen T.N.O." ermöglicht, der wir hier unseren Dank aussprechen.

XI. *Summary.*

According to a recent publication of Teunissen (1946) it appeared that the abnormally formed head-cap (acrosome-cap) in the bull leads to sterility.

From this observation it appeared that the head-cap plays a very important role in fertilization.

Therefore we investigated by electron-microscopy the head-caps in various animals, to get a better insight through comparison of different forms. We came to the following conclusions:

1. In the invertebrates and vertebrates investigated two structure-types can be distinguished:

a. The head-cap covers in cap-form part of the sperm-head (bull, horse, dog, Cavia).

b. The head-cap is separate, proximally standing out on the head (pathological head-caps of cock and bull, drone, squid).

2. It is possible to distinguish electron-microscopically in the head-caps of ripe sperms two parts, an inner (originating from the acrosome) and an outer part (originating from the acroblast). Morphologically (as parts with a certain form) as well as physico-chemically (by their behaviour in swelling) they show different structures.

3. According to its degree of differentiation, the head-cap of the squid is very simple, the head-cap of the drone very complex.

4. The pathological "acrosome-cap" of the bull sperm resembles, through inhibition of its development, much the simple-structured sperm of the squid. In both cases the inner part of the head-cap is embedded as a granulum in the substance of the outer part.

5. It was possible to investigate undifferentiated sperms in the ejaculate of the frog and the rock. The genesis of the homogeneous spiral body from the original free granular mitochondria could be followed step by step.

6. A description of a photo of an isolated double spiral of a bull sperm is given.

7. After application of ferric sugar the spiral body showed a strong selective adsorption of the ferric-ion; with this method the spiral body could be demonstrated very clearly.

8. A description is given of the structure of the undulatory membrane in the sperm of the newt.

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Fig. 6. Spermium vom *Hahn*. Mercurochrom. 1 : 18.000. 90 kV.
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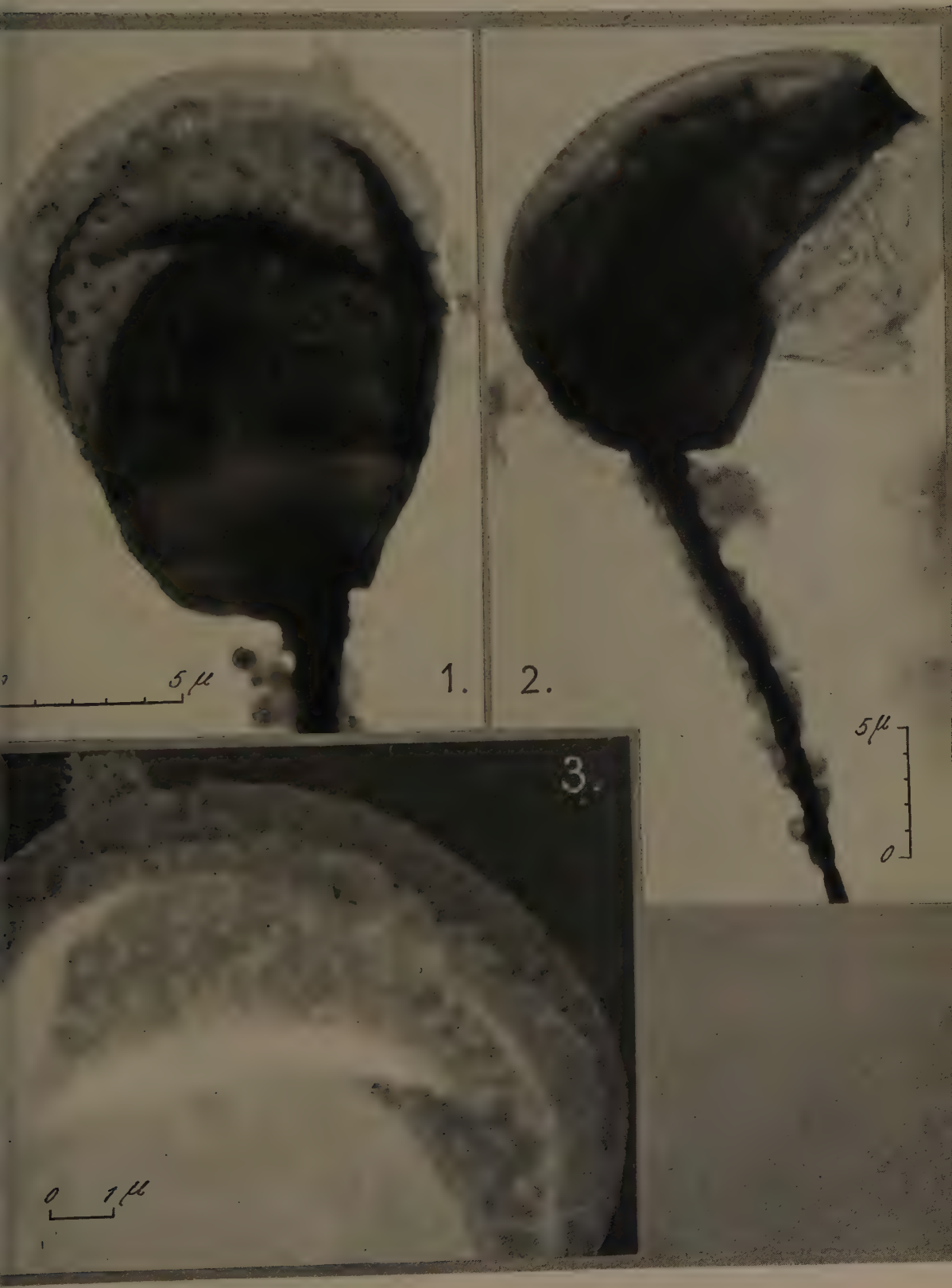
Fig. 28. Spermium von *Triton taeniatus*. Mercurochrom. 1 : 18.000. 100 kV.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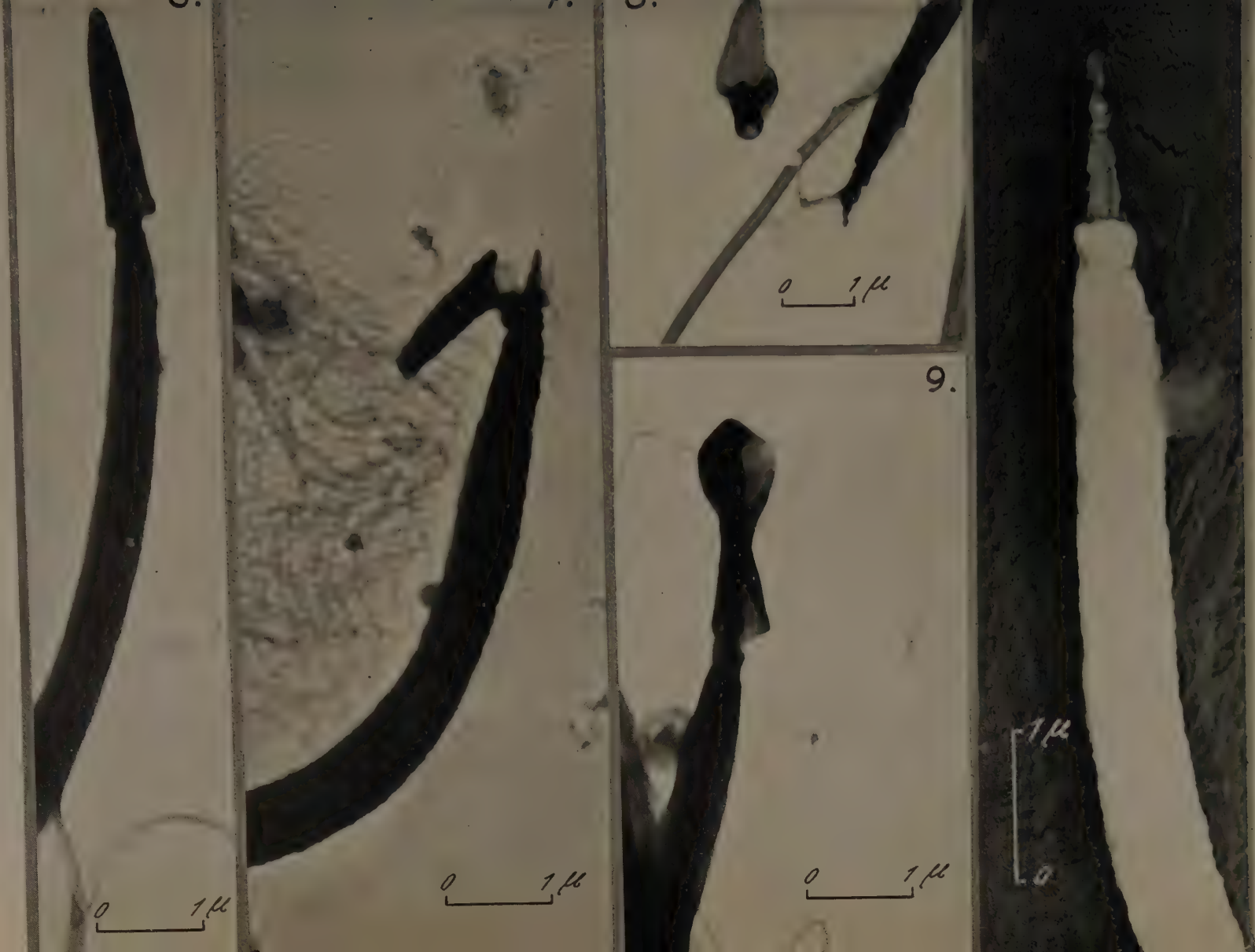
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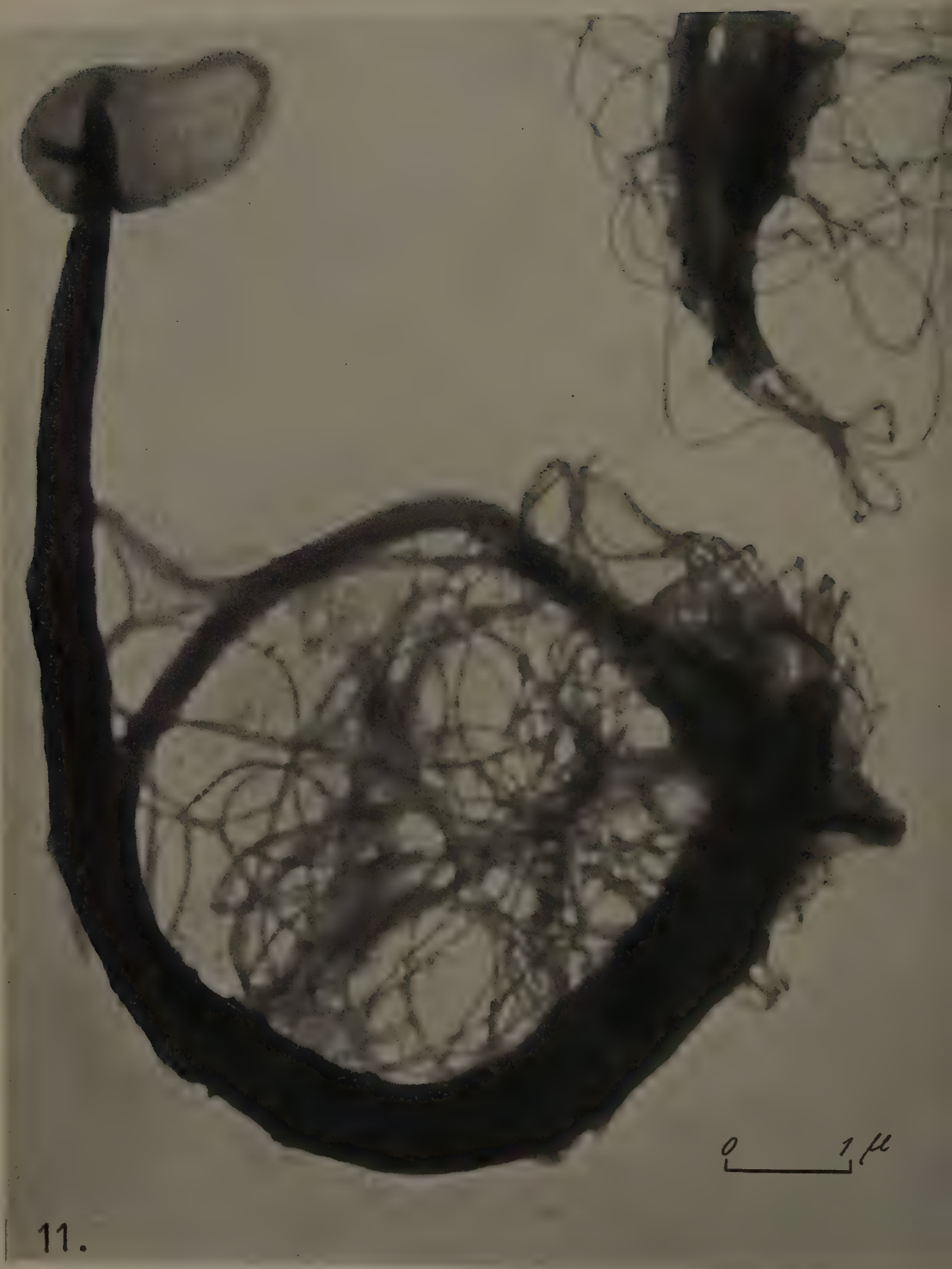

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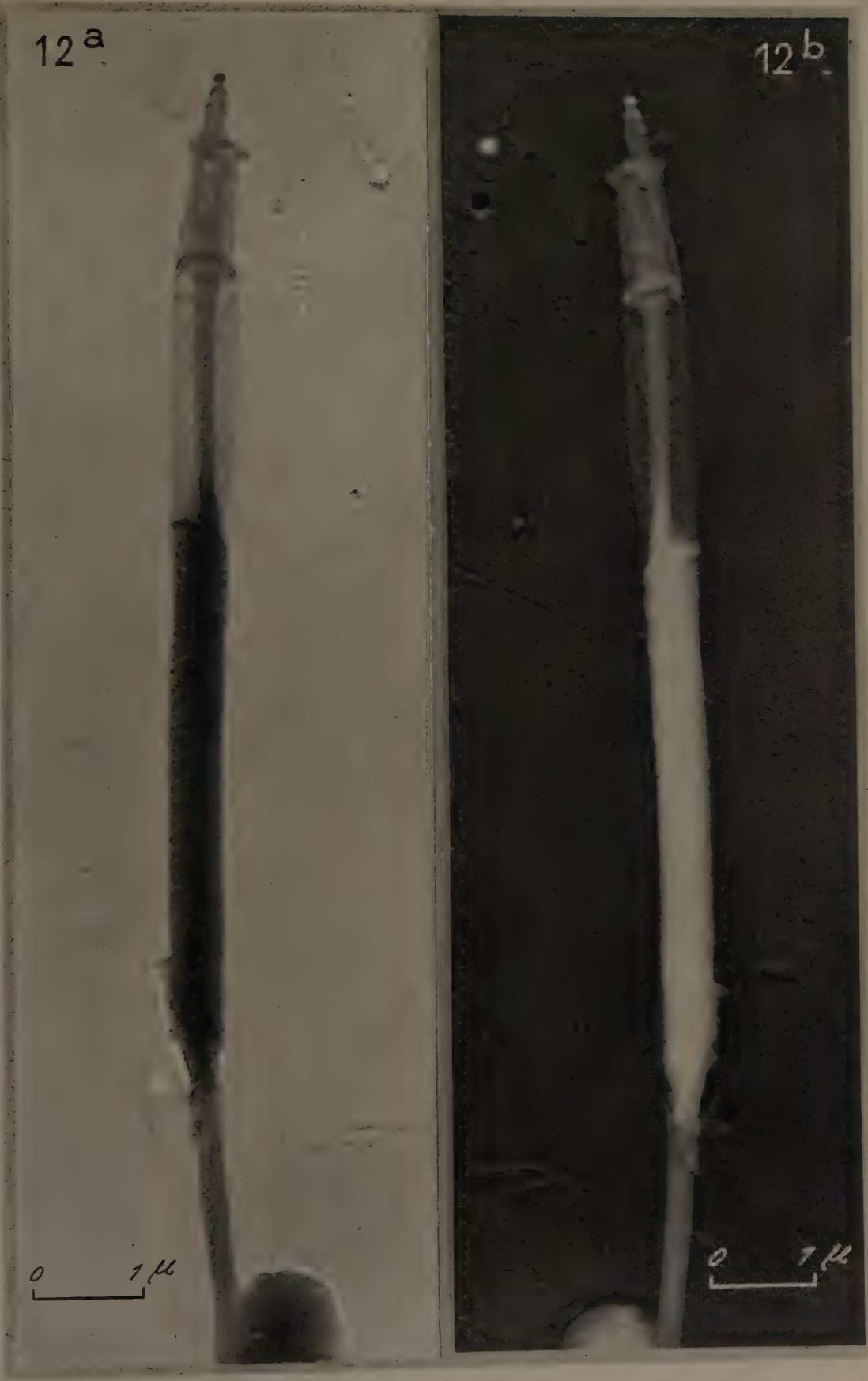
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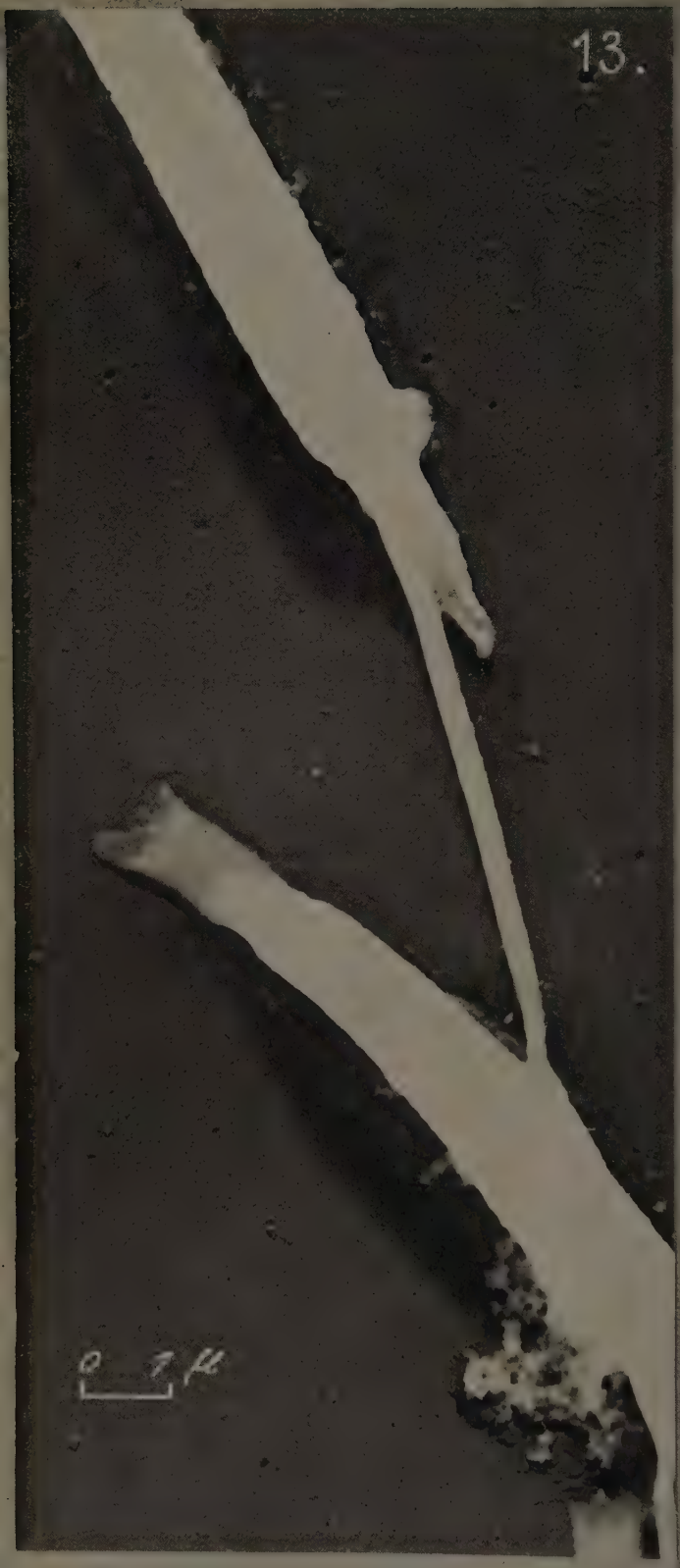
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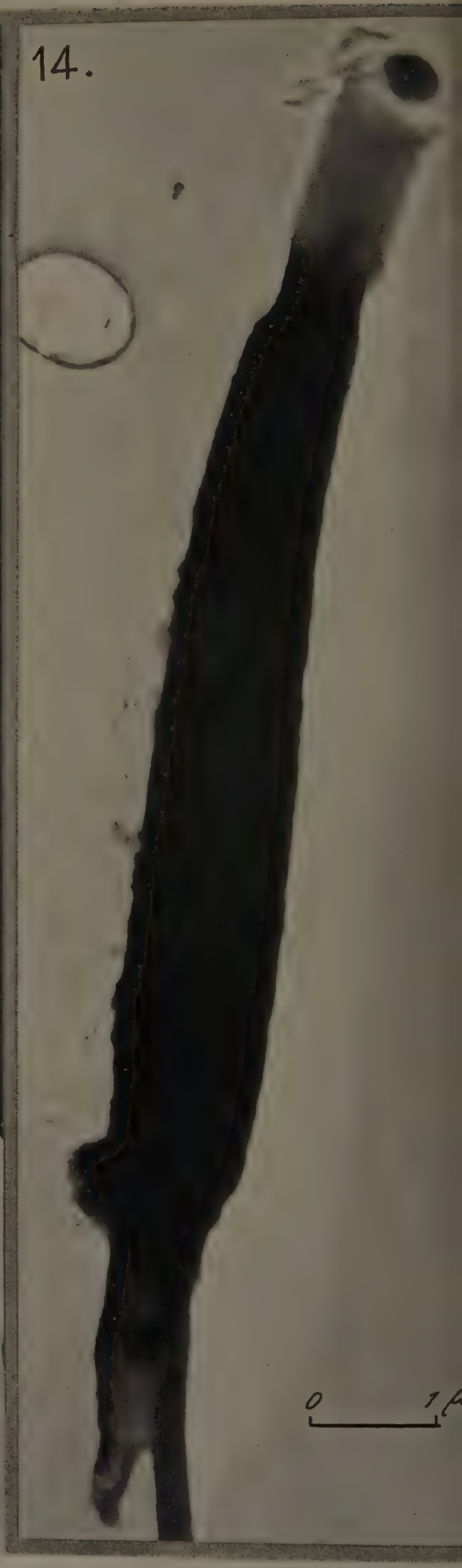
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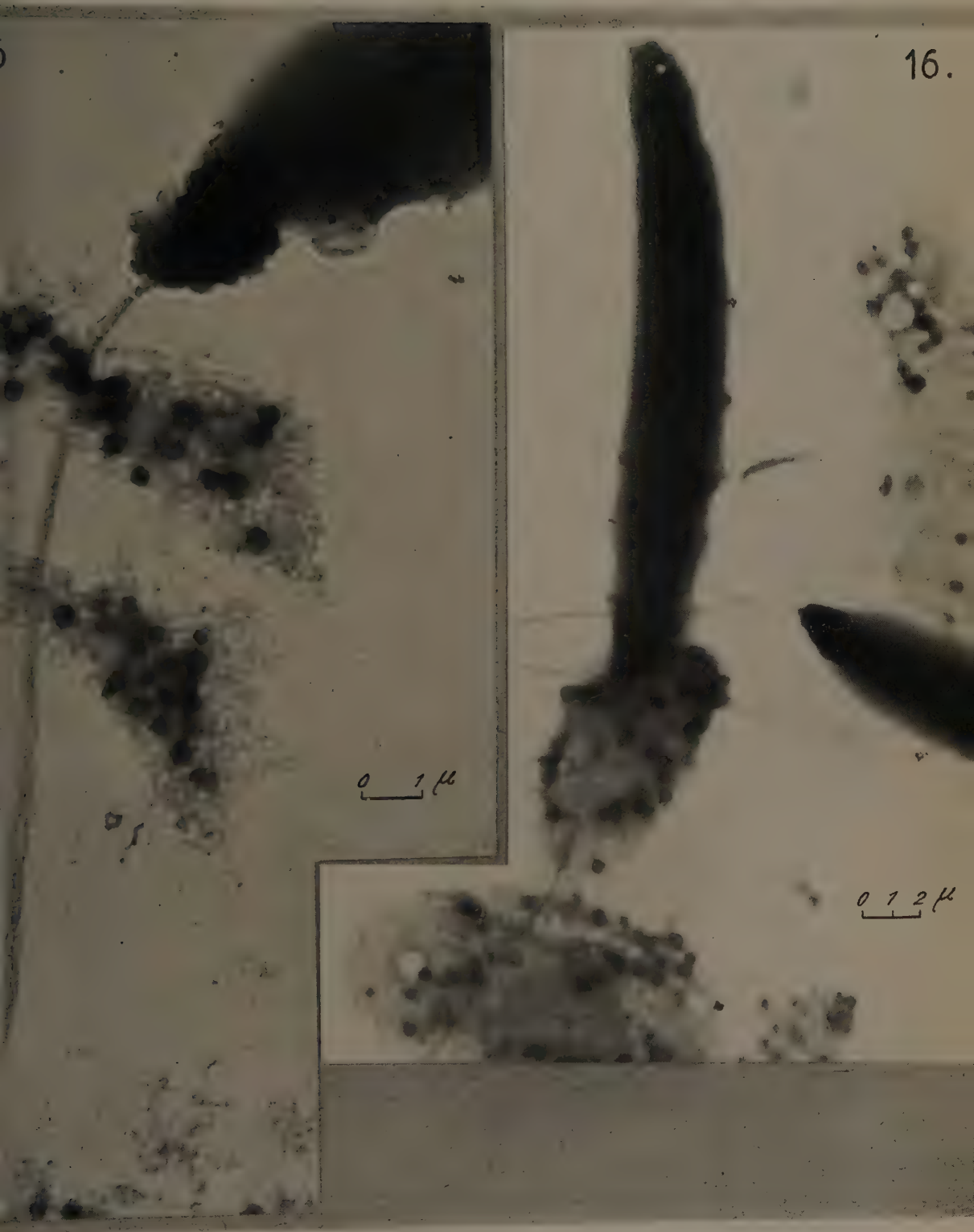
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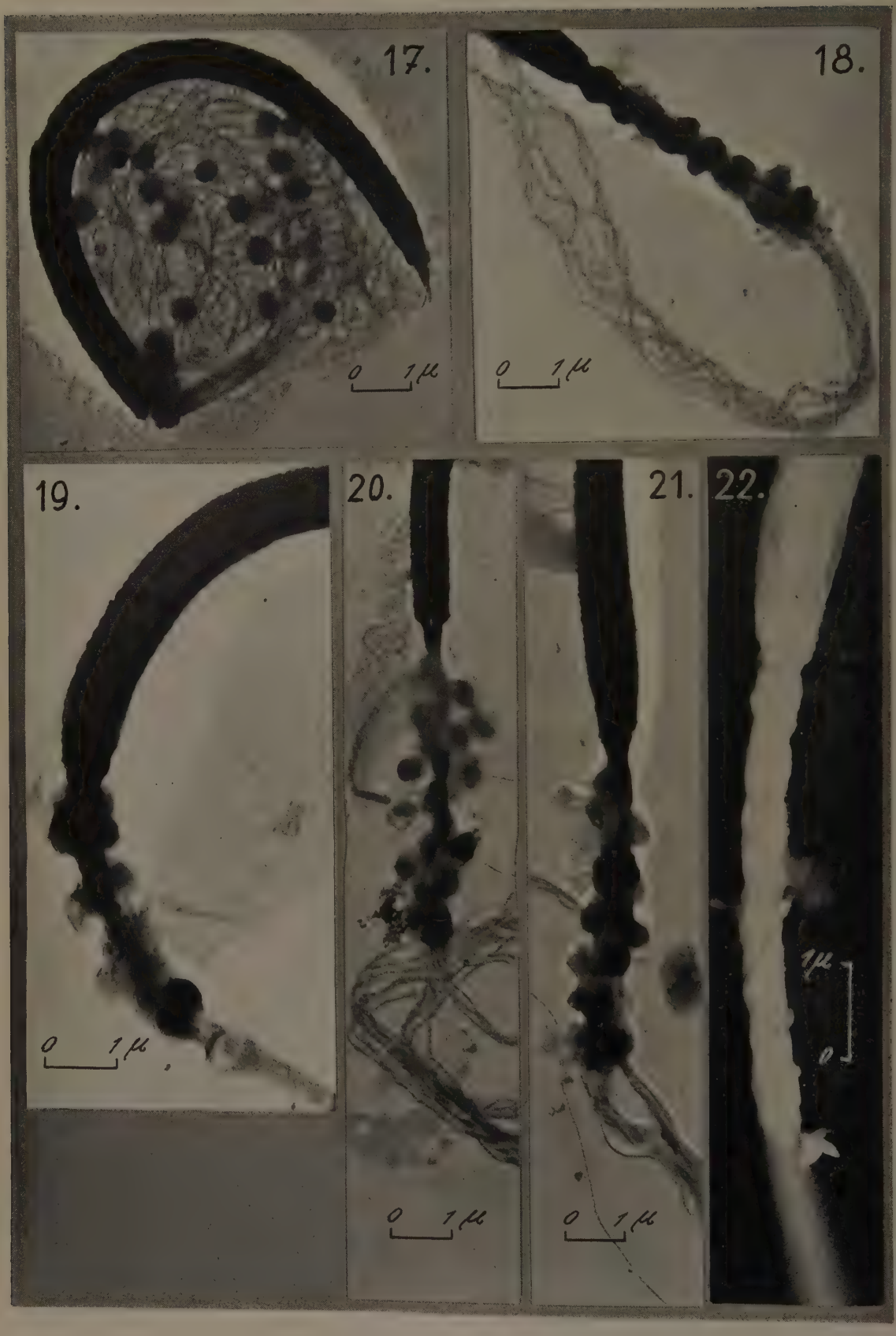
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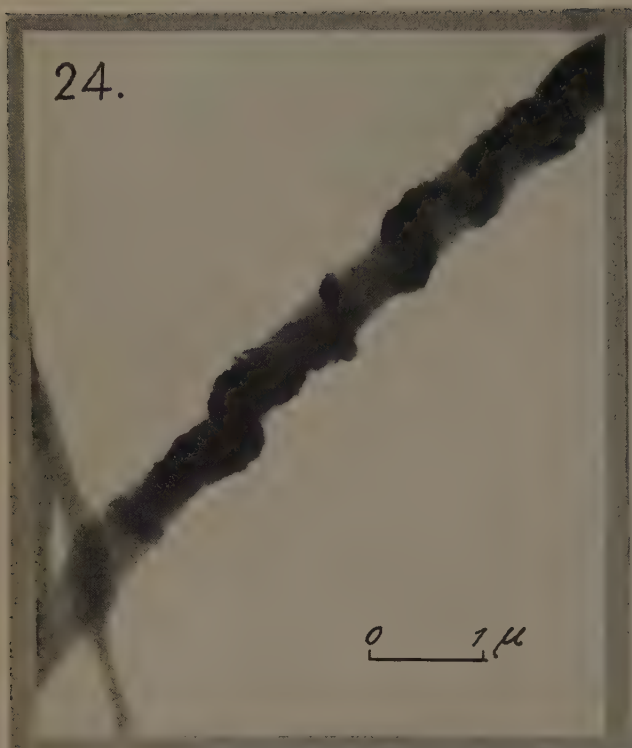
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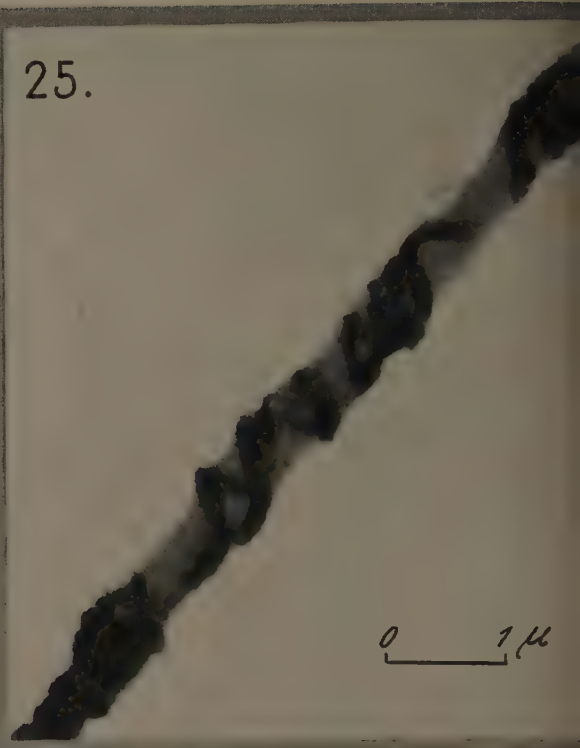
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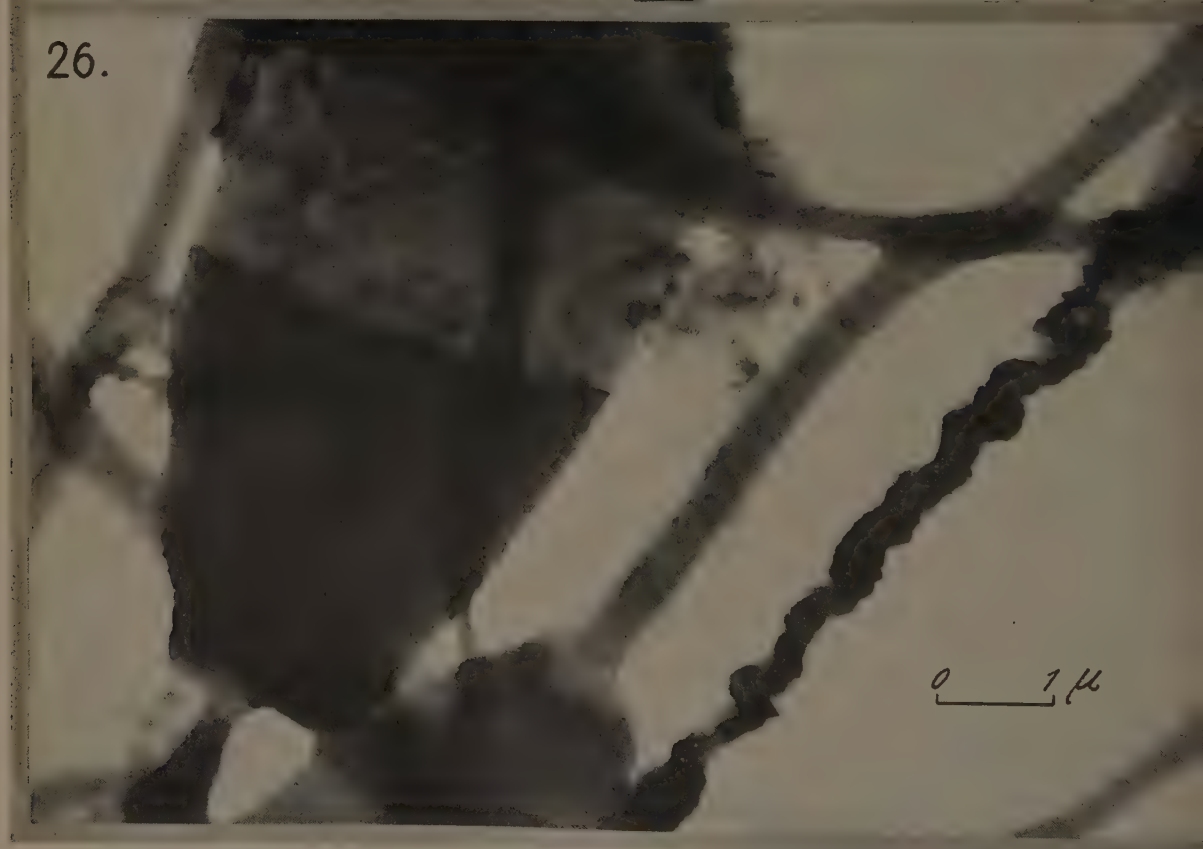
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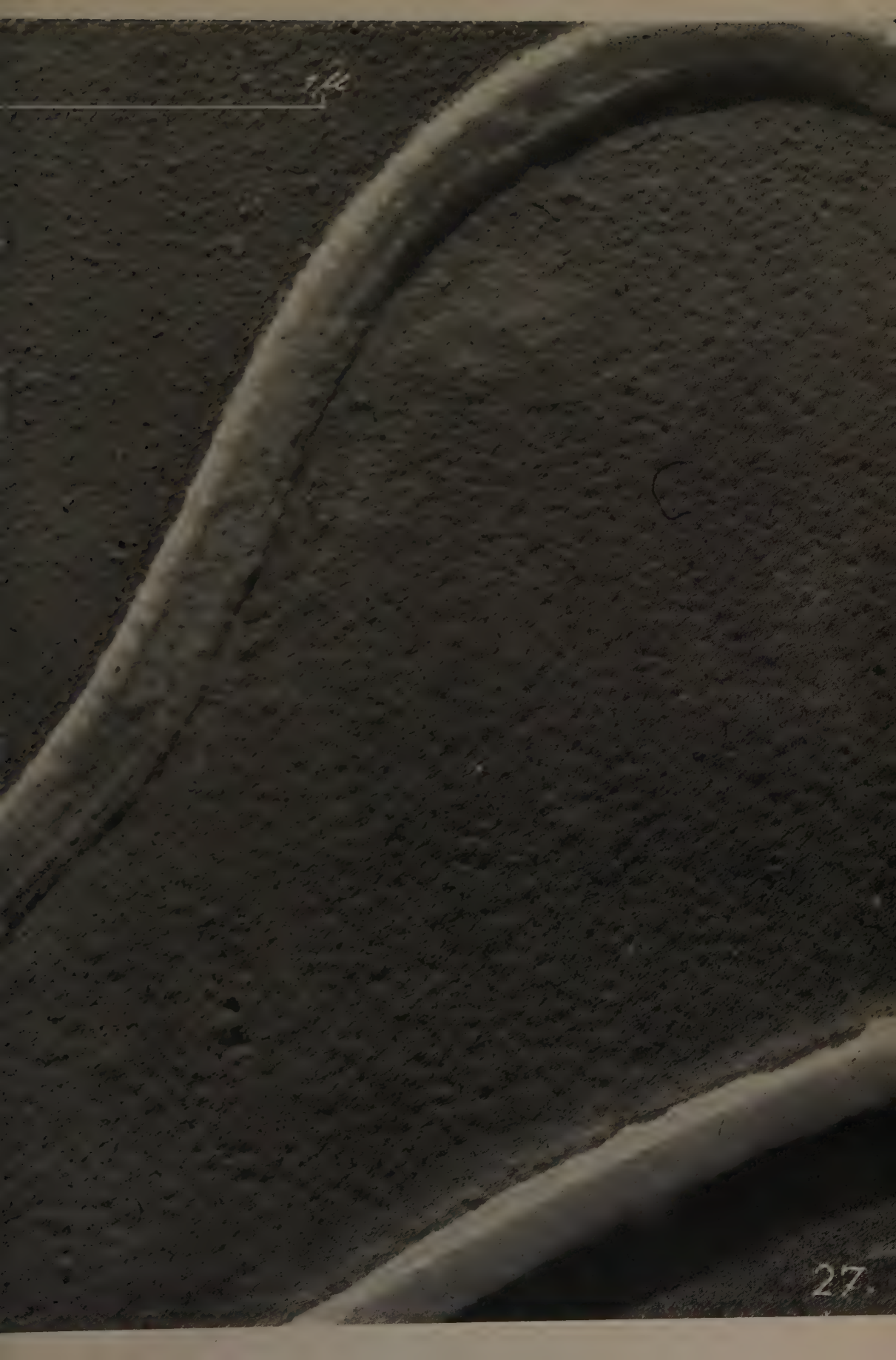

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TECTONICS OF A PRETECTONIC ORTHOGNEISS MASSIF NEAR ST. JEAN DU GARD IN THE SOUTHEASTERN CEVENNES, FRANCE. I

BY

D. DE WAARD

(Communicated by Prof. F. A. VENING MEINESZ at the meeting of Febr. 25, 1950)

1. Abstract
2. Introduction
3. Previous geologic work
4. Outline of the geology
5. Origin of the gneiss complex
6. Microtectonics
7. Age of the intrusion
8. Tectonic outline
9. The general structure of the area
10. Types of contacts
11. The structural behaviour near the contact
12. Tectonics of veins and dikes
13. The shape of the gneiss massif
14. The shape of the intrusion
15. The origin of the cleavage pattern
16. The history of the gneiss massif
17. Conclusions
18. Correlation with augen gneiss massifs in the Orthocevennes Complex
19. References

1. *Abstract.*

A 10 km long and 4 km wide augen gneiss massif near St. Jean du Gard in the southeastern Cevennes of the Central Massif in France is here shortly defined as "a small, lenticular, nuclear pluton, pretectonic intruded in epitype conditions as a compound laccolith of porphyritic granite or granite porphyry which is deformed in epitype conditions to a planar, orbicular gneiss, harmoniously infolded, partly concordantly and conformably, in the Orthocevennes Complex of the variscan orogen".

The scope of the investigations was the detailed description and the interpretation of the structures of a pretectonic pluton and the petrogenesis of the gneiss. An attempt has been made at revealing the origin of the structure pattern in massif and country rock, with its complications

near the contacts. These structures are interpreted as the co-operative effects of compaction, extension and shearing in the inhomogenious series of a relatively rigid igneous rock and a more plastic sediment, which reacted differently during the orogeny.

It is especially emphasized that it is the depth of the orogenic zone in which these structures originated, that is the main influence upon the form of the structure pattern. Each zone with its specific properties of plasticity, recrystallization and deformation possibilities generally, effects different structures from the same original components.

2. *Introduction.*

In the summer of 1948 a small area of about 60 square km between the villages of St. Jean du Gard and l'Estréchure, 20 km W of Alès, was investigated (fig. 1). This area, as a whole being well-exposed, is mountainous up to 750 m and intersected from W to E by the meandering river Le Gardon de St. Jean with many short tributaries in narrow V-shaped valleys.

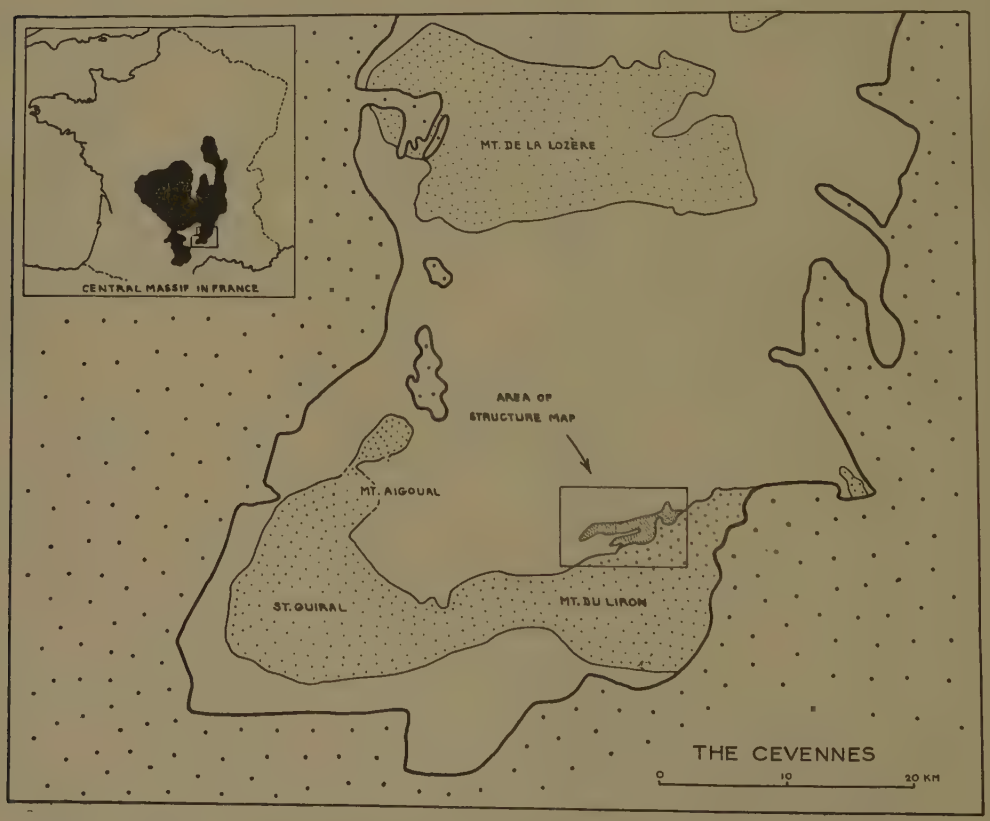


Fig. 1. Map of the southeastern part of the Central Massif showing the geographical position of the mapped area between slates (white) and posttectonic granite massifs (dotted). The surrounding Mesozoic strata are indicated by heavy dots.

The north of the area is part of the large slate complex of the Cevennes, which is intruded by several granite massifs. One of those, the Mt. Aigoual pluton, has been previously described (DE WAARD, 1949*a*, *b*, *c*, HEIM, 1949). A similar massif, that of the Mt. du Liron, stretching away to the E and W, borders the investigated area to the south. Near its northern contact a small elongated massif of augen gneiss is to be found, which has been the scope of the tectonic investigations.

The mapping has been done in team work with Messr. J. W. C. M. VAN DER SIJP and R. C. HEIM. The petrology of the area will be treated by Mr. VAN DER SIJP in a separate paper (VAN DER SIJP, 1950).

Grateful acknowledgements are made to "De Centrale Organisatie voor Zuiver Wetenschappelijk Onderzoek" (Dutch Organization for Scientific Research) for the royal grant which made this fieldwork possible as well as extensive comparative studies in similar geologic areas, elsewhere in France, in Switzerland, Spain and Finland.

3. *Previous geologic work.*

On sheet Alais (THIÉRY, 1933) this small gneiss area is indicated as a lenticular body of "gneiss granitoïde" with a broad aureole of "gneiss œillé". The colours and signatures used in the map suggest a gradual changing of the slates (grey) into the augen gneiss aureole (grey with red lines) and finally into the granitoid gneiss of the central part (red used for granite with identical darker red lines). This transition is also obvious by the symbols used for the rock types: x for slates, $x\gamma_1$ for augen gneiss, $\gamma_1 x$ for granitoid gneiss and γ_1 for granite. In the explanatory note of the map a gradual transition is mentioned between the slaty rocks and augen gneiss and between granitoid gneiss and granite.

Thus THIÉRY considers this small massif probably as paragneiss originated by metamorphism and migmatitization from the slates, a conception following the tradition of the French school of petrologists.

4. *Outline of the geology.*

In opposition to THIÉRY's idea an ortho origin of the gneiss massif proved to be evident.

As shown on the tectonic map at the back the gneiss massif forms an irregular lenticular body in the slate complex of the Cevennes. As a whole the orbicular texture is uniform throughout the massif. Locally feldspar augen may be much smaller and partly crushed to thin tails at both sides of the augen. This phenomenon has no regular distribution however, but may occur everywhere as fine-grained banks of the schistosity.

The boundary of the massif against the country rock is usually so sharp that it is possible to put one's nail on it (fig. 15). No gradual transition has been found. In most places along the contact a zone of

contact metamorphism in the slaty country rock has been observed. A hardening of the rock and a spotted appearance because of numerous biotite spots and possibly remnants of feldspars are the macroscopically noticeable alterations of the ordinary slates or phyllites of the Cevennes complex. This rock type is here characteristic for the contact zone; it has not been found elsewhere in the slates of the mapped region. Garbenschiefer and porphyroblastic slates occur locally in the slate complex, but they are not related to the contacts. The contact aureole forms a rather continuous zone around the contact. Where not indicated on the map the existence of contact metamorphism is not certain.

In many places in the contact zone boudined and ptygmatic veins and dikes of gneissose and acid crystalline rock have been observed. Deformed veins are also a common appearance within the gneiss massif.

East of the central part many blocks of slaty rock altered by contact metamorphism appear in the augen gneiss. Sometimes up to several metres long, they are roughly lenticular shaped and sharply contoured. Occurring in the central part of the massif the position of the blocks in a line with the penetrating tongue of country rock pointing E in the gneiss should be mentioned.

Most of the southern contact borders against the posttectonic granite of the Mt. du Liron massif. Like the Mt. Aigoual massif this porphyritic granite is evidently intrusif. Flow structures by planar and linear oriented feldspars, xenolithic blocks of augen gneiss, aplitic parts locally near the contacts, and a great number of granite-porphyritic, quartz-porphyritic, tonalite-porphyritic, lamprophyric, aplitic and pegmatitic dikes in the country rock throughout the augen gneiss and into the slates, have been found.

5. *Origin of the gneiss complex.*

Thus no support for a para origin of the augen gneiss as proposed by Thiéry could be found. On the contrary, the homogeneity of the massif, the sharp contacts, the contact aureole, the ptygmatic and boudined dikes and veins in and around the massif, as well as the sharply bordered blocks of country rock, are as many evidents supporting an ortho origin of the massif.

A porphyritic-granite mass, causing a contact aureole and followed by a number of dikes and veins penetrated a rock complex of argillaceous composition. The occupation of room has probably taken place in different parts separated by septa of which the blocks of country rock may have formed part. A following tectonic activity caused the schistosity of the slates and of the contact zone, the secondary orientation of feldspars in the augen gneiss massif and the boudinage structure and ptygmatic folding of veins and dikes.

6. *Microtectonics.*

The augen gneiss (photo 1) is an orthoclase-microcline-quartz-biotite-

muscovite-albite/oligoclase-rock, presumably of porphyritic granite origin. Under the microscope phenomena of strong mechanical metamorphism are abundantly presented. The parallelism of the gneiss is due to fracturing of the feldspars along cleavage planes which caused flattening and elongation of the crystals and to crushing of feldspars and quartz into layers and tails of breccia with larger crystal remnants.

The effect of the mechanical deformation differs locally in the massif. In parts of the schistosity the crystal faces of phenocrysts are very well preserved and only dislocations along cleavage planes of the crystals may then be shown. Elsewhere the crystals are brecciated into fine-grained layers with small nodules of quartz and feldspar. Microscopically a mortar structure dominates. Quartz aggregates show largely undulatory extinction. Feldspars are broken and penetrated by mortar. Recrystallization is usually shown in quartz particles of the mortar and occasionally by myrmekite.

The brecciated texture points to a relatively superficial zone of metamorphism in the earth's crust. Secondary minerals like sericite and chlorite, also point to epizonal conditions of metamorphism.

Especially sericite forms thin layers throughout the gneiss, coating cleavage planes. Those sericite planes are of tectonic interest, because of small folds or wrinkles which prove to be parallel to fold axes in the area. They are interpreted as small dragfolds originated during the deformation.

The alterations in the contact zone are twofold. Primary contact metamorphism in argillaceous material with growth of metacrysts of poikiloblastic biotite and feldspar and secondary growth of quartz. Secondly dynamometamorphism which changed the rock in oriented leptite. The metacrysts flattened into lens-shaped dots oriented in the general structural direction.

The slaty country rock shows a primary variety of composition ranging from ordinary slate to quartzite. In some places the primary stratification could be observed as millimetre thin bands crossing the cleavage planes. They are folded in small flat folds and sometimes bent towards the cleavage planes at both sides in opposite directions. The latter phenomenon may be interpreted as drag caused by motion along the cleavage planes during the orogeny.

7. *Age of the intrusion.*

The many crushing phenomena observed microscopically in the gneiss point to deformation of the original rock in solid condition. The consolidation of the massif must have been complete before the main phase of the orogeny started. Hence the massif is of pretectonic origin.

The age of the intrusion of the former porphyritic granite can thus be dated between the sedimentation of the argillaceous country rock and the local main phase of the orogeny. The age of the slates is not indicated

on sheet Alais (1923) of the geologic map of France; on sheet Le Vigan (Böhm and others, 1944) some kilometres south of the mapped area, the adjacent slates are mapped as Lower Ordovician, Potsdammian and Acadian (*pro parte*). The age of the main phase of the orogeny in the Orthocevennes complex is not certain (e.g. Demay, 1948). Usually the Middle Carbonian sudetic phase of the variscan orogeny is mentioned. Thus the intrusion must be dated probably somewhere in the long period after the Lower Ordovician and before the Middle Carboniferous.

8. *Tectonic outline.*

The area is affected by at least four mechanical events, each leaving more or less obvious traces.

1. The intrusion of the (now gneiss) massif in the argillaceous sediments was doubtless attended with a number of structural phenomena. The primary flow structures, dike directions, faults and joints in and around the pluton are however largely disturbed and faded by the following action. Boudined and ptygmatic dikes and veins, quartz-filled ptygmatic joints and occasionally relics of primary oriented feldspars are the obvious traces of intrusive tectonics, which are useless however for structural reconstructions.

2. The orogenic main phase originated the slaty cleavage, the parallelism of the gneiss, the folds, ribs and joints in the slates as well as in the gneiss, and probably a great part of the faults in the area. Dependent on the direction, the tectonic motion faded and possibly renewed pre-existing structures.

3. The intrusion of porphyritic granite of the Mt. du Liron massif in the south of the mapped area affected its country rock of which the gneiss massif is part. Similar diapiric structures as in the Mt. Aigoual massif (de Waard, 1949*a*, *b*, *c*) are observed. Concerning its northern country rock faulting and warping as parts of the upward movement of the intrusive mass took place. Dikes of the differentiation series of the granite penetrated joints. The effect on the country rock has largely been a renovation of pre-existing faults and joints.

4. Effects of younger movements, probably of alpine origin, are observed in the area. They are presented by some kilometres-long faults — usually characterized by slickensiding and metre-thick quarz-fillings — cutting the slates as well as the posttectonic granite.

Of all those events the movements of the orogenic main phase have been the most effective. They blurred the early orogenic granite tectonics whereas the posttectonic largely made use of the pre-existing structures. The few large faults indicate younger block movements.

A summary of the tectonic phenomena is given in the structure map at the back. In order to eliminate topographic distortion all data are transposed to a hypothetical peneplain at 400 m, being the average height in the mapped area. Most measurements have been made near this 400 m plane.

A close network of tectonic data, visible on the map by dip symbols, has been collected. Only strikes and dips, fold axes, ribs, contact planes and some of the faults concerned to the contact are given. The contact aureole and the dikes and veins, belonging to the gneiss, are drawn in with signatures.

9. *The general structure of the area.*

The strike symbols in the structure map have been produced by lines, thus giving a picture of the strike relations throughout the area. Though locally strong differences occur the general strike in the slates as well as in the gneiss is E—W, with a moderate dip to the north. The average of the cleavage in the mapped area has been found by plotting all 300 measurements in pole diagram fig. 2. The pole axes prove to be concentrated in a small area of the diagram with an average of N 84 E, 40 N.

Fig. 2. Pole diagram (300 poles) of the cleavage in the mapped area. Contours 53 – 27 – 16 – 7 %. Dots indicate the axes of ribs and minor folds. (All pole diagrams are projected in the southern hemisphere).

Usually the cleavage planes of the slates show small straight ribs of about 2 mm wave lenght. Identical phenomena are the wrinkles of mainly sericitic material on the schistosity planes of the augen gneiss. Moreover, minor folds locally occurring in the slates as well as in the gneiss and some small anticlines in the gneiss have been measured.

As pointed out before (DE WAARD, 1949*b*) there is an obvious tectonic relation between minor folds and the ribs of the slates. They must be considered as miniature drag folds parallel to the fold axis of the area. The anticlines and the wrinkles of the gneiss also proved to be tectonically related to the ribs and minor folds. Plotted in tectonic diagram fig. 2 no separation could be distinguished in their directions. The axes of ribs, minor folds, wrinkles and folds in the whole area are marked with points in the diagram. They are concentrated in the northwestern part of the diagram with an average of about N 125 E, 25 NW.

It may be of interest to compare diagram fig. 2 with the corresponding

pole diagram of the Mt. Aigoual area 20 km to the WNW (DE WAARD, 1949*b*, fig. 5). Both areas are part of the large isoclinal slate complex of the southern Cevennes (e.g. DEMAY, 1931*a*, 1948). In the western area an average of the cleavage planes of N 138 E, 34 NE was found, which gradually changes into N 84 E, 40 N in the eastern area. The average fold axis shifts from about N 140 E, 10 NW into N 125 E, 25 NW, from W to E. The fold axis thus slightly bends to the east while the dip increases, going from W to E. As the fold axes of both areas are about in line with each other this means the exposure of deeper parts of the slate complex in the eastern area. In general the slates in the eastern area appeared to be of more psammitic composition than in the west.

10. *Types of contacts.*

Strong divergences of the general E—W strike have locally been found, which are due to the inhomogeneity of the area. In the north of the area far from the gneiss contact, divergences have been mapped however, which can not be explained as caused by the vicinity of the gneiss massif. Those are either caused be hidden inhomogeneities or they are of regional origin and hence can not be interpreted with the observations in this small area.

The schistosity of the gneiss massif is as a whole parallel to that of the slates. As a result of the different structural behaviour of both rock types divergences appear especially near the contacts. Four different

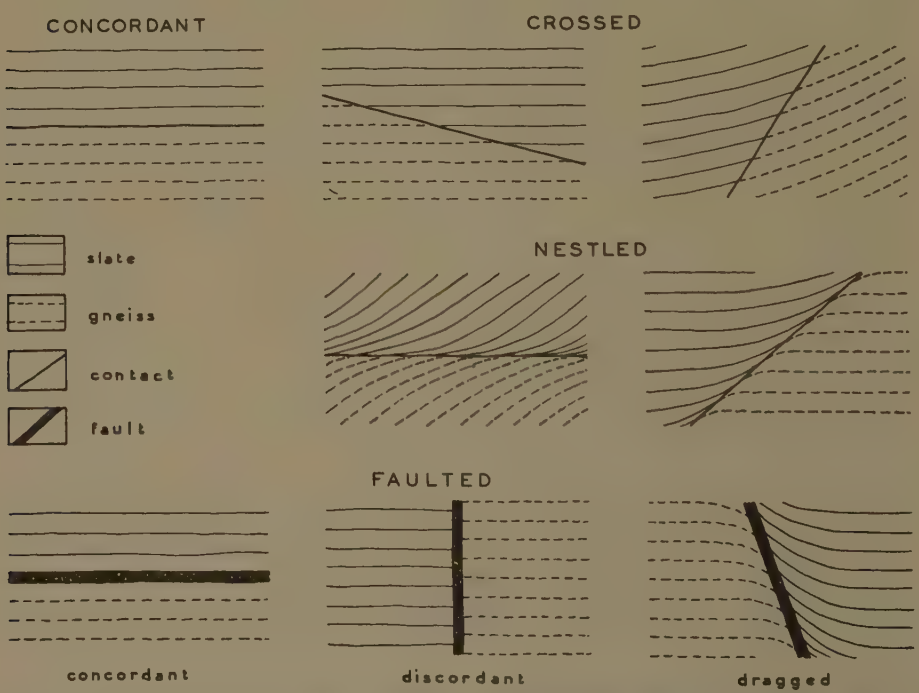
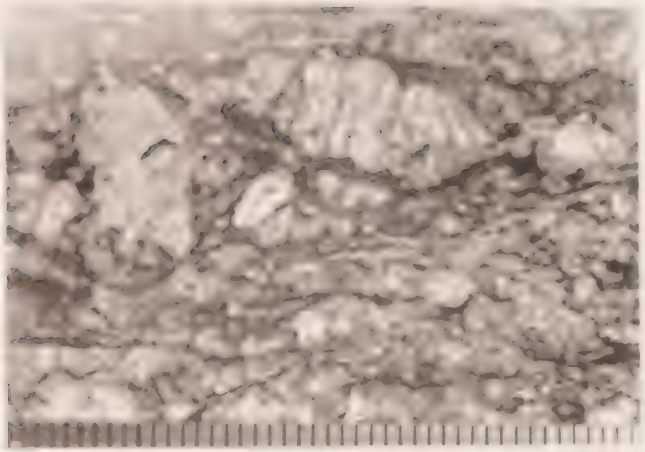


Fig. 3. Diagram of the different contact types observed around the gneiss massif.



types of contacts, viz. concordant, crossed, nestled and faulted contacts could be distinguished. They will be treated systematically.

Concordant contacts (fig. 3 and 4, photo 3) are characterized by a parallelism of the contact plane with the cleavage at both sides of the contact. As expected, the pole axes of these contacts in diagram fig. 4 are found near the area of the average cleavage.

Crossed contacts (fig. 3 and 5, photo 4) show a more or less perfect continuity of the cleavage across the contact in which the contact plane crosses the cleavage at a random angle. This is illustrated in pole diagram fig. 5 by a parallelism or a small gradual change of the cleavage directions at both sides of the contact and an arbitrary position of the concerning contact plane. Evidently the cleavage direction is not related to or affected by the direction of the contact plane.

Fig. 4

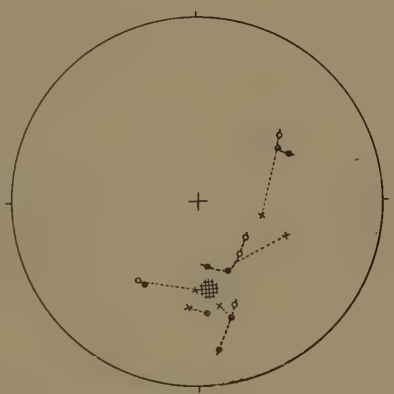


Fig. 5

Fig. 4. Pole diagram of the cleavage planes of gneiss (black) and country rock (white) near ten concordant contacts, compared with the average of cleavage. Contact planes coincide with cleavage at both sides.

Fig. 5. Pole diagram with the cleavage of gneiss (black) and country rock (white) and the interjacent contact plane (cross) of five crossed contacts. The contact plane is not related to the cleavage at both sides which has continuity.

Nestled contacts (fig. 3, 6 and 7, photo 3) are, contrary to the former type of contacts, strongly affected by the direction of the contact plane. The cleavage nestles at both sides of the contact up to the contact plane. The cleavage direction of the plastically behaving slates changes within a short distance towards the direction of the contact plane. The behaviour of the gneiss usually has been less plastic though an affinity to a parallelism with the contact plane is always distinct. Pole diagram fig. 6 shows the gradual change of the cleavage of the slate as well as that of the gneiss in the same direction from about the same starting point towards the pole axis of the contact plane. In one case the cleavage axes of the two rock types move from opposite directions, both nestling up to the contact

plane. Nestling from different directions is also illustrated in one of the two contacts of diagram fig. 7. The slate further away from the contact has a steeper dip than the gneiss. Towards the contact plane the dip gradually decreases.

Faulted contacts (fig. 3, 8 and 9) can be divided in three different types,

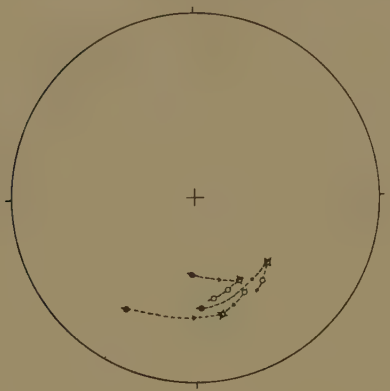


Fig. 6

Fig. 7

Fig. 6. Pole diagram with the cleavage of gneiss (black) and country rock (white) approaching the interjacent contact plane (cross) of three nestled contacts. The cleavage of both rocks gradually changes into the direction of the contact plane.

Fig. 7. Pole diagram of two nestled contacts.

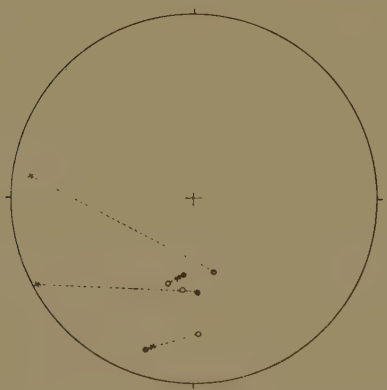


Fig. 8

Fig. 9

Fig. 8. Pole diagram with the cleavage of gneiss (black) and country rock (white) at both sides of contact plane (cross) of two concordant faulted contacts (almost coincidence of cleavage at both sides with faulted contact plane) and two discordant faulted contacts (coincidence of cleavage at both sides not related to random oriented and faulted contact plane).

Fig. 9. Pole diagram with the cleavage of gneiss (from G to G' at the contact) and country rock (from about C to S at the contact) at both sides of the fault (F) of a faulted contact with drag phenomena. CA, fault plane with striae in B. CF, theoretical zone of pole axes of dragged layers with fold axis in A, starting from C towards the fault plane F.

viz.: concordant, discordant and dragged. The faults are often marked by fault breccia and slickensides. In pole diagram fig. 8 two concordant faulted contacts are drawn in. The cleavage directions at both sides of the contact show little difference. The fault plane, about parallel to the cleavage, usually approximates the cleavage of the adjacent gneiss. In the same diagram (fig. 8) two discordant faulted contacts are given. As a result of the dislocation the cleavage at both sides of the contact abuts against the fault plane. In the diagram the cleavage of the gneiss and slate about parallel to each other, is cut off by a nearly perpendicular crossing fault. The third type is characterized by drag phenomena at both sides of the fault. One eloquent example is shown in fig. 9. In the ideal case drag may be described as a folding of the layers at both sides of the fault in opposite direction. In that case the fold axis is parallel to the intersection of the original cleavage or bedding and the fault plane. The pole axes of the dragged layers form a zone in the diagram in which the pole axes of the original layers and the fault plane are situated. Approaching the fault plane the pole axes of the bended layers move towards the pole axis of the fault either direct or, at first, away from each other. In the case in question the theoretical zone of the dragged layers (with fold axis in A) is drawn in between the fault (F) and the undisturbed cleavage (near C). The cleavage of the gneiss alters to G at 20 metres from the fault and to G' near the fault. The cleavage of the slate gradually changes till S near the fault. The measured pole axes of the dragged cleavage thus prove to be not far from the theoretical zone CF. This diagram illustrates the far more plastic behaviour of the slates compared with the gneiss. Near the contact the slate bent nearly parallel to the fault plane, whereas the gneiss curved only towards G' not yet halfway between C and F. A fault breccia of about 50 cm shows that the material gave way to the stress by crushing instead of folding. B in the diagram is the direction of the measured striae on the fault plane CA. Movement thus took place in the direction B, which is about the direction of the dip of the fault. Resolving this movement the direction C in the fault plane has been the active motion which caused the drag phenomena. A, perpendicular to C or the fold axis is the inactive resolved. As gneiss forms the western part of this contact, the western block has been warped down, according to the direction of bending from C to F direct.

11. *The structural behaviour near the contact.*

In the structure map at the back the four mentioned contact types are shown in the different parts of the gneiss massif. According to the map, large parts of the contact are either crossed, nestled or concordant, whereas faulted contacts are found irregularly distributed around the massif. However, the map shows merely the structural outline on this scale.

In detail, complicated structures near the contact are observed in

several parts. In some tens of metres all four contact types may be found. Frequently, small faults dislocate the contact decimetres or metres. The slates now nestle up to irregularities of the gneiss body, now lies concordant against the contact plane. In other parts the contact is mainly crossed. Transitions and combinations may be found within a few tens of metres; elsewhere one type dominates hundreds of metres.

It proved to be impossible to reproduce the sometimes rapid transitions of contact types on the structure map. As a rule cleavage lines have been drawn in according to the structures found at some distance from the contact plane. The map thus illustrates the general tendency of the cleavage in the contact area. An example of the difference between the general tendency and the detailed structure is shown in fig. 10. Though

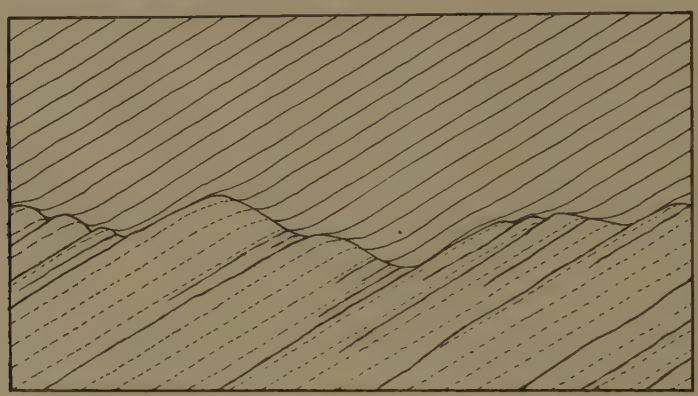


Fig. 10. Theoretical example of a compound contact with the general impression of a crossed contact which in detail is composed of faulted, concordant, crossed and nestled parts. Possible scales from millimetres to tens of metres.

the general picture of this contact is a crossed one, in detail faulted, crossed, nestled and concordant parts may be recognized. The scale of this figure — which might represent a section as well as a map — may be seen in actual size, in a few millimetres or in tens of metres.

As the gneiss body is lenticular shaped — more or less streamlined in the general direction of the cleavage — concordant contacts are predominant at the northern and southern borders. A general tendency to nestle is especally found northwest and south of the western part of the gneiss massif. Crossed contacts are a common appearance in the northeastern part (photo 4). Faulted contacts — of which only the most important have been drawn in — are found in many places along the borders of the massif. The compound contact of fig. 10 may be considered as a modification of the faulted contact type. Though it is observed in many places of the contact area, its appearance generally proved to be of local importance.

12. *Tectonics of veins and dikes.*

All observed veins and dikes, belonging to the magmatic sequence of the gneiss massif, are deformed by orogenic action. Ptygmatic folds, boudinage structures and such resembling or combined structures have been observed. The way of deformation proved to be dependent on the composition of the veins (aplite, pegmatite, quartz porphyry, granite porphyry, granitic, lamprophyre) as well as on the properties of the surrounding rock (gneiss, contact aureole, slates).

Pegmatitic veins in the gneiss are either broken, cut by sericitic cleavage planes and irregularly dislocated, or boudined. Aplitic veins in the gneiss are often ptygmatic folded, with preservation of distinct borders against the gneiss. Lamprophyric dikes have very irregular contacts with the gneiss. They show a reversed boudinage structure. Granitic, porphyritic and acid dikes in the slates are mostly boudined. The broken and seperated parts of the dikes have rounded off and pointed, ends. The slate surroundes the boudins plastically and fills the gaps between them. In the hardened rock of the contact aureole, both boudinage and ptygmatic folds are observed.

Some details of the structure round a boudined pegmatite dike in the contact aureole are shown in fig. 11. The upper part of the contact of

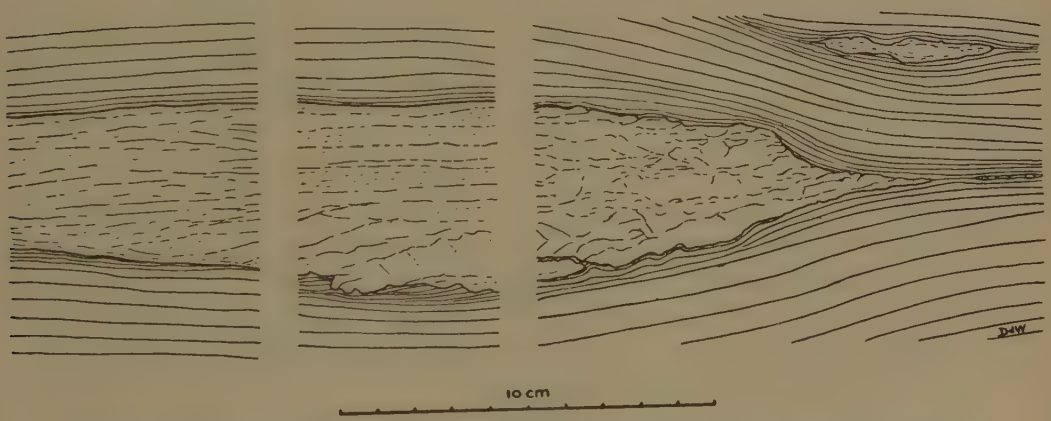


Fig. 11. Details of the boudinage structure of a pegmatite dike in the contact aureole. Mainly concordant and conformable contacts, with nestled and crossed parts.

the dike has concordantly developed. The lower contact shows crossed and nestled parts. At the end of the boudin, slates are nestled against the contact, whereas the cleavage planes in the gneissose pegmatite have a tendency to abut against the contact plane.

Details of reversed boudinage are given in fig. 12. The more competent gneiss determines the border of the dike, whereas the plastically behaving lamprophyre fills gaps and irregularities in this contact plane. The large quantity of biotite in the fine-grained lamprophyre is the main cause

of the flow possibility of this rock. The dike probably formed a zone of weakness in more resistent rock. Considerable motion along this zone and extensive flow within the lamprophyric rock has taken place.

In detail, flow phenomena are very well developed near the irregular contact with the gneiss. The material penetrated cracks and parts of

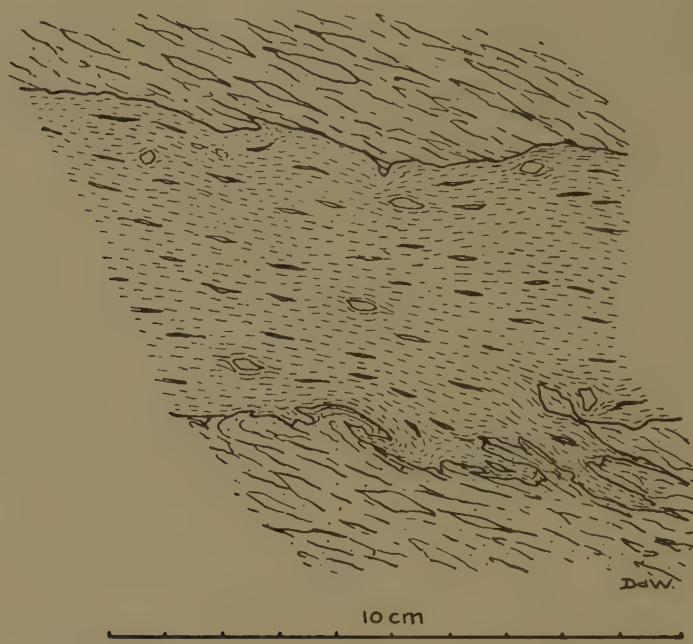


Fig. 12. Details of reversed boudinage of a lamprophyre dike in gneiss. Plastic behaviour of dike material in competent country rock.

gneiss have drifted away in the moving mass. The flow phenomena resemble in many ways the flow structures of rocks in magmatic condition. Microtectonic deformations, as strong undulatory extinction of the few quartz crystals and the displacement of broken biotites, however, are indications of plastic flow in solid state.

The origin of boudinage in dikes is generally considered as a purely mechanical deformation, which took place after the emplacement and the consolidation of the dike (Wegmann, 1932). Ptygmatic folds in some cases are believed to have been formed in the present shape, either by fluidal movements of the injecting mass (Sederholm, 1907, 1913, Suter, 1924) or by injection during plastic flowage of the host rock (Buddington, 1939) or by injection along tortuous fissures (Read, 1928). In most cases ptygmatic veins are believed to have been originally planar layers which contorted afterwards, either by plastic deformation in magmatic condition (Holmquist, 1920, 1921, Sederholm, 1926, Erdmannsdörffer, 1938) or by plastic deformation in solid state (Milch, 1900, Sander, 1914, Kuenen, 1938). In the gneiss of the mapped area

both boudinage and ptygmatic structures are found close together (photo 2). A more or less plastic deformation of originally planar veins and dikes in solid condition by postintrusive movements is in this case obvious.

The formation of boudinage as well as reversed boudinage and ptygmatic structures in veins of the same pretectonic age in the same host rock must be caused by difference of material and of their plastic behaviour, and by the direction of the original pretectonic dike with respect to that of the tectonic movements. Thus, boudinage develops best in a competent dike about parallel to the differential movements in more plastic surroundings. Tension joints broke the dike and separated the boudins in the direction of maximum extension of the tectonic unit (Wegmann, 1932, Walls, 1937, Cloos, 1946). Reversed boudinage develops in incompetent dike material and rigid country rock. The relation of the direction of the dike to that of the motion will be of great influence on the ultimate shape of the deformed dike. The maximum of flow will be found in a dike about parallel to the planes of differential movements.

If the mobility of dike rock and country rock is about the same, boudinage structures are not formed. Tectonic movements affected both rock types equally. Veins cutting the gneiss not parallel to the direction of motion will thus record the differential deformation of the rock during the alteration of the granite to gneiss. Though the plastic behaviour of the gneiss probably has not been constant throughout the massif, aplite appears to have a plasticity approximating that of gneiss, whereas pegmatite usually behaves more rigid and lamprophyre more plastic than gneiss. The local appearance of ptygmatic aplite together with boudined pegmatite can be explaned in this way (photo 2).

The mobility of both rocks depends largely on the tectonic zone of deformation in the earth's crust. The deformations in this area are mainly epitype, viz. formed by movements in the epizone of metamorphism.

A deformation by shearing dominates in the gneiss massif (sericitic shearing planes, crushed feldspars, broken pegmatite veins crossing the cleavage); locally a more plastic reaction occurred (ptygmatic aplite in some parts of the massif, flow in lamprophyric rock). An increase of general plasticity in deeper zones of the orogen, accompanied by increased flow conditions and possibilities of recrystallization, will probably result in variations in the appearance and in the formation of these structures.

THE INFLUENCE OF TEMPERATURE AND KIND OF FOOD ON THE INCREASE IN THE NITROGEN CONTENT OF THE YOUNG WORKER HONEYBEE (APIS MELLIFICA L.) [1])

BY

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(Communicated by Prof. CHR. P. RAVEN at the meeting of March 25, 1950)

Introduction.

During the first five days of the imago life of the worker honeybee, a considerable increase in the protein content of the body occurs, which amounts to nearly 50 %.

STRAUS (1911) was the first one to demonstrate this increase by determining the total nitrogen content of the bee body, but since the content of the digestive tract may vary greatly, so does the amount of nitrogen present there, thus the figures of STRAUS do not give an exact impression of the protein content of the body itself. HAYDAK (1934) and KELLER-KITZINGER (1935) gave a more detailed picture of the changes in the protein content. They determined the amount of nitrogen present in the whole body on successive days of the imago life after removal of the digestive tract. It was shown that the nitrogen increase occurs in the first five days after emergence and then comes to a standstill. According to KELLER-KITZINGER, no decrease takes place after the twentieth day.

The last mentioned author analysed the conditions necessary to achieve the normal protein increase. This happened only if young bees were fed honey and pollen from a comb (the so-called bee bread) at a temperature above 30 C. The same food did not cause an important increase if the bees were kept at 23°. If the bee bread was fed with sugar solution instead of honey, no increase took place, neither at 23° nor at 30 — 34°. Even with bee bread and honey at a temperature above 30° C, the normal increase was not achieved if the food was boiled previously. The enzymes present in honey and bee bread seemed to be indispensable for a normal protein increase. From these results it was concluded that young bees are able to digest proteins only at a temperature higher than 30° and that the normal level of the nitrogen content of bees more than

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5 days old, can be reached only if they are fed honey and bee bread. The findings of a complete lack of protein digestion and anabolism at a temperature only about 10° lower than under natural conditions seemed rather peculiar and a repetition of this observation was considered to be advantageous. Moreover, Haydak (1933, 1936, 1937), investigating the value of different nutritious substances as pollen substitutes for bees, found a normal nitrogen increase in the body in the absence of bee bread. From this it may be deduced that the enzymes of bee bread are not necessary.

In the present investigations, it is shown that neither the natural food (honey and bee bread) nor a temperature of more than 30° is a necessary condition for the protein digestion and anabolism of young bees.

Material and methods.

Young bees emerged from a comb with sealed brood in an incubator at 32 — 34° in September 1949. At intervals of 12 hours the bees were caught from the comb and transferred to small experimental cages (12 × 10 × 4 cm), about 50 bees per cage. (The cages were about equal to the Liebefelder experimental cages). The food consisted of sugar candy (4 parts sugar to 1 part tap water) mixed with the bee bread or casein, 20 and 10 % respectively (percentage of the dry sugar). Water was constantly available.

Two experimental groups fed respectively 20 % bee bread and 10 % casein were kept in an incubator at 30°; two other groups fed bee bread and casein respectively were kept in an incubator at 23°.

About 200 newly emerged bees were marked with a dot of paint and added to a colony in an observation hive. Samples of these marked bees taken at intervals served as controls. All experimental bees were descendants of the same queen. The changes in the protein content of the body were followed with nitrogen determinations on samples of 5 bees with the Kjeldahl method. After being killed with chloroform, the digestive tracts were removed, including the honey stomach and oesophagus. This may be done easily by pulling the sting out of the body, then the digestive tract comes out as well and may be removed as a whole. The nitrogen content of the digestive tract and of the rest of the body (including the sting) was determined separately .The bodies were weighed and then placed in an incubator at 80° C for 3 days to determine the dry weight. The destruction was performed with sulphuric acid, potassium sulphate and selenium mixture according to Wieninger. These reagents contained no appreciable amounts of nitrogen. The titrations were performed with 0.1 n sodium hydroxide and a mixture of methyl-red and methylene-blue as indicator. Determinations with ammonium sulphate, glycine and urea showed an error of less than 1.2 %, with an average of 0.5 %.

Results and discussion.

The nitrogen contents of the body and of the digestive tract on successive days are plotted in figures 1 and 2 respectively. Figure 3 shows the changes in the dry weight.

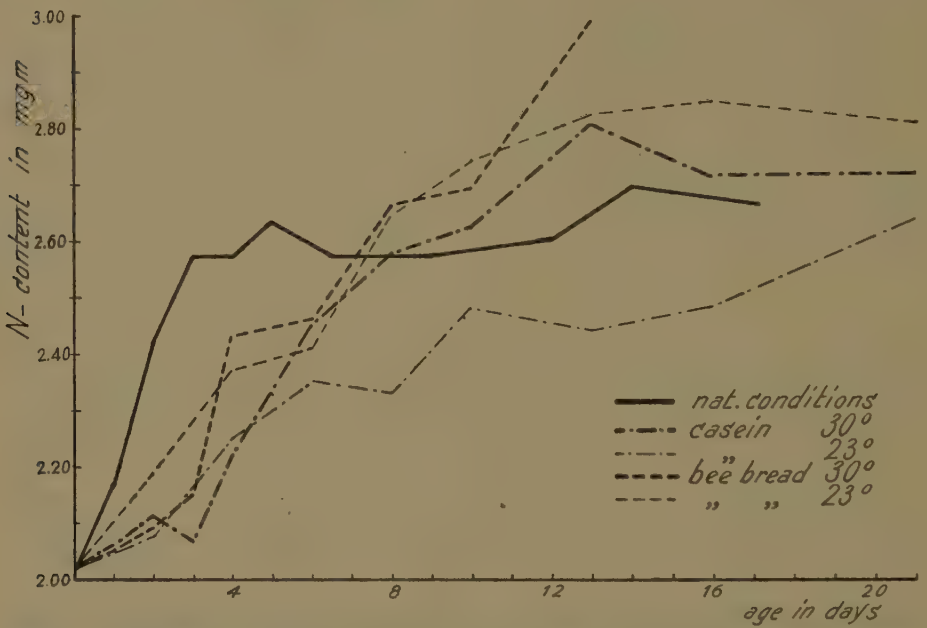

Fig. 1. Increase in nitrogen content of the body of young honeybees without digestive tracts under different conditions. Nitrogen content in mgm. per individual plotted against age in days.

Most of the curves show a rather irregular course, which may partly be attributed to the fact that each point is the result of only one determination. Even so, some important features are clearly demonstrated.

Figure 1 (drawn line) shows a steep rise in the nitrogen content of the body of bees kept in an observation hive which stops at the fifth day when a level of 2.63 mgm. has been reached. After the seventeenth day, a considerable decrease must occur, for the nitrogen content of field bees (descendants of the same queen and of more than 3 weeks of age) lies between 2.27 and 2.51 mgm. with an average of 2.41 mgm. This fact is not in agreement with KELLER-KITZINGER's statement that the nitrogen content remains constant after the twentieth day.

The other lines of figure 1 show a very important increase in the nitrogen content of the bees in each of the experimental cages, resulting in about the same level as under natural conditions (2.5 — 2.7 mgm. per bee) or even somewhat higher. This means that without honey and bee bread, even without a temperature of more than 30°, the normal nitrogen level is reached. This contradiction of the findings of KELLER-KITZINGER who could not obtain this estimation unless honey and bee

bread were fed at a temperature of 30 — 34°, is very striking. In the present investigation, it is proved that sugar and casein are adequate feeding substances to raise the nitrogen content of young bees to the normal level, even at a temperature about 10° below normal. The enzymes present in bee bread and honey appear to be unessential to reach the normal level.

Considering the rate of the nitrogen increase, one observes a considerably slower increase in the experimental bees than in the controls during the first two or three days. From the third till the eighth day however, the incline of the curves is nearly uniform, except for the casein-fed bees at 23°. This means that from the third day, the increase in the nitrogen content of the experimental bees is nearly as fast as under natural conditions.

With the exception of the casein-fed bees at 23°, the normal nitrogen level has been reached at the eighth day, being three days later than normal. This retardation is less pronounced if one considers the very slow rise during the first two or three days. From the second or the third, until the eighth day, the rate and amount of the nitrogen increase in the experimental bees is about equal to that of the controls during the first five days.

The discrepancy in the first three days may be due to the behaviour of the bees when first caged, when they remain very quiet, forming a little cluster and the food consumption is very low (fig. 2). KELLER-

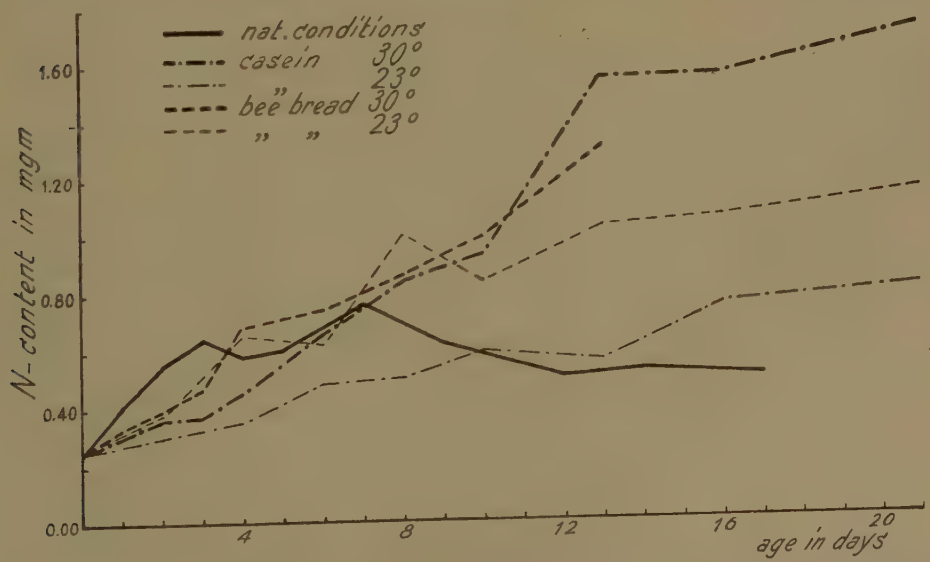

Fig. 2. Changes in nitrogen content of the alimentary canal of young honeybees under different conditions.

KITZINGER noticed that caged bees did not find the food on the first or second day. I am not sure if this is the real cause of the low food intake,

but the fact remains. If this phenomenon is taken into account, then the time in which the normal level has been reached in the experimental bees is only slightly longer than under natural conditions. Thus it seems justifiable to conclude that the retardation in reaching the normal level of the nitrogen content is mainly a consequence of the abnormal behaviour of caged bees and not primarily due to the kind of food or to the lower temperature.

The three curves for the casein-fed bees at 23° (fig. 1, 2 and 3) run somewhat separate from the others, showing a smaller food intake (fig. 2) and a slower increase in the dry weight (fig. 3) and nitrogen content (fig. 1). The bees exhibited a marked lack of vitality as compared with other groups. These differences may be due to the high casein concentration. The fact that a 10 % casein sugar mixture is not a suitable food for bees was shown in other observations in which the mortality was studied, when it was observed that the mortality was much lower if the food contained less casein. (These observations will be dealt with in a further paper. See also MAURIZIO 1950). It is possible that the damaging effect of the 10 % casein food is stronger at 23° than at 30° and this may be the cause of the smaller food intake and the slower increase in dry wieght and nitrogen content.

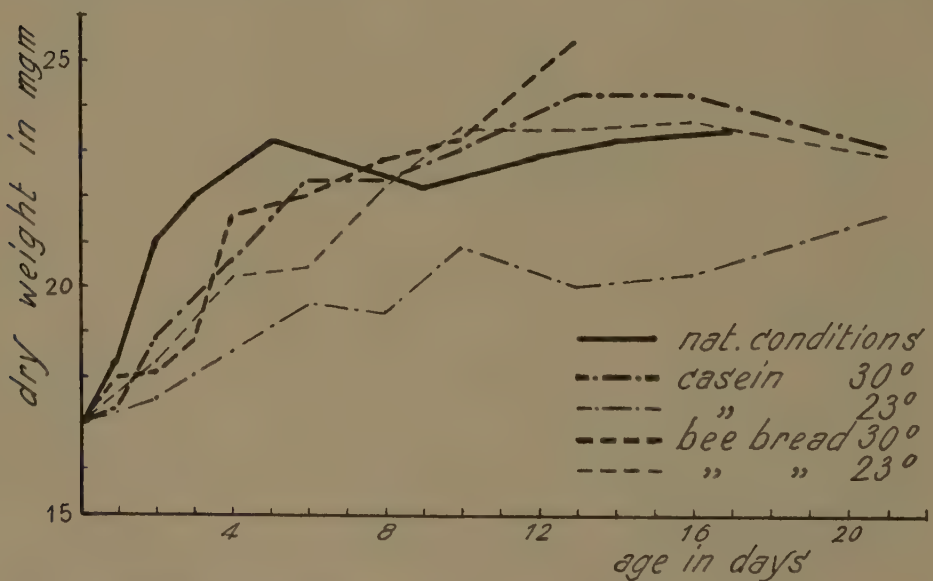

Fig. 3. Changes in dry weight of the body of young honeybees without digestive tracts under different conditions.

The slight influence of the lower temperature on the nitrogen increase is clearly demonstrated by the closely parallel course of the two bee bread series in figure 1. The increase shown by these curves is nearly equal during the first 12 days of the experiment, both in rate and amount. This means that the difference in temperature has no appreciable in-

fluence on the rate of the nitrogen increase: digestion, absorption and anabolism of proteins may occur as well at 23° as at 30°. From this it may be concluded that the temperature is a much less important factor in the metabolism of bees than was claimed by KELLER-KITZINGER.

Finally, it is important to draw attention to comparable figures of other authors. Although the normal nitrogen content of bees more than 5 days old agrees very well with that mentioned in literature, this is not the case regarding the nitrogen content and dry weight of emerging bees. HAYDAK (1934), KELLER-KITZINGER (1935) and LOTMAR (1939) found a nitrogen content of 1.6 — 1.74 mgm. and a dry weight of 14.3 — 14.9 mgm. In the present experiment, (performed in September and October 1949), the amount of nitrogen and the dry weight of 19 samples of newly emerged bees without alimentary tract was determined. The nitrogen content was found to be never less than 2.0 mgm. per bee with an average of 2.06 mgm. and the dry weight never less than 16.6 mgm. with an average of 17.2 mgm. These figures show important discrepancies with those of other authors and which cannot be attributed to a different preparation technique. 27 Samples of newly emerged bees taken from the same colony with the same queen, analysed in May and July, showed averages of 1.78 mgm. for the nitrogen content and 15.2 mgm. for the dry weight which is in good agreement with the figures of the authors mentioned. Thus considerable differences in the composition of the bee body may occur in the different seasons and it is obviously important to mention the season if figures of the body composition of bees are concerned.

Without further information, however, one is not justified in generalising these findings, but it is possible that the higher dry weight and nitrogen content in autumn is related to the preparation of the bee colony for hibernation.

Acknowledgement.

The author is indebted to Prof. S. DIJKGRAAF for providing the opportunity to work at his institute and to Prof. CHR. P. RAVEN for reading the manuscript. The author also expresses his appreciation to Miss CH. SEPRÉNYI for performing the nitrogen determinations and to Miss L. BOWEY for correcting the manuscript.

Summary.

1. Newly emerged honeybees, kept in confinement at temperatures of 23° and 30° C, were fed 20 % bee bread-sugar candy, or 10 % casein-sugar candy. The changes in the dry weight and in the nitrogen content of the digestive tract and the rest of the body, were followed on successive days up to the 21st. day and compared with those of bees living under natural conditions in an observation hive.

2. The statement of KELLER-KITZINGER (1935) that the normal nitrogen content of bees more than 5 days old is only attained if bee-bread and honey are fed at a temperature above 30° is proved to be incorrect.

3. An appreciable influence of the enzymes present in honey and bee-bread was not observed, from which it is concluded that these enzymes do not have an important part in digestion and anabolism of proteins. The mixture of casein and sugar appeared to be an adequate feeding substance for the normal nitrogen increase of young bees.

4. At a temperature of 10° below that of the brood nest of the bee colony, the normal nitrogen level may be reached at nearly the normal rate.

5. Attention is drawn to the considerable difference, both in dry weight and in nitrogen content, of emerging bees in summer and autumn, probably related to hibernation.

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ION-SECRETION INTO THE XYLEM AND OSMOTIC REGULATION OF EXUDATION

BY

R. VAN NIE, R. J. HELDER AND W. H. ARISZ

(Communicated at the meeting of March 25, 1950)

The phenomenon of exudation in plants has been the subject of many investigations. We shall confine ourselves here to those exudations of herbaceous plants where the sap contains exclusively salts. Some authors are of the opinion that the root acts as an osmometer. In consequence of the difference in osmotic concentration between exudated sap and outer solution, the sap in the xylemvessels is diluted. For the continuation of the exudation osmotically active substance must be secreted into the vessels. (EWART, ATKINS, PRIESTLY, SABININ, CRAFTS and BROYER, HOAGLAND). VAN OVERBEEK presumes that along with the osmotical action there is an active water secretion into the xylem. EATON discards this view and surmises that the mannitol used for compensation of the exudation pressure penetrates into the root-tissue. As a consequence too high a value will be found for the compensating osmotic value.

LUNDEGÅRDH described the exudation phenomena in a large number of papers. At first (1943) he considered these as an osmotic process in combination with an active anionsecretion into the xylemvessels. The possibility of polar watertransport was considered, too. In 1945 he assumed a secretion of a salt-solution by parenchymatic cells in the stele. In 1949 he stated that the secretion of cations into the xylemvessels was regulated by a glycolytical system. The anion-respiration, in his opinion, had only influence on the transport of anions in the protoplasm to the secreting cells.

The immediate cause of our investigation was the wish to examine the problem of active water-secretion more closely. Only a short outline of the results obtained will be given in this preliminary note.

Experiments.

After the tomatoplants had grown from ten to fifteen weeks, they were cut the evening before the experiment was carried out. The following day, during the experiment, either a tube with narrow bore was placed on the stump to collect the sap, or a potometer tube for rate determinations. The roots were placed in a large funnel in order to facilitate the changing of the solution. Especially the influence of changes in the osmotic value of the medium on the exudation process was investigated.

The different osmotic values were mostly obtained by adding different amounts of mannitol either to a Hoagland-solution or to water. In this way two kinds of experiments were made.

The first kind is illustrated by fig. 1. A root system is placed alternately on a Hoagland-solution and a similar solution of which the osmotic

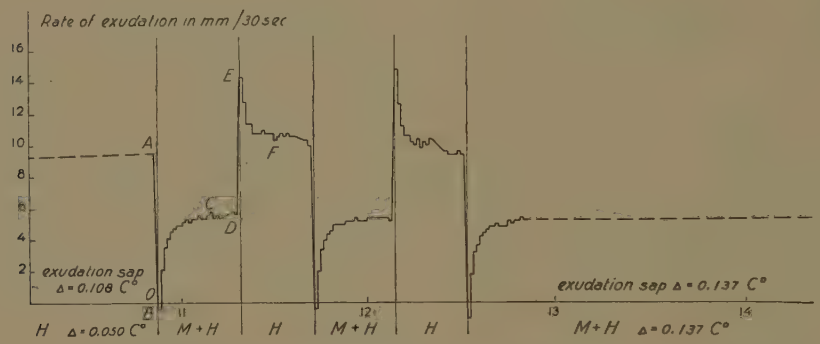

Fig. 1. Influence of changing the osmotic value of the medium on the exudation. The tomato plant is continuously on a Hoagland solution (H). The osmotic value is increased by adding mannitol. (M + H).

value was increased by the addition of mannitol. It is apparent that in changing the solutions a sudden strong change in the exudation occurs, followed by a more gradual change in the reverse direction. This slowly leads to a new equilibrium, which is attained after some 10 minutes. It is remarkable that the sudden decreases (fig. 1 *A B*) have the same value as the sudden increases of the exudation (fig. 1 *D E*).

During this experiment exudation sap was collected three times. Of these and of the outer solutions used the freezing point was determined (fig. 1). It is obvious that in placing a root system on a medium with a higher osmotic value the exudation decreases whereas the osmotic value of the exuded sap increases. The osmotic value is also a consequence of the salt-secretion, the strength of which can be calculated by multiplying the rate of the exudation by the osmotic value of the sap.

Here we start from the supposition that the concentration of the sap in the regions of the root system where these processes take place, equals the concentration of the exuded sap. In this experiment the salt secretion on the stronger osmotic medium was found to be smaller too.

Other investigators have found an identical course of the exudation. They ascribe the slow change in the exudation that appears after the sudden decrease or increase in changing the solution, to a permeation of the substance used. (Sabinin 1925, Eaton 1943). To test this supposition, an experiment was made as illustrated in fig. 1 but this time by increasing the osmotic value not only with mannitol but also with raffinose and thiourea. Though it may be surmised that these substances

will permeate in a different degree, no distinct differences were found.

The osmotic value of the medium was also altered in the experiments of the second kind. But in contrast with the first series we did not wait for the attainment of a new equilibrium. Thus it was possible to trace out, in the morning, as well as in the afternoon, the sudden effect of five solutions of different strength on the exudation. The result of such an experiment is represented in fig. 2. On the abscissa is given the osmotic

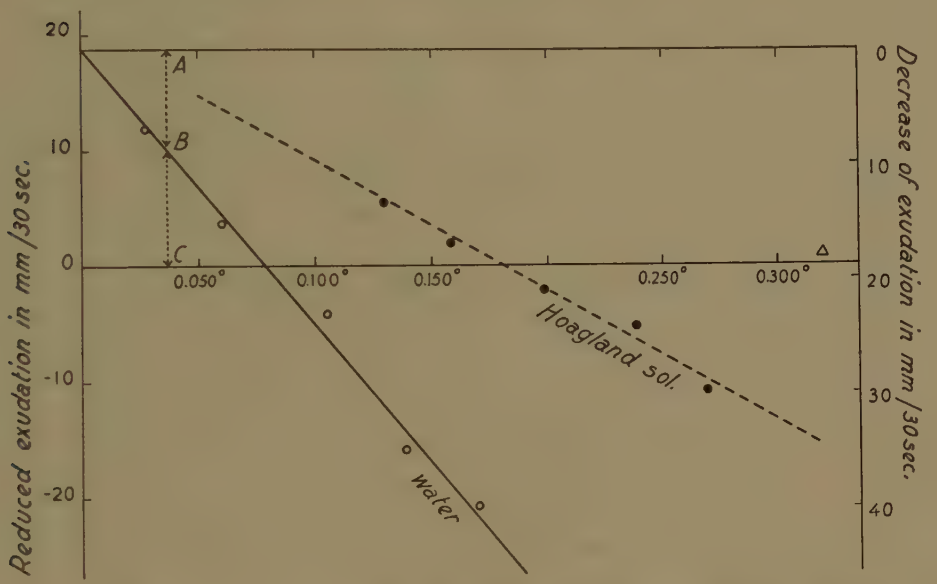

Fig. 2. Relation between osmotic value of the medium and the reduced exudation (left ordinate).

value of the mannitol solutions, on the ordinate the sudden decrease (AB) of the exudation resp. the reduced exudation (BC). The regression lines were computed. The figure shows a distinct rectilinear proportionality between the change of the exudation resp. the reduced exudation and the osmotic value of the medium. As well as for a root system on water, the regression line for the relation between reduced exudation and osmotic value of the medium is drawn for a root system on Hoagland solution (fig. 2, dotted line). The course of the regression line for the plant on water is steeper than that for the plant on Hoagland solution. This means that changes in the osmotic value of the medium have a greater effect with a plant on water: the resistance for the transport of water is smaller.

The point of intersection of the regression lines with the abscissa gives the osmotic value of the medium which brings the exudation momentarily to a standstill (compensating osmotic value). In comparing these osmotic values with those of the exuded sap we encounter the difficulty that

these values vary greatly during the day. It seems appropriate to confine ourselves to those experiments in which this periodicity is smallest, and which are therefore better to compare. We obtained the following results.

Osmotic value of the exuded sap	Compensating osmotic value	Difference
0.060	0.079	0.019
0.039	0.077	0.038
0.137	0.180	0.043
0.138	0.156	0.018

Average difference 27 %

As a result we find that the osmotic value of the exuded sap is about 0.5 atm. lower than theoretically from an osmotic standpoint could be presumed. It may be stressed that the difference is very variable and more than one atm. smaller than that found by VAN OVERBEEK.

Theory of the exudation.

We shall start our theoretical considerations from the osmotic view and will examine whether our experimental results are in accordance with this theory. So our working hypothesis is that the root system functions as an osmometer. We have according to SABININ (1925):

$$b = k(O_b - O_m) \tag{1}$$

when b represents the rate of exudation, O_b the osmotic value of the exudate, O_m that of the medium and k a factor of proportionality.

From this formula it follows first of all that the exudation stops when we make the osmotic value of the medium equal to that of the sap. We found, however, that the osmotic value of the sap was on the average 27 % lower than the compensating osmotic value of the medium.

It should be borne in mind that the experimental errors are rather great ones, and that there is a great variety in the differences.

Moreover, we may expect according to (1) that in changing the osmotic value of the medium O_{m1} to O_{m2} the exudation b_1 will change suddenly and in proportion to the osmotic value of the medium to b_2 according to.

$$b_2 - b_1 = k\,(O_{m1} - O_{m2}) \tag{2}$$

This actually happens (fig. 2) and in accordance with KRAMER we consider this to be a strong proof for the osmotic regulation between sap and medium.

However, the changing of the osmotic values of the medium will have still other consequences. As the osmotic value of the sap O_b results from the salt secretion S and the exudation b, the following relation obtains between these three factors.

$$S = b\,O_b \tag{3}$$

It is obvious that as soon as the exudation is suddenly altered (2), while the salt secretion continues, the osmotic value of the sap changes too (3). A period begins in which the exudation gradually alters in a direction counter to the preceding sudden change till a new level of equilibrium is reached. For these states of equilibrium we may derive from (1) and (3) that

$$b = -\tfrac{1}{2} kO_m + \sqrt{(\tfrac{1}{2} kO_m)^2 + k S} \tag{4}$$

It is apparent that according to the osmotic theory the exudation is dependent on the salt secretion, the osmotic value of the medium and the factor k, which is inversely proportional to the resistance of the root system for water-transport.

In order to form an opinion as to whether our results were in accordance with this formula we evaluated the salt secretion which pertains to the exudations at the state of equilibrium mentioned in fig. 1. The factor k can be derived with the aid of (2), the osmotic values of the medium are known. We can calculate the theoretical osmotic value of the exudation sap from the salt secretion and compare this value with the value found. The result is:

Medium	saltsecretion	osmotic value of the exuded sap		difference
		calculated	determined	
Hoagland	1.35	0.133	0.079	27 %
Mannitol + Hoagland	0.97	0.181	0.137	24 %

It is apparent that also for exudations in the state of equilibrium too low a value has been found for the osmotic pressure of the exudation sap and that the difference is of the same order as above.

We have mentioned already that there is, between the sudden decrease of the exudation and the more or less constant exudation at the state of equilibrium, a period of gradual increase. We can now by means of (1) and (3) arrange a formula for the relation between exudation and time. We have to start from the supposition that the salt secretion is constant during this period and we assume that as well as the water absorption it is limited to a region of the root, here designated as the active volume V.

The formula is as follows:

$$b = \tfrac{1}{2} kO_m + \sqrt{(\tfrac{1}{2} kO_m)^2 + k S} \cdot \operatorname{tgh} \left(\frac{\sqrt{(\tfrac{1}{2} kO_m)^2 + k S}}{V} \cdot t - C \right) \tag{5}$$

C = integrationconstant

A part of the experimental data on the exudation from fig. 1 has been plotted in figure 3 together with a curve giving the values of exudation as calculated according to (5). There is apparently an adequate agreement.

Summing up we can say that the course of the exudation in our

experiments can satisfactorily be accounted for. The level of the exudation is, however, higher than can be expected on the ground of the osmotic value of the exuded sap. This can be explained by assuming that the sap, during its transport to the stump, loses part of its salts or takes up extra water by osmosis. Either by one or by both processes a dilution effect results. In that way it is even theoretically thinkable that the osmotic value of the sap that is exuded, is lower than that of the medium. The difference may also be explained by accepting the view that there is

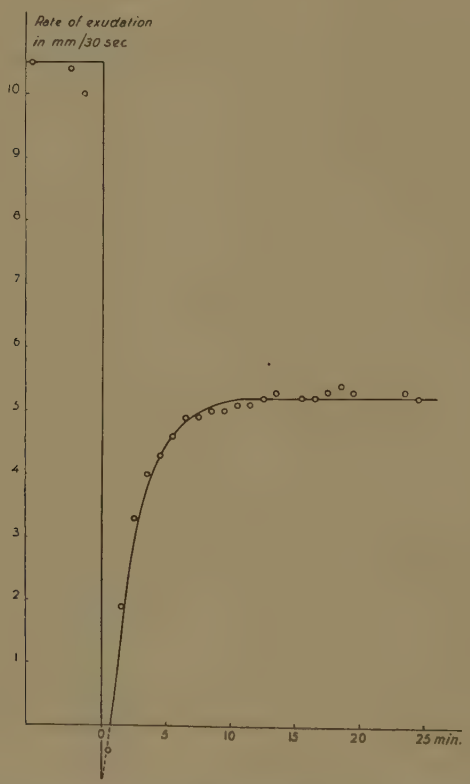

Fig. 3. Comparison between experimental data (○○○) and the calculated course of the exudation during the period in which the exudation increases by the continuation of the salt secretion.

still another process that transports water in a non-osmotic way. This does not mean the acceptance of an active water secretion. It seems possible that this process is bound in one way or another to the mechanism of the salt secretion, e.g. in the way supposed by LUNDEGÅRDH. Considering the importance of the osmotic regulation it does not seem very probable that such a process could produce a great difference between the osmotic values of the sap and the medium. Actually we found as largest difference that the freezing point of the exuded sap was only 0.009° lower than that of the medium. Further research, especially on

the relation of the salt secretion to the exudation, will have to decide which of the suppositions is the right one. So it can not yet be stated whether the exudation process is partly caused by a non-osmotic water transport; anyhow, it is certain that the osmotic regulation is of outstanding importance.

Discussion.

We have found that small changes in the osmotic value of the medium produce distinct changes in the exudation. This indicates a low resistance of the root system for water transport as mentioned also by other investigators. (KRAMER 1945, 1949; LUNDEGÅRDH 1945). Moreover, it points to an immediate osmotic regulation of the exudation as can be expected according to the formula of SABININ (1925): $b = k(O_b - O_m)$. LUNDEGÅRDH, in his latest paper, attaching only a very small importance to the osmotic value of the exudation sap, finds a relation between the square root of the osmotic value of the medium and the strength of the exudation. On this ground he gives the following equation

$$b = k\,(\sqrt{P_e} - \sqrt{P_m}).$$

The relation we found between changes in the osmotic value of the medium and of the exudation appears to be rectilinear, which is in close agreement with the formula of SABININ, but contrary to that of LUNDEGÅRDH.

We have to consider that we investigated momentary effects while LUNDEGÅRDH determined the exudation during longer periods. For this purpose we may make use of formula (4) giving the relation between the exudation in the state of equilibrium and the osmotic value of the medium. Calculation with this formula (4) shows us that between definite limits the exudations actually are proportional to the square root of the osmotic value. In using this formula we neglected a possible influence of the osmotic value of the medium on salt secretion. This result can explain the proportionality found by LUNDEGÅRDH with $\sqrt{O_m}$.

Just as VAN OVERBEEK we determined the compensating osmotic value of the medium and compared it with the osmotic value of the exuded sap. We had the same result and found the last one always higher than the first though the differences were much smaller. This can be explained by the difference in method. VAN OVERBEEK added the mannitol slowly by drops until the exudation was stopped for five minutes. Now we know that in this time the osmotic value of the sap will be increased. He obtained therefore too high a figure for the compensating osmotic value. Moreover too low an osmotic value of the sap exuded from the stump after being replaced in water, will be found.

EATON mentions also the gradually increasing exudation after the momentary sudden decrease. He explains it by the penetration of the

osmotic active substance from the medium into the root system. By means of substances which permeate in a different degree into the cell we could ascertain that permeation phenomena were out of the question. Moreover we concluded that it is superfluous to assume a separate process for this phenomenon. The salt secretion being the active force of the exudation, a sudden check of the exudation must be followed by a gradual increase, since the osmotic value of the exudation sap increases.

Summary.

The influence of a change in the osmotic value of the medium on the exudation of tomato-plants was investigated. An enhancement of this value is followed by a sudden decrease of the exudation passing into a gradual increase. A new but lower level is attained after about 10 minutes; the osmotic value of the sap is than higher. Lowering of the osmotic value of the medium gives opposite results. As it makes no difference whether mannitol raffinose or thiourea are supplied to the medium, this gradual change cannot be the result of permeation, as was supposed by SABININ and EATON. This phenomenon must be ascribed to the continuous salt secretion into the xylemvessels.

We found the sudden decrease of the exudation, caused by placing the plants on higher concentrated external solutions, proportional to the osmotic values of the medium. This agrees with the osmotic theory but is contrary to the theory and formula of LUNDEGÅRDH (1945, 1949). For prolonged experiments another formula could be derived, by which the conflicting results may be partly explained.

By means of the experiments just mentioned the osmotic value of the medium that momentarily stops the exudation could be determined. Comparing this value with that of the sap it appears that for the latter always too low a value is found. The differences are very variable, but always much lower than the differences found by VAN OVERBEEK, which could be ascribed to our improved method of determination.

The only deviations which we found in testing our theory, were the values of the exuded sap, which were too low. The course of the exudation found in our experiments was in perfect agreement with our theory, which indicated that the osmotic component of the exudation was of paramount importance for this process.

The too low values found for the exudation sap can be ascribed either to a difference between the exuded sap and the sap present in the active region of the root, caused by saltabsorption and water-dilution, or to a non osmotic water-transport. Whether this latter process really exists will have to be decided by further research, especially into the relation between salt secretion and exudation.

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DE F_5-ZAADGENERATIE VAN 1936 NA KRUISINGEN VAN TWEE ZUIVERE LIJNEN VAN PHASEOLUS VULGARIS. III.

DOOR

G. P. FRETS

(Communicated by Prof. J. BOEKE at the meeting of February 25, 1950)

Cl. 8. Er zijn *21 gevallen*, waar de formule van de uitgangsboon, l b th, cl 8, is (tab. 7). In 11 van deze gevallen is de formule van de gemiddelden van de bonenopbrengsten ook l b th, cl 8. We bespreken ze in de volgorde van de dalende lengte van de uitgangsboon.

Van bonen met de form. l b th, cl 8 zijn alle 3 afmetingen klein. Naar de indices hebben we ze nog nader onderscheiden. Van bonen met de form. l b th I, cl 8a komen de indices overeen met die van bonen van de I-lijn, van bonen met de form. l b th II, cl 8b, met die van bonen van de II-lijn. Van bonen met de form. l b th, cl 8c zijn de 3 afmetingen in dezelfde verhouding ten opzichte van hun gemiddelde waarden klein. Er zijn ook bonen met de form. l b th, cl 8 met indices, die overeenkomen met die van bonen van cl 5 en cl 6, form. l B Th en l B th, zij hebben een hoge L B-index. We hebben deze en nog andere, onder de bonen van cl 8c opgenomen (vgl. tab. 7).

Van *pl. 662* (fig. 7, K 4) is de uitg. boon van pl. 205. Zie ook pl. 663—665 (blz. 370). De bonenopbrengst van pl. 205 heeft zeer veel bonen in cl 2, form. L B th. Er zijn daarbij veel bonen met een hoge L B-index. Dit wijst er op, dat de lengte van deze bonen maar weinig boven de gemiddelde lengte is; ze naderen bonen met de form. l B th, cl 6 of l B Th, cl 5, die ook een hoge L B-index hebben. De bonen van cl 8a van pl. 205 hebben ook een nog al hoge L B-index. Hetzelfde geldt van de bonenopbrengst van pl. 662. Er is overeenkomst met bonenopbrengsten van de I-lijn, maar de gemiddelde L B- en L Th-indices zijn hoger. Volgens de classificatie zijn er bonen in cl 2, 4, 6 en 8 (met een hoge L B-index). De bonenopbrengst is gelijkmatig, behoort tot het gebied van cl 2. Er zijn geen bonen met een grote dikte. De uitgangsboon van pl. 205 heeft het genotype L B th, waarbij de afmetingen niet veel van de gemiddelde grootte verschillen; de lengte heeft ook l l-, de breedte b b-verbindingen, doch deze minder dan de lengte. Tot het gebied van bonenopbrengsten met de form. L B th, cl 2, behoort ook *pl. 1065* (fig. 6, K 10) zie ook blz. 578. Evenzo behoort *pl. 685* (fig. 7, K 4) tot het gebied van de I-lijn. De uitgangsbonen voor pl. 662, 1065 en 685 hebben de form. l b th I, cl 8a.

Pl. 648; de uitg. boon is van pl. 200 (fig. 6, K 10). Zie ook blz. 577.

De overige 4 bonen van de peul met 5 bonen van de uitg. boon hebben de form. l b Th, cl 7. Van de uitg. boon zijn „huid en boon gebarsten". De bonenopbrengst van pl. 200 heeft duidelijk het phaenotype van bonenopbrengsten van de II-lijn. De bonenopbrengst van pl. 648 is „goed". Er zijn veel bonen in cl 7 en 8, waarvan er door hun indices overeenkomen met bonen van cl 7 en van cl 6 en 5. De form. van de uitg. boon is l b th II, cl 8b en daaraan beantwoordt ook de bonenopbrengst van pl. 648.

Nog al samengesteld, doch overwegend als van cl 7 is ook de bonenopbrengst van *pl. 606* met uitg. boon van pl. 184 (fig. 6, K 10) en de form. l b th II, cl 8b.

Pl. 650. De uitg. boon is van pl. 200 (fig. 6, K 10 en tab. 5d en 5e). Zie ook blz. 576. Van de uitg. boon heeft de dikte de grenswaarde (th = 6.5 mm) en de form. l b th II, cl 8b. De 4 overige bonen van de peul behoren tot cl 7 (2 ×), 5 en 8b. Pl. 650 heeft kleine gemiddelden. De gemidd. indices komen overeen met die van de II-lijn, doch zijn iets te laag. Volgens de classificatie zijn alle bonen in cl 8 op één na (deze is in cl 5). Het zijn gave bonen. De bonenopbrengst beantwoordt aan het genotype l b Th met veel homozygotie en met ook enkele B B- en th th-verbindingen.

Ook van *pl. 633* (fig. 6, K 10) met uitg. boon van pl. 195, heeft de uitg. boon de form. l b th II, cl 8b. De bonenopbrengst van pl. 195 heeft zeer veel bonen in cl 5 en vele in cl 7 en 8. Slechts 5 bonen hebben een lengte boven de grenswaarde (l = 14.0, dan = 13.6 mm). Van pl. 633 heeft de gemidd. dikte de grenswaarde. Ze is kleiner dan van bonenopbrengsten van de II-lijn van 1936. De bonenopbrengst is „matig", „nog te groen". Er zijn geen bonen met een lengte groter dan 13.0 mm. De 25 bonen van cl 8 hebben een L B-index (L B = 66—71), die weinig verschilt van die van de 9 bonen van cl 7 (L B = 66—70). Dit geldt ook voor de L Th- (resp. L Th = 52—59 en = 52—61) en de B Th-index (resp. 78—87 en = 80—86). Pl. 633 behoort door bonenopbrengst en uitg. boon tot het gebied van bonen van cl 7 met de form. l b Th.

Van *pl. 624* (fig. 6, K 10) is de uitg. boon van pl. 192 van een peul met 7 bonen, die alle een zo kleine lengte hebben als we slechts bij uitzondering bij bonen van de II-lijn van 1935 vinden (b.v. pl. 24). De bonenopbrengst van pl. 624 met zeer veel bonen in cl 8 is „vrij goed". Ze behoort evenals haar uitg. boon tot het gebied van cl 7.

Pl. 632. De uitg. boon is van pl. 194 (fig. 6, K 10 en tab. 5d en 5e). Zie ook pl. 633, hierboven. De uitg. boon is zeer klein, evenals de overige 4 bonen van de peul. De bonenopbrengst van pl. 194 behoort overwegend tot cl 5. Pl. 632 heeft, volgens de classificatie, veel bonen in cl 7 en 5, de overige in cl 8. De grootste lengte van de bonen is l = 12.8 mm. De indices van de bonen van cl 5, cl 7 en cl 8 verschillen niet veel. De bonenopbrengst is als die van cl 5. Vele bonen van pl. 632, zijn in 1937 uitgezaaid. Ze leverden de pl. *436—462*, F_6-1937. Er is een goede overeenstemming tussen uitg. boon en bonenopbrengst (tab. 5e en 6e). De 6 bonenopbrengsten, waarvan de form. van de uitg. boon l B Th, cl 5 is, hebben meer bonen in cl 5, de 9 bonenopbrengsten, waarvan de form. van de uitg. boon l b Th, cl 7 is, hebben meer bonen in cl 7; van de 12 bonenopbrengsten, waarvan de uitg. boon tot cl 8 behoort, is de form. van de uitgangsboon, volgens de indices, meestal overeenkomend met bonen van cl 5,2 of (3 ×) met die van cl 7. De bonenopbrengsten hebben veel bonen in cl 5 en 7 en zeer vele in cl 8c (als cl 5). In één van de 2 gevallen, waar de B Th-index hoog is (form. l b th II, cl 8b) zijn er veel bonen in cl 5, in het 2de in cl 7. In enige van de 27 gevallen zijn er ook enkele bonen in cl 1 en cl 2. Bij de pl. 436—462 (27 pl.), is er een goede overeenstemming tussen bonenopbrengst en uitgangsboon.

In één geval, *pl. 578* (fig. 6, K 10), waar de uitg. boon de form. l b th heeft, is de formule der gemiddelden L B Th, cl 1. Zie ook blz. 576.

In *5* van de 21 gevallen, waar de uitgangsboon de form. l b th, cl 8 heeft, is de formule van de gemiddelden der bonenopbrengsten L B th, cl 2. Het zijn *pl. 1086* en *1079* (fig. 6, K 10) *822* (fig. 8, K 64), *753* en *687* (fig. 7, K 4). Volgens de indices behoren de uitgangsbonen tot cl 8a form. l b th I. In enige gevallen zijn ze iets te hoog. Er is een duidelijk verschil ten opzichte van de uitgangsbonen voor bonenopbrengsten met andere gemiddelden. Er is ook hier een goede overeenkomst van uitg. boon en bonenopbrengst. *Pl. 753* en haar uitgangsboon is een goed voorbeeld van een geval van cl 2.

Van de 21 gevallen met de uitgangsboon van cl 8, form. l b th, is de formule van de gemiddelden in 4 gevallen L b th, cl 4. Het zijn de pl. *1070* (fig. 6, K 10), *771* en *705* (fig. 1, K 33). *Pl. 771*; de uitg. boon is van pl. 287. De 5 bonen van de peul van de uitg. boon behoren alle tot cl 8a, form. l b th I; de lengte is nog al verschillend (l = 103—12.9 mm). De bonen van de bonenopbrengst van pl. 287 hebben „een ongelijke grootte" de peulen zijn „ingedroogd". Er zijn veel bonen met een zeer grote lengte (13 bonen in cl 2a en 1a). In de bonenopbrengst van pl. 771 zijn, volgens de classificatie, veel bonen in cl 4 en 8a, enige in cl 1a, 1b, 2a en 2b. De uitgangsboon is, nemen we aan, phaenotypisch klein, doch heeft genen ook voor grote lengte en grote breedte. Belangrijk is hier ook de ascendentie. *Pl. 1070* behoort ook door haar uitg. boon van pl. 174 tot het gebied van cl 4. Dit geldt in mindere mate voor de uitg. boon van pl. 219 voor *pl. 705*. De bonenopbrengst van *pl. 796* is samengesteld. De gemiddelde afmetingen liggen dichtbij haar gemidd. grootten. De uitgangsboon is van pl. 322 en heeft door haar indices overeenkomst met bonen van cl 3.

Hiermede zijn alle gevallen van cl 8 besproken. Waar, bij bonen met kleine afmetingen, de niet-erfelijke variabiliteit groot is door ongunstige milieu-invloeden, treft toch ook de grote betekenis van de erfelijkheid: we vinden bijna steeds een overeenstemming van de gemiddelden van de bonenopbrengst en de uitgangsboon.

Summary F_5-1936.

We found the heredity as correlation of the initial beans of F_4-1935 and the averages of the beanyields of F_5-1936. See tab. 1. The positive correlation-coefficients found with the aid of the formula of BRAVAIS are fairly great, somewhat greater than those of F_3-1934 and F_4-1935 and than those of F_2-1933 and F_3-1934; the latter are irregular and often insignificant (tabs. 2 and 3).

The variation-range of the length of the beans of F_5-1936 is 8.7—18.1 mm. That of the beans of the I- and the II-lines together of 1936 is 8.0 (followed by 8.5)—19.2 (preceded by 18.1) mm.

The greatest mean length of the beanyields of F_5-1936 is $l_m = 16.2$ mm, the greatest mean breadth is $b_m = 10.0$ mm. Of the beanyields of the I-line of 1936 the greatest mean length is $l_m = 15.5$ mm, the greatest mean breadth is $b_m = 9.5$ mm. So there are beanyields of F_5-1936 with a greater mean length and breadth than we find among beanyields of the I-Line of 1936. The smallest mean length and breadth of beanyields of F_5-1936 do not exceed the smallest mean length and

breadth of beanyields of the II-line of 1936 ($l_m = 10.5$ and $b_m = 7.5$ mm against $l_m = 10.1$ and $b_m = 7.5$ mm).

The variability of the beans of F_5-1936 as expressed by standard deviation and variation-coefficient is not greater than of beans of the I-line and the II-line (tab. 4).

The material is classified under the 8 classes of our tetrahybrid scheme (1947, these Proc. Vol. 50, p. 798). The F_6-seed-generation 1937 beanyields, grown in 1937 have also been included in the descriptions. We want especially to bring to light how far in the later generations after crosses hereditary variations are formed. This is best investigated for beanyields of cl 2 and of cl 7 because they can be compared with resp. beanyields of the I- and of the II-line.

In one case (*pl. 783*, fig. 1, K 33 and tabs 5d and 6d) the form. of the initial bean and also that of the averages of the beanyield is $L_1\ L_2$ B Th, cl 1a. According to the classification of pl. 783 all beans but one belong to cl 1, form. L B Th. The mean length and the mean breadth of the beanyield of pl. 783 are greater than the greatest mean length and mean breadth of beanyields of the I-line of 1936. We have here an instance of transgressive variability. *Pl. 776* (fig. 1, K 33 and tabs. 5d and 6d) is a good instance of a beanyield with the form. $L_1\ L_2$ B Th, of which the initial bean of pl. 295 with the form. $L_1\ L_2$ B Th, has this formula in a homozygous form to a high degree. The beanyield of *pl. 441* (fig. 3, K 15 and tabs. 5d and 6d) shows heredity of beans with the form. L B Th and of great dimensions ascending (tabs. 5c and 5b; 6c and 6b) as well as descending (tabs. 5e and 6e).

Of *pl. 785* (fig. 1, K 33 and tabs. 5d and 6d) the initial bean of pl. 118 has the form. $L_1\ L_2$ B Th, cl 1a and the form. of the averages of the beanyield is $L_1\ l_2$ B th, cl 2b. It has a great many beans in cl 2 (p. 215). Of *pl. 438* (fig. 3, K 15 and tabs. 5d and 5e) the formula of the averages is also $L_1\ l_2$ B th, cl 2b. It greatly resembles the beanyields of the I-line of 1936 (fig. 4). The thickness of the initial bean, however, is too great. Of one bean of pl. 438, which was sown in 1937 and produced pl. 277, F_6-1937 (tabs. 5e and 6e) the beanyield (that is pl. 277) does not entirely resemble beanyields of the I-line of 1937. The initial bean of pl. 118 for pl. 785 is not homozygous for the form. L B th of beans of the I-line.

There are 34 cases where the initial bean of F_4-1935 for F_5-1936 has the form. $L_1\ l_2$ B Th, cl 1b. Of 5 beanyields the form. of the averages is $L_1\ l_2$ B Th, cl 1b. *Pl. 489* (fig. 3, K 15 and tabs. 5d and 6d) has the phaenotype L B Th, cl 1b but as appears from the composition of pl. 332, F_6-1937 (tabs. 5e and 6e), which grew from an initial bean of pl. 489, the initial bean of pl. 126 for pl. 489, is not homozygous for the form. L B Th. The beans of cl 1, form. L B Th are chiefly bastard beans, beans with much heterozygousness. The beanyield of pl. 485 (tabs. 5d and 6d) is more compound than that of pl. 489. It contains a few segregates with the form. l b Th, cl 7.

Of the pls. *1068*, *1069* and *580* the initial beans are from pl. 174 (fig. 6, K 10). The beanyields greatly resemble beanyields of the I-line. Of pl. 580, 21 beans were sown in 1937; they produced pls. 489—509, F_6-1937. These beanyields point to a not entirely homozygous composition of the beanyield of pl. 580.

Of *pl. 533* (fig. 2, K 13 and tabs. 5d and 6d) the formula of the averages is L b th, cl 4 and the formula of the initial bean is L_1 l_2 B Th, cl 1b. Also according to the classification pl. 533 belongs in the domain of cl 4 and cl 2.

An important case is *pl. 503* (fig. 3, K 15 and tabs. 5d and 6d). This beanyield contains only beans in cl 7 and cl 8b. Such averages as those of pl. 503 are not found for comparison-beanyields of the II-line of 1936: in respect of the mean length, the mean breadth and the mean thickness are too small, the mean L B- and B Th-indices too low (comp: Proceed. 1949, tab. 4b). Among the individual beans there are some that entirely resemble beans of the II-line of 1936. Two beans of pl. 503, F_5-1936 were sown in 1937; they produced pls. 349 and 350, F_6-1937 (tabs. 5e and 6e). Such small mean breadths and thicknesses as pl. 349 has, are not found for the averages of the beanyields of the II-line of 1937. According to the classification all the beans of the beanyield of pl. 349 belong to cls. 8b and 8c. We are dealing with a beanyield from the domain of cl 7, form. l b Th, but with remarkably small dimensions of the beans. There is transgressive variability of minus-variations. Of pl. 350, F_6-1937 the small and bad beanyield (tabs. 5e and 6e) belongs under cl 7. It has many beans in cl. 8 (according to the indices they are beans of cl 6 and 5).

For the pls. *1066* and *1061* (fig. 6, K 10) the initial beans are from pl. 174 (p. 219); they have the form. L B Th and the form. of the averages of the beanyields is l b th. They belong under cl 2.

For *pl. 504* the initial bean is from pl. 131 (fig. 3, K 15 and tabs. 5d and 6d) as it is for pl. 503 (p. 580) which it greatly resembles.

The many cases we have discussed, where the initial bean has the form. L B Th (cls. 1a and 1b) and the formula of the averages is L B Th or other formules also, point to heredity of beans with these formulas as well as to their frequent affinity with beans of cl 2, form. L B th, but also to the high degree of heterozygousness peculiar to beans with this formula.

In 12 cases the formula of the initial bean is L_1 L_2 B th, cl 2a. In 5 of these cases the formula of the averages of the beanyields is L_1 l_2 B th, cl 2b. Among them are beanyields which greatly resemble those of the I-line of 1936 (tabs. 5d and 6d, pls. 666 and 450; tabs. 5e and 6e, pl. 1046).

In 35 cases the formula of the initial bean is L_1 l_2 B th, cl 2b. In 15 of these cases the formula of the averages of the beanyields is also L B th. In one of these cases (pl. 674) the formula of the averages is

L_1 L_2 B th, cl 2a. Here too there are several beanyields which entirely or almost entirely resemble beanyields of the I-line of 1936. In tab. 8 these beanyields have been compared with beanyields of the I-line of 1936. It appears that among the beanyields that resemble the I-line — as compared with 64 beanyields of the I-line of 1936 — there are too many with great mean thicknesses and high mean L Th-indices such as only rarely occur in the I-line. In a few cases only (pls. 1066, 674, 675, 765 and 562) the phaenotype resembles random beanyields of the I-line. So far as of the 28 beanyields beans were sown in 1937 it appears, that they are not entirely homozygous for the characters that we investigated. Of pl. 765 and its ascendance the characterograms are given (fig. 9). We see from them the heredity of beans with the form. L B th, cl 2.

There are 8 beanyields with L B th, cl 2 as the formula of the initial bean, of which the formula of the averages is l b th, cl 8. Of one of these beanyields the averages are very small and of the beanyield of F_6-1937, grown from one of the beans of the beanyield of F_5-1936, the averages show transgressive variability of minus-variations.

Our material does not contain a good example of the group cl 3, form. L b Th.

Pl. 729 (fig. 7, K 4 and tabs. 5d and 6d) is a beanyield of which the initial bean is to a high degree homozygous for the form. L b th, cl 4.

Pl. 1087 (fig. 6, K 10 and tabs. 5d and 6d) belongs to the beanyields of cl 5, form. l B Th. The beans in the other classes are milieu-variations of beans with small genotypical differences with regard to beans of cl 5.

Pl. 1089 is related to pl. 1087 and is a good example of a beanyield of cl 6.

Interesting beanyields are also those of pls. *499* and *497* (fig. 3, K 15 and tabs. 5d and 6d). See p. 369. The beanyield of pl. 497 has to a high degree the phaenotype of cls 7 and 5.

Pl. 480 (fig. 3, K 15 and tabs. 5d and 6d). A bean of pl. 480 produced in 1937, pl. 325, F_6-1937 (tabs. 5e and 6e). It has the phaenotype of cl 7 fairly well but does not altogether conform to it, however.

Of the 15 cases where the formula of the initial bean of the beanyield is l B Th, cl 5, most beans belong to the domain of cl 5 and cl 7. Beanyields which entirely resemble beanyields of the II-line of 1936 we do not find among them.

In the material there are 2 good cases (pl. 806 and pl. 1089, tabs. 5d and 6d) of cl 6, form. l B th.

In 18 cases the formula of the initial bean is l b Th, cl 7 and in one of them the form. of the averages of the beanyield (pl. 652) is also l b Th, cl 7. The beanyield is regular but does not entirely resemble beanyields of the II-line of 1936. The initial bean is to a high degree homozygous for a genotype that differs slightly from that of beans for

the II-line; it does not have all the Th Th-pairs of factors of the beans of the II-line and not exclusively those of l l-pairs of factors.

One of the 3 cases where the form. of the averages of the beanyield (pl. 645, fig. 6, K 10, tabs. 5d and 5e, fig. 10) is a good instance of heredity in ascendance and descent of beans of the domain of cl 7. This also appears from the other cases included in tab. 5d and 6d.

There are 21 cases where the form. of the initial bean is l b th, cl 8. In 11 of these cases the form. of the averages of the beanyield is also l b th, cl 8. According to the indices of the beans some cases belong to the domain of cl 2, that is of the I-line; others to that of cl 7, that is of the II-line. For the beans of cl 8, form. l b th we distinguish those of cl 8a, l b th I and cl 8b, l b th II. Beans of which all 3 dimensions are uniformly small we distinguish as beans of cl 8c, form. l b th. In this group are also included beans with high L B-indices as of cl 5 and cl 6, (comp. tab. 7). Occasionally we find transgressive variability of minus-variations. Where, for beans with small dimensions, the nonhereditary variability is great, owing to unfavourable environment influences, the great significance of heredity still is striking; we nearly always find a resemblance of the averages of the beanyield and the initial bean.

For this publication we studied the material of F_5-1936, compared the beanyields with their initial beans of F_4-1935 and also included a few data regarding ascendance and descendance.

The investigation of many beanyields of F_5-1936, grown from crosses in 1933 of 2 pure lines, line I and line II, shows us a great measure of heredity, which is a confirmation of the statistical researches, whereby we find a high positive correlation-coefficient.

Some cases show a close resemblance to the parent form, an entire resemblance for the investigated characteristics we did not find.

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TABLE 5. Tab. 5d contains cases of F_5-1936, i.e. of initial beans of F_4-1935 and the beanyields of F_5-1936. Tab. 5c—5a contain the ascendance of the cases of tab. 5d and tab. 5e the descendance.

Explanation of tab. 5 and 6. Tab. 6 belongs to tab. 5. Tab. 5 and 6 are subdivided in the tab. a—e, that are relative to the generations F_2—F_6, 1933—1937.

We start from tab. 5d; it contains cases of F_5-1936, which represent initial beans of F_4-1935 with the mean dimensions, weights and indices of F_5-1936.

To clear up the connection of the tab. a—e we give an example.

The first case of tab. 5d is the beanyield of pl. 783, F_5-1936; its initial bean is of pl. 297, F_4-1935. We find the beanyield of pl. 297, F_4-1935 back in tab. c with the initial bean of pl. 272, F_3-1934. The beanyield of pl. 272, F_3-1934 occurs as first case in tab. b with the initial bean of K 33, F_2-1933. In tab. 5a, to conclude, the beanyield of K 33, F_2-1933 is mentioned with the cross that was the starting point of these generations. So, in the tab. 5d—a the complete ascendance is given of the beanyields of F_5-1936, that tab. 5d contain.

A 2nd example. The second case of tab. 5d is pl. 784. We can pursue pl. 784 of tab. 5d ascending in tab. 5c, 5b and 5a. We see furthermore that the first case of tab. 5e represents the initial bean of pl. 784, F_5-1936 for pl. 1065, F_6-1937 with the averages of the beanyield of pl. 1065, F_6-1937.

We can follow in the same manner all beanyields of F_5-1936, that tab. 5d contains, ascending in the tab. c. b and a and descending in tab. e.

TABLE 5a

Cl	Pl	bean	L	B	Th	W	L B	L Th	B Th	Pl	col	n	L	B	Th	W	L B	L Th	B Th
		Crosses. F_1-1932											F_2-1933. Averages						
		$I_1 \times II_4$	149	81	65	55	54	44	80	K 33	v [2]	49	138	86	67	56	62	49	79
		$II_2 \times I_2$	133	89	72	56	67	54	81	K 15	w [3]	38	134	79	63	45	59	47	79
		$I_4 \times II_{19}$	152	89	67	62	59	44	75	K 10	v	44	141	82	65	51	59	46	79
		$I_2 \times II_5$	142	88	64	57	62	45	73	K 4	v	44	148	86	66	57	58	45	77
		$I_4 \times II_4$ [1]	147	93	68	61	63	46	73	K 64	v	44	only photographed						
		$I_5 \times II_{14}$	148	84	66	56	57	43	79	K 13	v	46	139	83	65	52	60	47	79
		$II_2 \times I_3$	145	90	77	65	62	53	86	K 16	w	32	138	81	64	49	59	47	79

[1]) A bean of a pod with 4 similar beans (1934b, p. 311). [2]) v = violet. [3]) w = white.

TABLE 5b

Cl	Pl	bean	L	B	Th	W	L B	L Th	B Th	Pl	col	n	L	B	Th	W	L B	L Th	B Th
		Initial beans of F_2-1933 for F_3-1934											F_3-1934. Averages						
7	K 33	10 p 5 b	127	81	66	47	64	52	82	272	v	65	157	98	68	75	62	44	70
8a	K 15	1 p 4 b	124	73	59	37	59	48	81	159	v	48	146	94	64	64	65	44	69
8c	K 15	2 p 5 b	122	74	63	38	61	52	85	160	v	63	124	84	71	52	68	57	84
1b	K 10	1 p 2 b	151	88	68	60	58	45	77	177	v	54	141	89	68	59	63	48	76
3	K 4	2 p 5 b	137	76	67	46	56	49	88	189	v	52	135	88	65	54	66	48	75
3	K 10	4 p 2 b	142	85	68	57	60	48	80	179	w	50	132	86	68	53	65	52	79
3	K 4	3 p 3 b	147	85	68	58	58	46	80	191	w	82	141	87	66	57	62	47	77
4	K 10	5 p 2 b	142	83	65	51	58	46	78	180	v	61	125	88	70	54	70	57	80
	K 64 [1]									359	v	45	122	85	62	44	69	50	73
4	K 13	1 p 3 b	137	83	65	49	61	47	78	168	v	43	140	89	72	62	64	51	80
8b	K 16	1 p 5 b	119	68	62	34	57	52	91	153	v	42	117	81	61	42	68	52	76

[1]) Initial bean of K 64, has not been measured.

TABLE 5c

Cl	Pl	bean	L	B	Th	W	L B	L Th	B Th	Pl	col	n	L	B	Th	W	L B	L Th	B Th
		Initial beans of F_3-1934 for F_4-1935										F_4-1935. Averages of beanyields							
1b	272	2 p 5 b	151	95	67	71	63	44	71	297	v	41	141	93	66	64	66	47	71
8a	272	5 p 4 b	127	81	55	38	64	43	68	300	v	44	153	98	68	75	64	45	70
1a	272	1 p 3 b	168	109	69	94	65	41	63	295	—	43	148	93	63	61	63	43	69
1a	159	1 p 4 b	163	97	66	78	60	40	68	118	v	66	153	96	68	72	63	45	71
7	160	1 p 3 b	123	82	70	48	67	57	85	126	—	51	132	87	71	57	66	53	81
1b	177	1 p 1 b	140	89	72	59	64	51	81	153	v	46	149	91	66	66	61	44	72
8b	160	9 p 4 b	104	75	61	33	72	59	81	131	—	54	123	84	64	46	69	52	76
1b	189	2 p 2 b	147	98	72	73	67	49	74	206	—	45	147	93	63	63	64	43	68
1a	159	3 p 1 b	159	98	72	80	62	45	74	119	v	51	163	103	69	85	63	42	67
1a	159	14 p 1 b	159	100	68	79	63	43	68	124	v	54	148	96	63	64	65	43	65
2b	159	8 p 1 b	150	90	62	62	60	41	69	120	v	57	143	92	59	55	64	41	64
1b	159	8 p 3 b	155	95	75	71	61	48	79	121	v	50	158	100	67	76	64	43	67
1b	179	2 p 2 b	139	93	70	62	67	50	75	174	w	58	137	85	63	53	63	46	74
2b	191	10 p 1 b	144	87	65	57	60	45	75	228	—	22	140	88	66	58	63	48	76
5	180	2 p 2 b	130	94	75	62	72	58	80	185	v	43	119	84	67	48	71	56	80
7	160	3 p 1 b	114	83	71	47	73	62	86	129	w	52	123	81	69	48	66	56	84
7	160	2 p 3 b	115	82	70	46	71	61	85	128	—	51	111	81	67	43	73	60	82
1b	160	1 p 1 b	137	88	75	63	64	55	85	125	—	53	128	83	69	52	65	54	83
8c	359	7 p 5 b	112	82	55	34	73	49	67	333	v	42	126	87	64	50	69	52	74
7	180	14 p 1 b	125	90	76	59	72	61	85	200	v	75	121	85	69	51	71	58	82
7	168	4 p 3 b	123	91	68	56	74	55	75	136	w	72	138	90	68	61	66	50	76
7	160	1 p 5 b	125	88	76	59	70	61	86	127	—	51	113	80	68	44	71	60	85
8c	153	3 p 1 b	111	78	54	33	70	49	69	112	w	68	120	82	64	44	69	54	78
1b	180	7 p 2 b	146	93	68	64	64	47	73	194	v	44	113	82	64	43	73	57	78

TABLE 5d

Cl	Pl	bean	L	B	Th	W	L B	L Th	B Th	Pl	col	n	L	B	Th	W	L B	L Th	B Th
		Initial beans of F_4-1935 for F_5-1936										Beanyields of F_5-1936. Averages							
1a	297	7 p 2 b	175	105	77	102	60	43	73	783	—	30	158	100	73	83	63	46	72
1a	300	1 p 2 b	167	104	77	105	62	44	74	784	—	42	153	97	68	73	64	45	70
1a	300	3 p 2 b	169	109	73	101	65	43	67	785	—	43	151	97	64	63	64	42	66
2	300	9 p 2 b	139	86	55	47	62	40	64	786	v	40	133	90	59	48	68	45	66
1a	295	4 p 3 b	164	99	67	74	60	41	68	776	—	40	151	96	68	63	64	46	71
1a	118	8 p 4 b	158	104	72	82	66	46	69	441	v	24	148	96	66	64	64	44	69
2a	118	1 p 3 b	164	101	77	85	62	47	76	438	v	26	148	93	62	59	63	42	67
1b	126	8 p 2 b	135	87	71	58	64	53	82	489	v	23	141	95	69	65	67	49	74
1b	126	1 p 1 b	134	91	72	62	68	54	79	485	v	25	136	87	70	59	64	51	80
1b	153	6 p 3 b	151	88	67	65	58	44	76	533	w	25	137	84	60	48	61	44	72
1b	131	11 p 1 b	143	88	69	60	62	48	79	503	w	24	115	79	66	42	69	58	83
1b	131	11 p 2 b	136	86	71	57	63	52	83	504	w	35	112	77	63	—	68	56	83
2a	206	6 p 2 b	159	102	63	72	64	40	62	666	—	24	155	93	63	66	60	42	69
2a	119	10 p 1 b	192	115	65	108	60	34	57	450	v	28	162	99	67	74	61	42	68
2b	124	4 p 1 b	152	101	59	66	66	39	58	478	v	25	142	90	61	56	64	43	67
2b	120	4 p 4 b	133	88	54	45	66	41	61	454	—	24	148	94	63	63	64	42	67
2b	121	11 p 1 b	145	91	65	58	63	45	72	463	w	16	160	100	70	80	63	44	71
2b	174	11 p 3 b	142	87	64	56	61	45	74	1075	w	24	138	82	60	52	59	43	73
4	228	4 p 2 b	133	85	63	50	64	47	74	729	w	25	137	80	61	50	58	45	77
5	185	1 p 1 b	128	88	68	54	69	53	77	1087	—	34	126	90	70	57	71	55	79
5	185	1 p 3 b	122	86	67	50	71	55	78	1089	v	29	123	86	63	46	70	51	73
5	129	8 p 1 b	128	87	74	56	68	58	85	499	w	28	130	84	66	49	65	52	78
5	128	12 p 1 b	122	92	68	53	75	56	74	497	v	24	116	84	68	46	72	59	81
5	125	1 p 3 b	128	89	68	53	70	53	77	480	v	27	116	80	63	41	68	55	80
6	333	1 p 3 b	116	86	65	43	74	56	76	806	w	40	121	82	61	41	68	51	75
7	200	6 p 2 b	117	83	67	47	71	57	81	645	v	30	126	87	68	53	69	54	78
7	136	17 p 2 b	126	84	68	51	67	54	81	514	w	24	116	80	59	39	69	51	74
7	127	5 p 2 b	125	83	70	51	66	56	84	492	v	24	105	74	64	35	71	62	87
7	112	2 p 3 b	120	82	67	48	68	56	82	425	—	26	120	78	58	40	66	49	74
8b	200	7 p 4 b	115	82	65	43	71	57	79	650	v	32	109	75	60	33	68	55	80
8b	194	2 p 2 b	97	71	58	28	73	60	82	632	v	27	114	82	65	42	71	57	79

TABLE 5e

Cl	Pl	bean	L	B	Th	W	L B	L Th	B Th	Pl	col	n	L	B	Th	W	L B	L Th	B Th
		Initial beans of F_5-1936 for F_6-1937										Beanyields of F_6-1937. Averages							
1a	784	1 p 2 b	170	106	69	92	62	41	65	1065	v	23	130	83	55	37	64	42	66
2b	785	1 p 4 b	153	101	57	63	66	37	56	1079	v	25	154	98	68	64	64	44	69
1a	441	1 p 1 b	174	101	72	82	58	41	71	279	v	29	144	92	68	68	64	48	75
1b	438	2 p 3 b	146	94	66	62	65	45	70	277	v	28	146	90	64	64	62	44	71
1b	489	2 p 1 b	149	100	72	71	67	48	72	332	v	24	138	86	65	59	62	48	77
1b	485	3 p 1 b	145	96	75	74	66	52	78	328	v	26	137	86	68	60	63	49	79
7	503	3 p 3 b	115	82	66	43	71	57	81	349	w	25	110	75	60	37	68	54	80
7	503	6 p 1 b	114	84	69	45	73	61	82	350	w	19	121	82	57	35	68	47	70
2a	666	8 p 1 b	156	92	57	55	59	37	62	1046	w	25	147	86	60	50	58	41	70
4	666	7 p 1 b	132	83	51	39	62	39	61	1044	w	25	144	87	59	47	61	41	67
1b	666	3 p 2 b	148	92	68	68	62	46	74	1039	w	16	158	91	67	64	58	43	73
8c	666	7 p 3 b	114	76	51	29	67	45	67	1045	w	18	127	77	48	31	61	38	63
1a	450	3 p 1 b	168	98	72	77	58	43	73	289	v	25	163	98	67	67	60	41	69
1a	450	6 p 3 b	180	110	78	102	61	43	71	292	v	25	165	96	70	73	58	43	74
1b	478	3 p 2 b	149	92	66	64	62	44	72	323	v	29	148	90	63	67	61	43	71
1b	463	1 p 3 b	151	101	67	74	67	44	66	317	v	19	155	99	66	69	64	42	66
5	499	2 p 3 b	129	87	72	54	68	56	83	347	w	27	122	78	63	44	64	52	80
5	499	4 p 1 b	146	91	71	64	62	49	78	348	w	30	122	77	61	43	63	50	79
8b	480	1 p 2 b	106	72	59	32	68	56	82	325	v	27	122						
1b	645	1 p 3 b	132	86	69	57	68	52	77	832	v	25	120	81	68	42	68	57	84
5	645	1 p 2 b	124	90	70	57	73	57	78	831	v	25	121	82	62	39	68	51	76
7	645	1 p 5 b	123	84	67	49	68	55	80	834	v	25	115	78	64	37	68	57	83
8c	645	2 p 3 b	130	84	64	51	65	49	76	837	v	25	124	82	68	42	66	55	83
8c	514	3 p 4 b	113	80	58	38	71	51	73	356	w	25	124	85	67	46	69	54	79
8b	492	3 p 5 b	104	77	65	36	74	63	84	344	v	26	101	71	61	34	71	60	85
8b	492	3 p 7 b	98	70	61	29	71	62	87	346	v	30	104	74	62	36	71	61	85
8c	425	4 p 3 b	127	84	65	49	66	51	77	274	w	29	124	81	62	48	66	50	76
8c	632	1 p 3 b	104	74	58	31	71	56	78	438	v	29							
8b	632	4 p 6 b	112	77	62	38	69	55	81	456	v	25							

TABLE 6. The classification of the beanyields of tab. 5. Tab. 6d. The classification of the beanyields of tab. 5d; tab. 6c—a the classification of the beanyields of tab. 5c—a and tab. 6e the classification of the beanyields of tab. 5e.

TABLE 6a

Pl	n	$L_1 L_2$ B Th	$L_1 l_2$ B Th	$L_1 L_2$ B th	$L_1 l_2$ B th	L b Th	L b th	l B Th	l B th	l b Th	l b th I	l b th II	l b th
		1a	1b	2a	2b	3	4	5	6	7	8a	8b	8c
Beanyields of F_2-1933. Classification													
K 33	49	1	18		2	18	4			3	3		
K 15	38		3			3	19				4	1	8
K 10	44		5	1		18	19				1		
K 4	44	4	11	7	4	13	4			1			
K 13	46		6		5	15	17				3		
K 16	32		1		4	7	16			1		3	

TABLE 6b

Pl	n	$L_1 L_2$ B Th	$L_1 l_2$ B Th	$L_1 L_2$ B th	$L_1 l_2$ B th	L b Th	L b th	l B Th	l B th	l b Th	l b th I	l b th II	l b th
		1a	1b	2a	2b	3	4	5	6	7	8a	8b	8c
Beanyields of F_3-1934. Classification													
272	65	33	15	6	10						1		
159	64	6	17	1	19		1		2		2		
160	64		3			2	1	22		33		2	1
177	54		34		10	6	3				1		
189	52		25		7	4	5				9	1	1
179	50		21		1	4	1	1	1	10	1		10
191	82	1	38		11	6	18	1		2	3	2	
180	62		11					34	2	9	3	3	
359	45		3		2	2		3	9	7	15	3	
168	43	2	25			5		7	1	3			
153	42									3	6	5	28

TABLE 6c

Pl	n	$L_1 L_2$ B Th	$L_1 l_2$ B Th	$L_1 L_2$ B th	$L_1 l_2$ B th	L b Th	L b th	l B Th	l B th	l b Th	l b th I	l b th II	l b th
		1a	1b	2a	2b	3	4	5	6	7	8a	8b	8c
Beanyields of F_4-1935. Classification													
297	41	6	15	1	10		3	1	1	1	3		
300	42	16	12		13		1						
295	43	6	4	4	25		2				2		
118	66	22	28	2	11		2		1				
126	51		31		1	1		4		13		1	
153	46	7	13	4	14	2	6						
131	54		7		4	4		8	1	5	17	8	
206	45	5	12	2	23		2		1				
119	51	32	7	5	5		2						
124	54	1	17	1	34					1	1		
120	57	4			40		7		1		5		
121	49	24	1	2	20		1		1				
174	58		11		19		20				8		
228	22	1	7	1	5		6			1	1		
185	43		2			1		15	1	17	2	5	
129	52		4			2		7		35	2	2	
128	51							11		27	1	12	
125	53		9		2	10	1	3		23	3	2	
333	42		7		4			7	8	6	6	4	
200	75		8					25	2	27	1	9	
136	72	2	44		10	5	4	1	2	2		2	
127	51							8		32		11	
112	68		2		1			8	6	17	33	1	
194	44							16	3	1		24	

TABLE 6d

Pl	n	$L_1 L_2$ B Th	$L_1 l_2$ B Th	$L_1 L_2$ B th	$L_1 l_2$ B th	L b Th	L b th	l B Th	l B th	l b Th	l b th I	l b th II	l b th
		1a	1b	2a	2b	3	4	5	6	7	8a	8b	8c
Beanyields of F_5-1936. Classification													
783	30	20	9		1								
784	42	14	18		10								
785	43	7	5	3	26		2						
786	40		2		24		1		4		9		
776	40	17	8	1	8	1	2						3
441	24	4	9	1	10								
438	26		3	6	17								
489	23		22		1								
485	25		16			3		1		4		1	
533	25				4	18					1		2
503	24									19		5	
504	35									6	4	25	
666	24	11	1	4	4		2				2		
450	28	17	1	4	6								
478	25		2	1	17		3		1		1		
454	24		1		23								
463	16	9	5		2								
1075	24						23						1
729	25						21						4
1087	34		10			1		16		5			2
1089	29		1		1			4	8	2		1	12
499	28		7		4		2	3			1		
497	24							6	1	10		3	4
480	27						1	2		5		5	14
806	40				2		1		7	1	3	3	23
645	30		9		1			9		9			2
514	24										5	2	17
492	24									5		19	
425	26						1				5		20
650	32							1				15	16
632	27							6		9		2	10

TABLE 6e

Pl	n	$L_1 L_2$ B Th	$L_1 l_2$ B Th	$L_1 L_2$ B th	$L_1 l_2$ B th	L b Th	L b th	l B Th	l B th	l b Th	l b th I	l b th II	l b th
		1a	1b	2a	2b	3	4	5	6	7	8a	8b	8c
Beanyields of F_6-1937. Classification													
1065	23				12		4		1				6
1079	25	9	15		1								
279	29	4	12		7	1		4					1
277	28	9	4	2	5	1	2				3		2
332	24		10		3	2	6				1		2
328	26	1	13		1	2	3			3		1	2
349	25											16	9
350	19				4		1		1			1	12
1046	25			3	9		13 [1]						
1044	25			2	17	1	4						1
1039	16	7		5	1	1	2						
1045	18						7				11		
289	25	17		7			1						
292	25	19	2	2	1	1							
323	29	2	1	5	13	1	5		2				
317	19	6	3	2	8								
347	27						1			7	2	4	13
348	30					3	1			1		8	17
325	27					1				21	1		4
832	25							5		9		10	1
831	25		1				1	5	1	2	2	3	10
834	25									13		7	5
837	25		1		1			2		16	1	2	2
356	25						1	8	1	8			7
344	26									2	1	22	1
346	30									6		19	5
274	29				1		2			4	2	3	17
438	29							10	1	2	1	4	11
456	25									6		11	8

[1] One bean belongs to cl 4a, $L_1 L_2$ B th.

TABLE 7. Examples of beans of the various classes in 1936

Cl and form.	Pl	bean	L	B	Th	W	L B	L Th	B Th
1a, L_1 L_2 B Th	783	8 p 1 b	179	109	84	108	61	47	77
	784	2 p 5 b	163	101	76	87	62	47	75
1b, L_1 l_2 B Th	485	3 p 1 b	145	96	75	74	66	52	78
2a, L_1 L_2 B th	459	1 p 1 b	164	92	65	68	56	40	72
2b, L_1 l_2 B th	666	1 p 4 b	145	87	60	52	60	41	69
3a, L_1 L_2 b Th	1066	7 p 1 b [1]	159	85	66	53	53	42	78
3b, L_1 l_2 b Th	825	2 p 4 b	143	80	66	56	56	46	83
cl 4a, L_1 L_2 b th	161	2 p 5 b [2]	156	85	62	64	55	40	73
4b, L_1 l_2 b th	343	1 p 1 b	151	80	56	50	53	44	70
	517	2 p 3 b	143	84	62	59	52	43	74
5, l B Th	524	3 p 2 b	127	95	72	58	78	57	76
	1087	3 p 3 b	129	97	72	63	75	56	74
6, l B th	806	3 p 1 b	118	86	62	43	73	53	72
7, l b Th	652	3 p 3 b	116	82	66	48	71	57	81
	647	2 p 1 b	110	76	66	40	69	60	87
8a, l b th I	771	1 p 1 b	128	81	55	38	63	43	68
8b, l b th II	606	5 p 3 b	108	74	60	35	69	56	81
8c, l b th	650	8 p 2 b	116	76	57	36	66	49	75
8c, l b th	624	2 p 3 b	113	84	63	42	74	56	75

[1]) This is a bean of F_6-1937; [2]) a bean of F_4-1935.

TABLE 8. The beanyields of F_5-1936 that wholly (or almost wholly) correspond to beanyields of the I-line of 1936. The mean thicknesses and L Th-indices of these beanyields and for comparison those of the I-line of 1936

28 beanyields of F_5-1936				64 beanyields of the I-line of 1936			
th_m	n	per 64	L Th_m	n	per 64	th_m	L Th_m
4.9	0	0	37	0	0	1	2
5.0	1	2.3	38	0	0	1	4
5.1	1	2.3	39	0	0	2	13
5.2	1	2.3	40	2	4.6	4	22
5.3	1	2.3	41	5	11.4	5	33
5.4	1	2.3	42	13	29.4	5	52
5.5	1	2.3	43	19	43.4	8	59
5.6	3	6.9	44	21	48	12	62
5.7	3	6.9	45	25	57.1	18	63
5.8	5	11.4	46	25	57.1	28	64
5.9	11	25.1	47	25	57.1	48	64
6.0	13	29.4	48	27	61.7	47	64
6.1	17	38.8	49	27	61.7	53	64
6.2	20	45.7	50	27	61.7	57	64
6.3	25	57.1	56	28	64	63	64
6.4	26	59.4				64	
6.5	27	61.7				64	
6.7	28	64				64	

HISTOLOGY OF THE CORPORA LUTEA IN BLUE AND FIN WHALES OVARIES

BY

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Introduction.

One of the most important questions with which whale research is concerned is the problem of deciding the absolute age of a whale. MACKINTOSH and WHEELER (1929), WHEELER (1930) and LAURIE (1937) showed that the total number of old corpora lutea in both ovaries of a whale might be a clue, provided that the average number of ovulations in one year is known, as the old corpora lutea or corpora albicantia remain visible in the ovaries for a considerable time, probably for the rest of the whales' life. Now, the average number of ovulations annually can only be estimated very roughly and there is probably a great individual variation, depending among other things on the time of the year each individual has its first ovulation and the time of the year at which the animal becomes pregnant.

PETERS (1939) thought it possible to decide whether an old corpus luteum was one of pregnancy or one of ovulation only. The two kinds of old corpora lutea, according to this author, differed in macroscopical texture, colour, in amount of connective tissue, etc. If this is true, it would be possible to decide the number of corpora lutea of pregnancy in both ovaries of a whale, that is the number of pregnancies the animal has had. This idea induced us to study the histology of the corpora lutea and corpora albicantia in detail.

Materials and methods.

The ovaries of blue and fin whales (Balaenoptera musculus L. and Bal. physalus L.) were collected on board the Dutch floating factory "Willem Barendsz" during the Antarctic whaling seasons 1946 — 1949 by the biologists Dr E. J. SLIJPER, Dr W. VERVOORT, Mr W. G. BRAAMS and the author. The corpora lutea were weighed, measured and then slices were preserved in formalin 12 %. Small pieces of fresh material were also fixed in Helly and afterwards chromated in a mixture of chromalum and potassiumbichromate. Paraffin sections of Helly fixed

material were stained for mitochondria according to ALTMANN, whereas paraffin sections of the formol fixed material were stained with haemalum-eosin, haematoxylin-picrofuchsin (VAN GIESON), orcein, etc. The freezing microtome sections were treated with fat stains (Sudan III; Sudan Black, etc.).

The corpus luteum of pregnancy.

The corpus luteum of pregnancy is a conspicuous spherical body with a mean diameter of 20 cm and a mean weight of 2600 g in blue whales (maximum and minimum weight among 50 corpora lutea 7500 and 800 gram resp.) and a mean diameter of 14.3 cm and a mean weight of 930 gram in fin whales (with a maximum and minimum weight among 25 corpora lutea of 2800 and 400 gram respectively). It protudes far out of the ovary often connected with the latter only by a narrow stalk or hilus. The place where the follicle burst is indicated by a dimple with a small pit in the centre (MACKINTOSH and WHEELER). The outer layer of the yellow body consists of connective tissue from which centripetal strands penetrate into the pale yellowish brown luteal tissue. The centre often contains a cavity filled with a liquor and loose connective tissue from which centrifugal strands radiate.

The luteal tissue itself is mainly made up of two kinds of cells which according to their probable origin will be named granulosa and theca lutein cells (fig. 1).

The granulosa lutein cells are large oval cells (mean size 50 by 30 micra) with an eccentric nucleus (mean diameter 10 micra). The cytoplasm contains mostly in its centre a cloud of numerous small fuchsinophile granules and furthermore many long threadlike or shorter clavate mitochondria.

The granulosa lutein cells lie in radial strands or groups, surrounded by the theca lutein cells.

In the course of pregnancy some cytological changes occur which are probably connected with a functional change. Nothing is known however of the functional activity of whales' corpora lutea in the different months of pregnancy. MACKINTOSH and WHEELER report a slight decrease in weight of the corpora lutea during the later months of pregnancy but I do not think this is significant. If there is a decrease it can only be very slight and no conclusion can be drawn from it as to the functional activity of the gland. JACOBSEN (1941) determined the hormone content of blue whale corpora lutea but he did not mention the length of the foetuses. On the other hand the cytological pictures are not always reliable if one takes account of the difficult conditions under which the material is fixed and of the long time which often elapses between the death of the animal and the fixation of the gland.

MACKINTOSH and WHEELER have noted that the granulosa lutein

cells of young corpora lutea of whales contain many vacuoles. The same has been observed by HARRISON (1949) in the corpora lutea of the Ca'aing whale Globiocephala melaena. In our material I found one corpus luteum in which the large luteal cells were highly vacuolated. In this case no foetus could be found in the rather swollen uterus. Either the foetus was so small that it was overlooked or the corpus luteum was one of ovulation. The animal concerned was a blue whale caught on December 15th. The corpus luteum weighed 4250 grams, that is well above the average. In the other ante-partum corpora lutea only small vacuoles were found or vacuoles were entirely absent. The small acidophil granules stained slightly with Sudan Black but not with fat stains like Sudan III or scarlet red.

The cells of the interstitial tissue are supposed to be of thecal origin. HARRISON describes how thecal cells invade the granulosa tissue from the penetrating bloodvessels together with other adventitial cells and fibroblasts. They are mostly spindle-shaped and often contain clavate or clublike mitochondria. No collagenous nor elastic fibres are formed in this interstitial tissue, but the granulosa luteal cells become embedded in a network of reticulin fibres.

In the interstitial tissue immediately adjoining the granulosa lutein cells — probably in the intercellular spaces — small lipid droplets are often found: these droplets are birefringent, loosing their birefringence on heating up to approximately 60° C and regaining it when cooled. They show a rather weak SCHULTZE-LIEBERMANN reaction. It is uncertain whether these droplets of cholesterol have anything to do with the production of the steroid hormone(s) in the corpora lutea. No correlation was found between the amount of cholesterol droplets and the age of the corpora lutea (see CLAESSON and HILLARP 1947).

The bloodvessels are inconspicuous if compared with the bloodvessels in the ovarian stroma. The arterioles, venules and capillaries are unusually long. Nevertheless I cannot agree with the supposition put forward by DEMPSEY and WISLOCKI (1941) who investigated the corpus luteum of the humpback whale Megaptera nodosa, that the greater length of the capillaries by hampering the bloodflow in the gland would induce the central cavity. If this were true, all the corpora lutea of blue and fin whales, being much larger than those of the humpbacks, would have a central cavity, whereas only half the corpora lutea we investigated had any cavity at all.

The regression of the corpus luteum.

Immediately after parturition the corpus luteum begins to degenerate. No signs of regression have been found in corpora lutea of whales with large foetuses. In the corpora lutea of the lactating whales caught, degeneration had progressed so far already that one could speak of a corpus albicans: from the granulosa lutein cells only fat droplets remained,

the theca lutein cells were already quite strongly collagenized. Later regression follows at a much slower speed, taking probably 3 or 4 years before degeneration is completed.

The two kinds of lutein cells undergo a degeneration very different from each other.

The granulosa lutein cells become vacuolated, they obtain a "fumy" appearance. The vacuoles contain a fatty (sudanophile) substance. In the course of the degeneration the light yellow colour of the droplets changes into a yellowish brown; the substance can still be coloured with Sudan Black then but no longer with Sudan III or scarlet red. From some remaining capillaries there arise in the mean time endothelial networks in which many macrophages and lymphocytes appear. These macrophages take up the brown granules and after the disappearance of the networks the macrophages lay scattered between the large bloodvessels.

The theca lutein cells dedifferentiate already at the end of pregnancy and attain a fibroblastlike appearance whereafter they begin to form collagenous fibres. After the degeneration of the granulosa lutein cells they form the bulk of the remaining, slightly collagenous, yellowish brown "luteal tissue". This luteal tissue diminishes and at the end of the regression of the corpus luteum no remains of it can be found except for the macrophages mentioned above.

The connective tissue of an old corpus albicans is mainly derived from two sources: the connective tissue already present in the active corpus luteum and the adventitial cells accompanying the bloodvessels. Among the first are the tunica albuginea on the outside and the connective tissue nucleus or the tissue surrounding the central cavity on the inside. Both contain only few elastic fibres but many collagenous fibres. The connective tissue arising from the adventitial cells contains many elastic fibres.

The collagenous tissue does not play an important part in the regression of the corpus luteum. It is gradually reduced in size and only a few narrow strands of collagenous tissue remain between the bloodvessels. (fig. 2.) The tunica albuginea becomes a part of the ovarian stroma as the corpus albicans sinks beneath the general surface of the ovary.

The capillaries disappear for the greater part but a few of the larger ones remain. They become surrounded by smooth muscle cells and adventitial cells. DAWSON (1946) finds for the corpus luteum of the cat that the adventitial cells play a big part in the postpartum induration. He describes how these adventitial cells penetrate from their loci into the luteal tissue, replacing the old luteal cells and forming a rather faint collagenous stroma. In whales this role is played by the theca lutein cells whereas the adventitial cells remain all around the bloodvessels. In their further development the small bloodvessels acquire some resem-

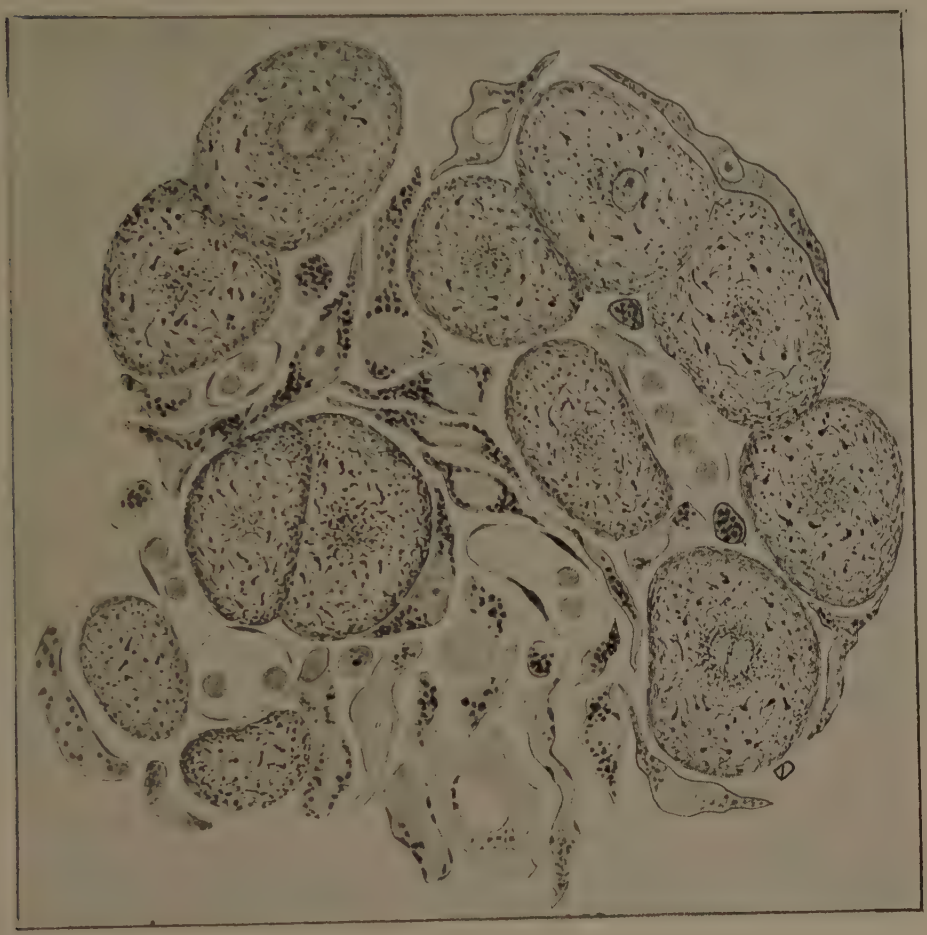

Fig. 1

EXPLANATION

Fig. 1. Balaenoptera physalus. Corpus luteum graviditatis. Large granulosa lutein cells with numerous threadlike or clavate mitochondria. Smaller spindleshaped or polymorphous theca lutein cells with more granular mitochondria. Note the large intercellular space. HELLY-ALTMANN. 600 ×.

Fig. 2. Balaenoptera physalus. Corpus albicans. Between the enlarged bloodvessels only narrow strands of connective tissue are left. Haemalun-eosin. 35 ×.

Fig. 3. Balaenoptera physalus. Corpus albicans. The twisted bloodvessels are surrounded by strands of luteal tissue containing numerous macrophages. Haemalum-eosin 35 ×.

Fig. 4. Balaenoptera musculus. Cross-section of a corpus albicans with a small central cavity and radial connective tissue strands. Natural size.

Fig. 5. Balaenoptera musculus. Cross-section of a corpus albicans of the branched type. To the left a few small follicles. Natural size.

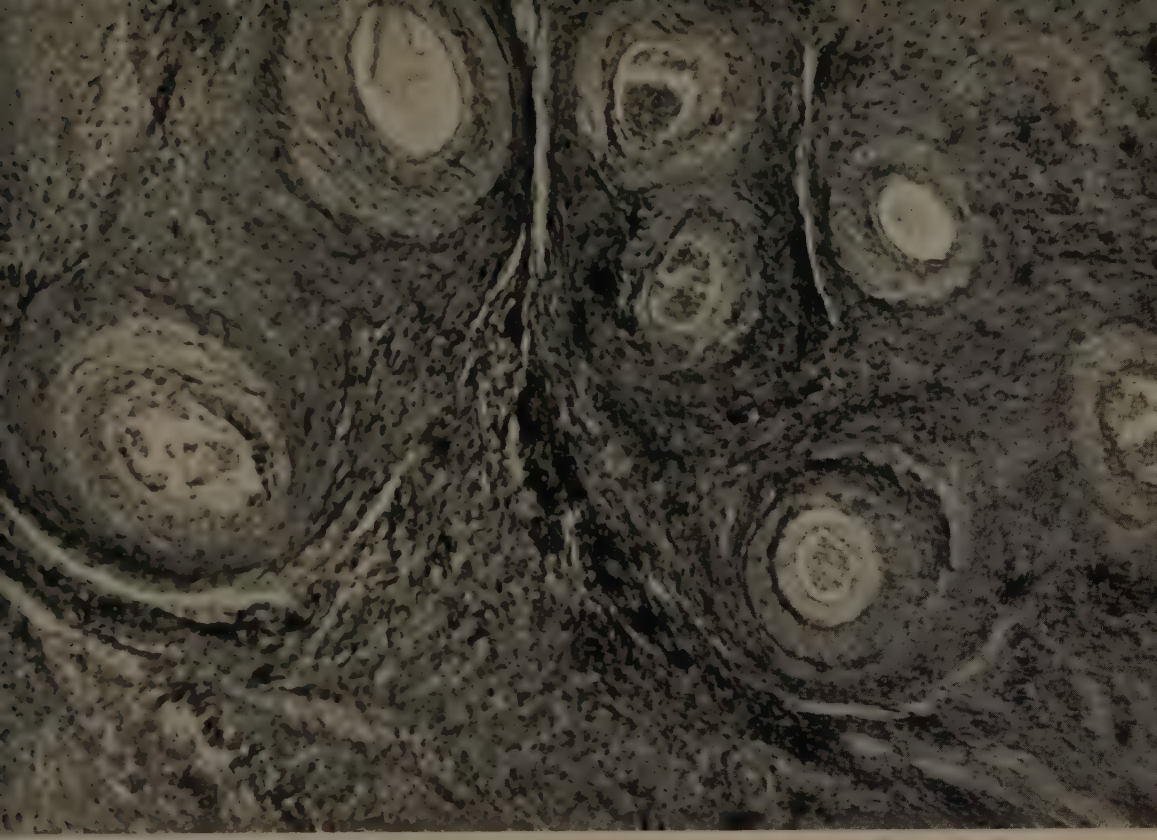

Fig. 2

Fig. 3

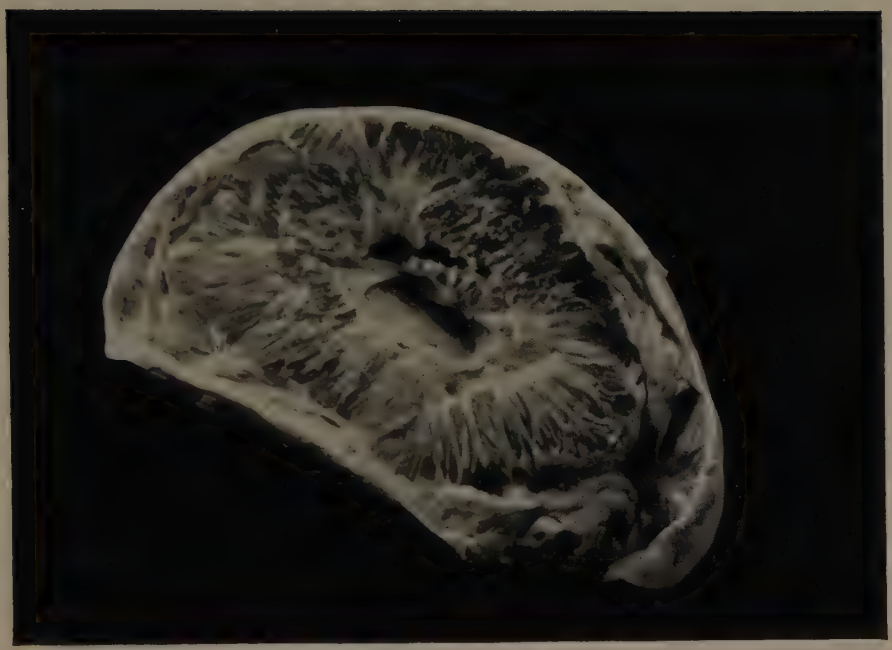

Fig. 4

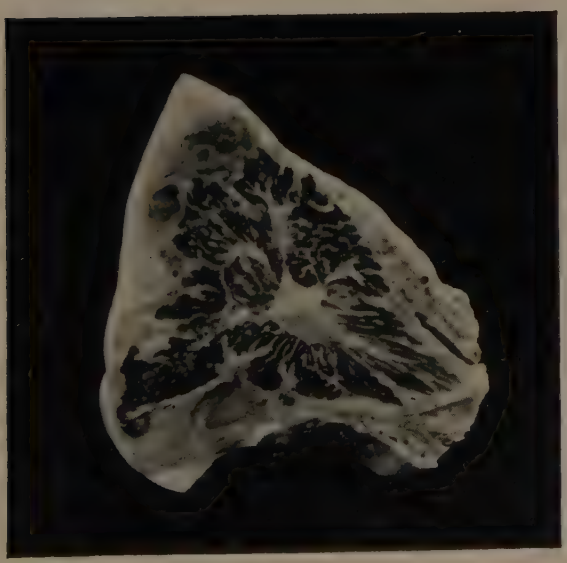

Fig. 5

blance to arterio-venous anastomoses which have been found in the ovaries of other mammals (MATHIS and EGLITIS 1936, WATZKA 1936 a.o.). They become twisted corkscrewlike (fig. 3), which development stands probably in connection with the reduction in size of the corpus albicans and the original length of the bloodvessel; the lumen is narrow and on the inside of the membrana elastica interna there is often a thick layer of smooth muscle cells, comparable with the "Gefäss-sperren" which WATZKA described.

The tunica media consists of a few muscle cells and collagenous tissue. In the first stages of development of the bloodvessels the tunica media is the most dominant of the three layers but in the end the external layer (membrana elastica externa) becomes far more important, so that a fully regressed corpus albicans consists almost only of elastic tissue.

Though it could not be proved owing to technical difficulties it is probable that the bloodvessels described are at least for a part true arterio-venous anastomoses as only very few capillaries remain in an old corpus albicans.

Discussion.

From the results of the histological study of the old corpora lutea no indication can be obtained that the old corpora lutea of pregnancy follow in their regression a pattern different from that of the corpora lutea of ovulation. The differences in texture PETERS described were certainly only differences in age of the corpora albicantia.

Yet another difference can be seen in young corpora albicantia: the trabeculae of connective tissue strands run in some corpora albicantia straight from periphery to centre (fig. 4); in others they are branched with often numerous ramifications (fig. 5). Most, if not all, corpora albicantia of the first type are probably derived from corpora lutea of ovulation as only young corpora lutea of pregnancy have been found to possess the same structure and may therefore be supposed to become corpora albicantia of the second (branched) type. This does not necessarily mean that all corpora albicantia of the branched type have been corpora lutea of pregnancy as corpora lutea of ovulation may very well be of a complicated structure but I had no opportunity to study corpora lutea of ovulation as the oestrous cycle takes place during the southern winter outside the season of antarctic pelagic whaling.

The mean diameter of a corpus albicans might give us an indication of its probable origin. According to MACKINTOSH and WHEELER the average diameter of corpora lutea of ovulation from blue and fin whales are 11 and 8 cm respectively and of corpora lutea of pregnancy 14 and 12 cm respectively. As corpora lutea of pregnancy contain more connective tissue than corpora lutea of ovulation the latter can be expected to be smaller as a rule than corpora lutea of pregnancy in the same stage of regression. Meanwhile the sizes of corpora lutea of ovulation and of

pregnancy overlap each other, so that the size of the corpora albicantia cannot be decisive.

The fact that notwithstanding careful sectioning of hundreds of ovaries no corpora albicantia smaller than $1\frac{1}{2}$ cm diameter have been found and that these small corpora albicantia consist practically only of the thick elastic walls of bloodvessels makes it highly probable that the corpora albicantia never completely disappear during a whale's lifetime.

The same pattern of regression of corpora lutea as described above was found in the ovary of a porpoise (Phocaena phocaena L.). Comrie and Adam (1938) described some corpora lutea of the false killer whale Pseudorca crassidens but they give no histological details.

A discussion of the probable average annual number of ovulations and of the rate of regression of the corpora lutea will be given in a later paper.

Summary.

1. A histological description has been given of the corpora lutea of pregnancy of blue and fin whales.

2. A detailed account is given of the process of regression of the corpora lutea. During the regression the luteal tissue disappears and the lutein pigment is taken up by macrophages. An old corpus albicans consists solely of the thick elastic walls of the twisted bloodvessels.

3. It is highly probable that old corpora albicantia remain visible as tough lumps, 1 to 2 cm in size, during the whole life of a blue or a fin whale.

4. Two possible means of distinguishing between old corpora lutea of pregnancy and old corpora lutea of ovulation are suggested.

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ON THE INFLUENCE OF DISTILLED WATER AND LITHIUM CHLORIDE UPON THE EGGS OF LIMNAEA STAGNALIS L. AT THE 2-CELL STAGE

BY

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RAVEN (1942) exposed eggs of *Limnaea stagnalis* to weak solutions of LiCl. Characteristic malformations were induced by this treatment: part of the eggs developed to exogastrulae, whereas other eggs yielded embryos with malformations in the head region.

RAVEN, KLOEK, KUIPER and DE JONG (1947) showed that a maximum of sensibility for the production of exogastrulae exists at the time of second cleavage. Lithium treatment immediately after shedding or in the 24-cell stage (RAVEN and RIJVEN 1948) produces mostly head malformations.

DE GROOT (1948) exposed decapsulated eggs immediately after shedding to higher concentrations of LiCl, in order to study the direct effects of this treatment. Hypertonic LiCl solutions caused disturbances of the nuclear cycle of maturation and fertilization, which lead to a standstill of development at different stages, dependent on concentration, stage of treatment, temperature and susceptibility of the eggs. In isotonic solutions, the nuclear phenomena, as far as could be observed, proceeded normally, but the cytoplasmic components of the egg showed interesting abnormalities; the distribution of the subcortical plasm was quitec abnormal. The animal pole plasm did not develop either in hypertoni or isotonic solutions.

RAVEN and ROBORGH (1949) treated decapsulated eggs with isotonic and hypotonic LiCl solutions. In these eggs, the telophase chromosomes of the first maturation division show a reversible swelling into karyomeres. At the same time the sperm nucleus swells and migrates towards the animal pole. The spermaster shows an increased hydration after the second maturation division; its disappearance is delayed. The amoeboid mobility of the eggs is considerably increased after the second maturation division. The first cleavage is delayed. Various abnormalities of the first cleavage mitosis occur. The results indicate that the primary action of the lithium ions consists in a change of the state of hydration of the protoplasmic colloids, causing changes of water equilibrium between various components of the egg.

Finally, treatment of decapsulated *Limnaea* eggs at the 24-cell stage

with LiCl solutions leads to an increase in the size of the nuclei and a decrease of their distance from the cell surface (RAVEN and DUDOK DE WIT 1949). Both phenomena are most pronounced in 0.05 % LiCl and diminish in intensity with increasing concentrations of LiCl, but they are also less pronounced in distilled water.

In the present investigation the direct effects of LiCl solutions at the 2-cell stage, when the eggs show a maximum sensibility for the production of exogastrulae, have been studied.

Material and Methods.

When the eggs of an egg-mass had developed to a 2-cell stage with beginning flattening of the blastomeres after their maximum rounding off (about stage 5 according to RAVEN 1946), the egg-mass was divided into 3 equal parts. The eggs of one part were decapsulated and transferred for 1, 2 or 3 hours to a 1.0 %, 0.4 % (hypertonic), 0.2 % (isotonic), 0.1 % or 0.05 % (hypotonic) LiCl solution. Those of the second part were likewise decapsulated and put for the same time in distilled water. Both groups of eggs were fixed immediately after treatment. The eggs of the third part remained in their capsules till the end of the experiment, then they were also decapsulated and fixed simultaneously with the other eggs.

41 Egg-masses have been treated in this way. 188 Li-treated eggs have been studied cytologically, together with 156 controls and 186 distilled-water eggs.

All eggs were fixed in BOUIN's fluid. The 7.5 μ thick sections were stained with iron haematoxylin and saffranin.

Results.

1. *Controls.* In the control eggs, which develop inside their capsules, after 1 hour the second cleavage has begun. On an average, they have reached a 4-cell stage with rounded blastomeres and reconstituting karyomere nuclei (stage 12 of RAVEN 1946). After 2 hours, they have developed to a stage just preceding 3d cleavage; as a rule, the cells contain metaphase or early anaphase spindles (stage 15). Finally, 3 hours after the beginning of the experiment, the controls mostly have reached an 8-cell stage with wide cleavage cavity and interphase or early prophase nuclei (stage 19). As a matter of course, development proceeds quite normally in these eggs; they show a regular cycle of formation and extrusion of fluid in the cleavage cavity, with well-developed secretion-cones, as described by RAVEN (1946). The animal and perinuclear plasm is quite dense, whereas the rest of the cytoplasm is moderately to strongly vacuolar, the vacuoles having arisen by swelling of the γ-granules (fig. 2, 4).

2. *Distilled water.* In most eggs kept in distilled water, after 1 hour the second cleavage has begun, but a fairly large proportion of them

is still in the 2-cell stage (41 % over against 16 % in the controls). On an average, they have reached the stage of just beginning 2nd cleavage (stage 10). Hence, it seems that their development is slightly retarded. In the 2-celled eggs, the flattening of the cells against each other is often incomplete and the cleavage cavity is lacking or incompletely developed.

After 2 hours, retardation of development is still more distinct. On an average, the eggs are in stage 13 (4-cell stage with polymorphic resting nuclei); numerous eggs have reached stage 14 (prophase nuclei), but in no case the spindles of 3d cleavage have been formed in these eggs, whereas this has occurred in 81 % of the controls. Moreover, it now becomes clear that in a certain proportion of these eggs (about 10 %) development is blocked in the 2-cell stage; in other cases, 2nd cleavage has taken place in one of the blastomeres only, so that 3-cell stages are formed. The flattening of the blastomeres is mostly incomplete, and the cleavage cavity is reduced or is lacking altogether. Vacuolisation of the cytoplasm is somewhat stronger than in the controls. The dense animal pole plasm is well-developed in these eggs.

After 3 hours, it is evident that all eggs which have passed the second cleavage are blocked in stage 13 — 14 (4-cell stage with interphase or prophase nuclei); in no case the spindles of third cleavage have been formed. The flattening of the blastomeres is incomplete, the cleavage cavity is lacking or rudimentary. The nuclei are often big and have a swollen and abnormal appearance. The vacuolisation of the cytoplasm is much stronger than in the controls; often, the structure of the cells has become quite abnormal by confluence of the vacuoles to irregular cavities, while the cytoplasm has been reduced to a mesh-work of slender strings between these cavities. In this condition the cell is on the verge of cytolysis, which has occurred, as a matter of fact, in some cells. The animal pole plasm and perinuclear cytoplasm do not take part in this vacuolisation and show a rather normal structure till the end.

3. *LiCl 0.05 %*. Development in 0.05 % LiCl corresponds, generally, to that in distilled water. Again, development is retarded as compared with the controls, and comes to a standstill at stage 13 — 14. Vacuolisation of the cytoplasm is strong, but in these eggs no confluence of the vacuoles with subsequent cytolysis has been observed after 3 hours. The nuclei are big and vesicular; in some cases, their swelling even exceeds that in distilled water. Flattening of the blastomeres against each other is stronger than in distilled water, but the cleavage cavity is often reduced or does not form at all. There is a well-developed animal pole plasm and the dense perinuclear protoplasm is very distinct and shows a concentric arrangement of granules.

4. *LiCl 0.1 %*. In this concentration, after 1 hour all eggs are still at the 2-cell stage. The nuclei are markedly big and swollen, the cytoplasm is moderately vacuolar. The cells show a nearly normal flattening and the cleavage cavity has often formed in a normal way.

After 2 hours, all eggs have passed through 2nd cleavage; most of them are in stage 13 — 14 (fig. 1), one (out of 11) has formed the spindles of 3d cleavage (stage 15). The eggs are much smaller than the controls and those in distilled water, but the nuclei correspond in size to those of the distilled-water eggs. The cytoplasm has a rather dense structure and is only weakly vacuolar. Flattening of the cells is good, but the cleavage cavity is often lacking (fig. 1).

After 3 hours, all eggs are still in stage 13 — 14; evidently, development is blocked at this stage. The nuclei are still big and swollen. The cytoplasm is moderately vacuolar. The perinuclear protoplasm is very dense and exhibits a conspicuous concentric arrangement of granules. In most eggs no cleavage cavity is present.

5. *LiCl 0.2 %*. In this LiCl solution, which is isotonic with the freshly-laid *Limnaea* egg, but, in consequence of the swelling of the eggs which has meanwhile taken place, presumably must be regarded as hypertonic to 2-celled eggs, after 1 hour all eggs are still at the 2-cell stage. Flattening of the cells has occurred and a lenticular cleavage cavity is present in most eggs. The cytoplasm is rather dense and only weakly vacuolar.

After 2 hours, in most eggs development has made no progress. The cytoplasm has become still more dense, but the nuclei are big and vesicular (fig. 3). In some eggs, big round sharply-delimited cavities have appeared in the cytoplasm. A few eggs show a beginning of 2nd cleavage, which is, however, quite abnormal, the blastomeres being very unequal in size.

After 3 hours, part of the eggs are still in the 2-cell stage, but in most eggs an abnormal second cleavage is taking place. All kinds of cleavage anomalies have been observed: unequal cleavage of both cells, in extreme cases leading to the splitting off of very small cells by both blastomeres; formation of non-nucleated cytoplasmic lobulations at the animal end of the cells, which may likewise be pinched off; cleavage of one blastomere only, so that 3-celled eggs are formed. All eggs possess a very dense deeply-staining cytoplasm, in which only few minute vacuoles have formed around γ-granules.

6. *LiCl 0.4 %*. In this strongly-hypertonic solution development is still more inhibited. No more cleavage takes place, even after 3 hours. The eggs are much reduced in size. The cytoplasm is very dense and contains only a few very small vacuoles around γ-granules. The nuclei, as a rule, have an abnormal appearance, and contain some heavily-staining chromatin masses in a clear karyolymph. The cleavage cavity between the cells is often very wide, while the cells are reduced to narrow crescents bordering this cavity. Apparently, a strong dehydration of the cytoplasm has taken place.

7. *LiCl 1 %*. The same phenomena, but still more pronounced, appear in 1 % LiCl solutions. Apart from flattening of the blastomeres

against each other and formation of a cleavage cavity between them, no further development has taken place. The cytoplasm is extremely dense and stains deeply with saffranin; no vacuoles are present. The nuclei in most cases are very abnormal, with condensed and clotted chromatin often forming a coarse network. They have a markedly shrivelled appearance. Dehydration of the cells is extreme; the cleavage cavity is often very wide and the cells are narrow crescents. In other cases, however, no cleavage cavity has been formed and the cells are small and more or less spherical.

Discussion.

1. *The inhibition of development by distilled water.*

Raven and Klomp (1946) showed that in eggs transferred to distilled water shortly after shedding the first cleavage nearly always takes place, although often somewhat delayed as compared with the controls. In some cases development comes to a standstill at the 2-cell stage; mostly, however, the 2nd cleavage also takes place. Development does not proceed beyond the 4-cell stage, even when the eggs are transferred to distilled water after the 1st or even the 2nd cleavage.

Raven (1949*b*) and Raven and Roborgh (1949) observed that eggs put in distilled water immediately after oviposition develop normally till 1st cleavage. Both the maturation divisions and the formation of a cleavage spindle take place in an entirely normal way; no differences with normal eggs developing in their capsules can be observed.

The first signs of a detrimental effect of distilled water on the *Limnaea* egg occur at the time of 2nd cleavage. A small part of the eggs are blocked at the prophase of this cleavage; in other eggs only one of the blastomeres divides. Most eggs cleave, however, without external signs of injury, but with some delay as compared with the controls. Reconstitution of the nuclei takes place, and in most cases the prophase stage of the next division is reached. None of the cells, however, is able to form the spindles of 3d cleavage. Soon, the first signs of impending cytolysis appear by the confluence of cytoplasmic vacuoles.

It may be concluded from these observations that the eggs are not very susceptible to distilled water at the uncleaved stage. Susceptibility increases, however, towards the end of the 2-cell stage, and reaches a maximum at the 4-cell stage. The changes provoked in the egg by this medium affect especially the mechanism of division, preventing the dissolution of the prophase nuclei and the formation of metaphase spindles. Whether the observed swelling of the nuclei is directly responsible for this effect, or the primary point of attack is on the cytoplasm, cannot be decided from these observations.

Cleavage in distilled water is abnormal; the flattening of the cells against each other and the formation of a cleavage cavity are prevented

or reduced (RAVEN and KLOMP 1946). One might think that this latter circumstance be responsible for the inhibition of development, since the extrusion of fluid from the egg through the cleavage cavity, as takes place in normal eggs (RAVEN 1946), is not possible. However, the addition of a small quantity of $CaCl_2$ to the medium leads to normal cleavage, but still development stops at the 4-cell stage; however, cytolysis is somewhat delayed as compared with eggs in pure distilled water (RAVEN and KLOMP 1946).

On the other hand, the block of development at the prophase of 3d cleavage may be broken by the addition of egg capsule fluid or of crushed eggs to the distilled water (HUDIG 1946).

The same holds true of 0.0004 M $CaCl_2$, 0.0016 M $MgCl_2$ and 0.0125 M NaCl and KCl solutions (GRASVELD 1949). These observations, and especially the quantitative difference between the effects of bivalent and monovalent kations, point to some specific action of the ions; this may be due to their influence on the water permeability of the egg cortex or the state of hydration of the cytoplasmic colloids.

2. *The effects of lithium.*

Contrary to NaCl and KCl, LiCl in no concentration with certainty breaks the block of development at the prophase of 3d cleavage. As a matter of fact, GRASVELD (1949) observed a few cases, in which the eggs in 0.2 % (0.05 M) LiCl developed to an 8-cell stage. We found among our material one egg that had formed the spindles of 3d cleavage in 0.1 % LiCl. These cases remain exceptions, however; in general, development in hypotonic LiCl solutions agrees with that in distilled water.

In 0.05 % LiCl, the state of the cytoplasm does not differ very much from that in distilled water. Vacuolisation is strong, as it is both in the controls and in distilled-water eggs. On the contrary, in 0.1 % LiCl the cytoplasm has a rather dense structure and is only weakly vacuolar; moreover, the cells are much smaller than those of the controls (fig. 1 and 2). This is still more pronounced in 0.2 % solutions, where the eggs possess a very dense deeply-staining cytoplasm with only a few minute vacuoles (fig. 3 and 4). Finally, in 0.4 % and 1.0 % LiCl the dehydration of the cytoplasm becomes extreme.

On the contrary, the nuclei have a swollen appearance not only in 0.05 % and 0.1 %, but also in 0.2 % LiCl solutions; in the former, in some cases they even exceed the nuclei of distilled-water eggs in size. Even in 0.4 % LiCl solutions, they appear not to be markedly smaller than in the controls; in 1 % solutions, however, they are small and shrivelled down. We may conclude, therefore, that lithium affects differently the cytoplasm and the nuclei. It provokes a swelling of the nuclei, which is only prevented by strongly hypertonic solutions. This corresponds to the observations of RAVEN and DUDOK DE WIT (1949) and RAVEN and ROBORGH (1949). On the other hand, lithium has a

dehydrating effect on the cytoplasm. The observation that this is obvious already in 0.1 % solutions, which are hypotonic and have an osmotic pressure which is, presumably, only slightly higher than that of the egg capsule fluid, indicates that it is no mere osmotic phenomenon. Moreover, we have compared the structure of eggs in 0.2 % (0.05 M) LiCl with that of eggs in isosmotic (0.1 M) urea and glucose solutions. Since these eggs did not come from the same egg-mass, and had not been fixed and stained simultaneously with the other ones, no absolute value may be attached to this comparison. The experiment has to be repeated with greater precautions. Still, the preliminary results indicate that the density of the cytoplasm in LiCl is markedly greater than that in isosmotic non-electrolyte solutions.

Hence, it appears that the conclusion drawn by Raven and Roborgh (1949) from their experiments viz. that the primary action of the lithium ions consists in a change of the state of hydration of the protoplasmic colloids, causing changes of water equilibrium between various components of the egg, is correct. Notably the nuclei and the cytoplasm react in a different manner to these ions.

Our results, therefore, support the views upheld by Ranzi (e.g. 1947*a* and *b*) according to which the influence of lithium upon embryonic determination in various animals depends on its tightening action upon filamentous proteins.

Eggs in 0.05 % and 0.1 % LiCl solutions agree with those in distilled water in possessing greatly swollen nuclei. The state of the cytoplasm, on the other hand, especially in 0.1 % LiCl, differs markedly from that in distilled water. It seems probable, therefore, that the block of development at the prophase of 3d cleavage, which occurs in all these eggs, may rather be connected with the excessive swelling of the nuclei than with the state of the cytoplasm. On the other hand, the delayed and atypical 2nd cleavage observed in 0.2 % LiCl, and the immediate inhibition of development in stronger LiCl solutions, may be due to the dehydration of the cytoplasm in these media.

According to de Groot (1948) and Raven and Roborgh (1949), eggs transferred to 0.2 % LiCl immediately after shedding show a rather normal development up to first cleavage. This cleavage occurs in most of the eggs, though with some delay; part of the eggs exhibit disturbances in the formation of the cleavage spindles, however. Also a second cleavage may take place, but, according to de Groot, nearly always a 3-cell stage is formed at this cleavage. No further development has been noted by this author. Hence, development comes to a final stop at the same stage, whether the eggs are transferred to the solution immediately after oviposition or not before the 2-cell stage. Evidently, the susceptibility to the direct effects of lithium chloride is low during the uncleaved stage, and increases markedly towards the end of the 2-cell stage. The same has been found above to be the case with regard to the action of distilled

water. Previous experiments have shown that the eggs at the time of 2nd cleavage are also highly susceptible against the action of urea (RAVEN and KLOMP 1946), thiourea (SOBELS 1948) and various chlorides (RAVEN and SIMONS 1948).

According to DE GROOT (1948), in eggs transferred to 0.2 % or stronger LiCl solutions immediately after shedding, the distribution of cytoplasmic substances is abnormal and the animal pole plasm does not form at all. In the eggs put in 0.05 % and 0.1 % LiCl at the 2-cell stage, the animal pole plasm and perinuclear plasm are well-developed and quite distinct even after 3 hours. In the eggs kept in 0.2 % or higher concentrations, these plasms are not clearly distinguishable on account of the great density of the whole egg; in many cases, however, even in these eggs the animal pole plasm is indicated by its somewhat darker staining. It seems, therefore, that lithium does not suppress these plasms once they are formed. The difference in density between these plasms and the rest of the cell, which is striking in normal eggs, is diminished by the general dehydration of the cytoplasm, however. Perhaps this attenuation of the density contrast between the animal and the vegetative side of the egg may bear a relation to the depression of the polar gradient field which is apparent from the morphogenetic results of lithium treatment (RAVEN 1949*a*).

Summary.

1. The direct effects of distilled water and lithium chloride solutions upon 2-celled eggs of *Limnaea stagnalis* have been studied.

2. Development in distilled water is blocked at the prophase of 3d cleavage.

3. In hypotonic (0.05 % and 0.1 %) LiCl solutions development is blocked at the same stage. In 0.2 % LiCl 2nd cleavage is abnormal. In hypertonic (0.4 %, 1.0 %) LiCl solutions development stops nearly immediately.

4. Lithium provokes a swelling of the nuclei, which is only prevented by strongly hypertonic solutions.

5. Lithium has a dehydrating effect on the cytoplasm.

6. The primary action of the lithium ion consists in a change of the state of hydration of the cytoplasmic colloids.

7. At the time of 2nd cleavage the eggs are very susceptible against various injurious agents.

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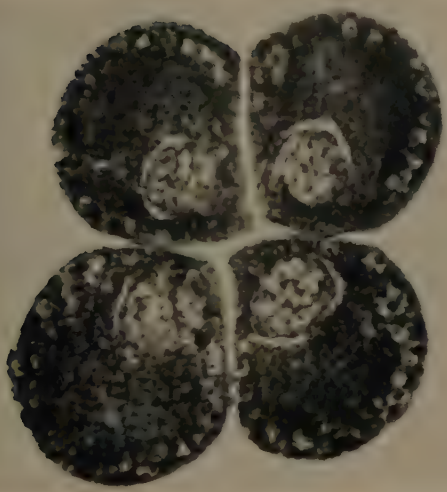

Fig. 1

Fig. 2

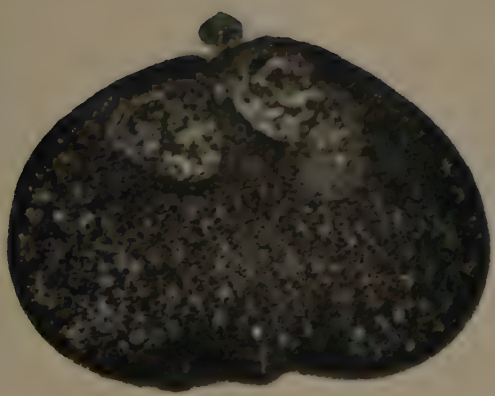

Fig. 3

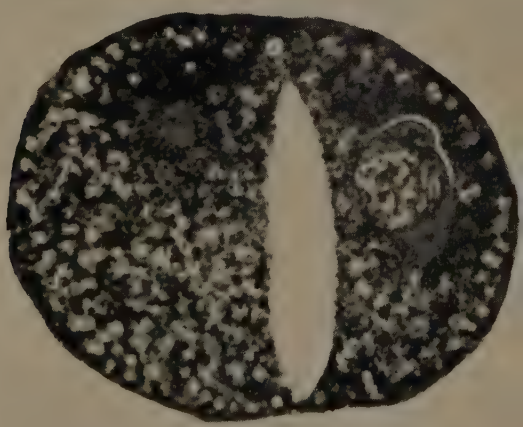

Fig. 4

Fig. 1. Egg of *Limnaea*, 2 hours in 0.1 % LiCl. Stage 14.

Fig. 2. Normal egg at the same stage.

Fig. 3. Egg of *Limnaea*, 2 hours in 0.2 % Li Cl. Stage 8.

Fig. 4. Normal egg at the same stage.

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RADIATIVE ELECTRON INTERACTION. II

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5. HIGH ORDER TRANSITIONS.

5. 1 *Building up.* The more-electron transitions can be built up from those dealt with in 4 by successively intercalating further electrons or electron pairs into the loops. If by the same arguments as in 4. 4 we disregard processes which knot the loops, the only effects of the new electrons are those of screening as in 4. 2 and exchange as in 4. 3.

The calculation of the summations over small regions and thin shells proceeds in the same way as in 4 and will not be repeated here. When successively integrating the small regions over the various thin shells one has to observe the order in which they are met with in circumscribing the loop.

The term in the transition probability corresponding to a certain loop is the product of a factor

$$\gamma_{\varrho\varrho'}\,|t_r - t'_r| \tag{42}$$

for the central transition $\varrho' \to \varrho$, a factor

$$dr\,\chi\,(r;|\xi_{\varrho\varrho'}|) \tag{43}$$

for each shell $r; dr$ with an upward transition, a factor

$$dr\,\chi\,(r;-|\xi_{\varrho\varrho'}|) \tag{44}$$

for each shell $r; dr$ with a downward transition and a factor

$$-dr\,\chi\,(r;|\xi_{\varrho\varrho'}|)+\chi\,(r;-|\xi_{\varrho\varrho'}|) \tag{45}$$

for each screening shell $r; dr$ with a virtual transition. The central transition occurs during the time interval $t'_r t_r$, all connected transitions occur in 4-dimensional time-space between the light cones through $(\vec{x}_r, t'_r)$ and $(\vec{x}_r, t_r)$, i.e. in 3-dimensional space successively from outside inwards (ingoing phase) until the central transition occurs and then outwards again (outgoing phase). The time lags between the various transitions correspond to a travel velocity c. A certain thin shell during the ingoing phase and the same shell during the outgoing phase can be treated as two different shells (in-shell and out-shell).

5. 2 *Atom in universe.* We consider the central atom r inside the spheric symmetric distribution of matter. This distribution can be divided into thin shells. Let r_0 be the radius of the sphere inside which there are no other atoms than the central r. The only thing we have to do is to account for the effect of transitions in all these shells on the central transition $\varrho' \to \varrho$ of r.

5. 21 *Unexcited surroundings.* We take a start by supposing for a moment that all the matter around r, but not the atom r itself, is in its ground state. Then none of the other electrons can make a down transition

$$\chi(r;\xi) = 0 \quad \text{for} \quad \xi < 0. \tag{46}$$

In this case exchange is excluded and only screening is left.

Take the central atom r and a shell $r_1; dr_1$. The probability of a (down) transition $\varrho' \to \varrho$ of r during $t_r' t_r$ together with an (up) transition in the shell is according to (42) and (43) or (25)

$$\gamma_{\varrho\varrho'}\, dr_1\, \chi(r_1; -\xi_{\varrho\varrho'})\, |t_r - t_r'|. \tag{47}$$

The screening effect of another shell $r_2; dr_2$ multiplies according to (45) or (35) this probability by a factor

$$\exp[-dr_2\, \chi(r_2; -\xi_{\varrho\varrho'})] \tag{48}$$

if $r_2 < r_1$ and leaves it unaffected if $r_2 > r_1$. Now if more-electron processes are taken into account every shell inside r_1 gives rise to an analogous factor, so that the total screening of all shells multiplies (47) by

$$\exp\left[-\int_{r_0}^{r_1} dr\, \chi(r; -\xi_{\varrho\varrho'})\right]. \tag{49}$$

The result can be written

$$-\gamma_{\varrho\varrho'}\, dr_1 \frac{\partial}{\partial r_1} \exp\left[-\int_{r_0}^{r_1} dr\, \chi(r; -\xi_{\varrho\varrho'})\right] |t_r - t_r'|. \tag{50}$$

The ultimate probability of a (down) transition $\varrho' \to \varrho$ during $t_r' t_r$ together with any (up) transition of any other electron under influence of all possible virtual transitions of all other electrons is found by summing (50) over all thin shells. This gives

$$\gamma_{\varrho\varrho'} \cdot \sigma(r_0; -\xi_{\varrho\varrho'})\, |t_r - t_r'|, \tag{51}$$

where

$$\sigma(r_0; -\xi_{\varrho\varrho'}) = 1 - \exp\left[-\int_{r_0}^{\infty} dr\, \chi(r; \xi_{\varrho\varrho'})\right]. \tag{52}$$

All virtual transitions as well as the final up transition occur in the outgoing phase.

If the integral in the exponent of (52) diverges, i.e. if the total oscillator strenght of the universe for the upward energy jump $-\xi_{\varrho\varrho'}$ is ∞, the universe will be said to be perfectly black for this energy jump. We shall

see in 6. 4 that we are forced to assume this to be the case for all up jumps, because otherwise our description is certainly inconsistent. With this assumption we obtain for (52)

$$\sigma(r_0; -\xi_{\varrho\varrho'}) = 1, \tag{53}$$

so that the transition probability (51) becomes

$$\gamma_{\varrho\varrho'} |t_r - t'_r|. \tag{54}$$

The probability (54) has to be equivalent to the spontaneous transition probability of r from ϱ' to ϱ during $t'_r t_r$ in ordinary radiation theory [7]). (54) appears $^3/_2$ times as large as the latter. This is as should be hoped for, because we have used the mutilated interaction (14) and therefore should compare with a mutilated radiation theory in which besides the 2 transverse photon states also the 1 longitudinal state takes a part in the radiation. That amounts to a factor $^3/_2$ indeed. If instead of using (14) we had been using the original (12), the screening (and exchange) effect of a thin shell would not have been independent of its radius and that would frightfully have complicated the calculations.

If the exponent in (52) would be treated as small and the expansion of the exponential into a power series in e cut off after the 2nd order term, the resulting expression would be that one, which has been rejected in 4. 15. In a black universe this 2nd order term instead of being small would diverge (and the higher order terms as well). The power series expansion is not appropriate to reveal that the remote atoms are sufficiently strongly screened off by the nearby ones, so that at large distances they have hardly an influence on r.

This converging effect of screening is also of importance in problems of vacuum polarization. That belongs to problem Q_3 put in 11. 42 and 21. 3.

5. 22 *Excited surroundings.* In unexcited surroundings the ingoing phase did not act a part. In excited surroundings it does, but we can still artificially separate the parts of the ingoing and outgoing phases.

5. 221 *Transition excess.* As the transition excess at a certain concentric sphere in the ingoing or outgoing phase we take the excess number of those preceding down minus up transitions (or those succeding up minus down transitions), which could be in resonance with the central transition (the latter inclusive). The transition excess has to correspond to the number of photons present in the state, which in the dualistic radiation theory is connected with the central transition.

In particular we shall need the transition excess n_{in} at the sphere r_0 in the ingoing phase and n_{out} at r_0 in the outgoing phase, i.e. just before and after the central transition. If the latter is downward we have $n_{out} = n_{in} + 1$, if it is upward $n_{in} = n_{out} + 1$.

5. 222 *Outgoing phase.* First we suppress all down transitions in the outgoing phase by provisionally putting in all out shells $\chi(r; \xi) = 0$

for $\xi < 0$, leaving it unchanged for $\xi > 0$. We leave the in transitions and the central transition as they were. Suppose among them are n_{out} more down transitions than up ones. Then there are just n_{out} up transitions in the provisional outgoing phase. The contribution of each up transition is just as in 5. 21 given by (52). The contribution of the n_{out} transitions is the n_{out}th power of (52). In a black universe this amounts to 1. There is an extra factor due to the combinatorics of different arrangements of the thin shells, which will be dealt with in 5. 3.

Now restore in the innermost out shell $r_0; dr_0$ the original $\chi(r_0; \xi)$ for $\xi < 0$. In each of the $2n_{out}$ links of the loop which cut the out shell this can intercalate a virtual or direct down transition. In the latter case a new up transition at a later time in a further outward shell has to be intercalated in the loop as well. For each of the links the transition probability becomes multiplied by a factor

$$\exp\left[-\tfrac{1}{2} dr_0\, \chi(r_0; -|\xi_{\varrho\varrho'}|)\right] \tag{55}$$

for the new virtual down transition and a factor

$$\exp\left[+\tfrac{1}{2} dr_0\, \chi(r_0; -|\xi_{\varrho\varrho'}|)\ \sigma(r_0 + dr_0; |\xi_{\varrho\varrho'}|)\right] \tag{56}$$

for the new direct down transition in the shell. The factor σ in the exponent accounts for the new up transitions outside the shell and the additional screening in the same way as has been found in 5. 21. In a black universe we have again $\sigma = 1$. In that case the product of (55) and (56) is 1, so that restoring the possibility of the down transitions in the innermost thin out shell has no effect on the preceding transitions.

The same argument can successively be applied to every next surrounding thin out shell, so that we finally see that it will suffice to consider only n_{out} up transitions in the outgoing phase and that the latter contributes to the transition probability a factor 1 also in excited surroundings, provided they are black.

5. 223 *Ingoing phase.* For the ingoing phase we follow a similar procedure by first suppressing the up transitions provisionally putting in all shells $\chi(r; \xi) = 0$ for $\xi > 0$, leaving it unchanged for $\xi < 0$. We leave the out transitions and the central transition as they were. If among them are n_{in} more up than down transitions, there are just n_{in} down transitions in the provisional ingoing phase.

A down transition in a shell $r_1; dr_1$ contributes in the probability a factor (44) multiplied by a screening factor, which in the provisional ingoing phase is

$$\exp\left[-\int_{r_0}^{r_1} dr\, \chi(r; -|\xi_{\varrho\varrho'}|)\right. \tag{57}$$

analogous to (49). Summing the product over all in shells $r_1; dr_1$ gives a factor

$$\sigma(r_0; -|\xi_{\varrho\varrho'}|) \tag{58}$$

analogous to (52). For n_{in} down transitions we get the n_{in}th power of (58).

In this case we shall not be forced to assume the divergence of the integral in the exponent for the energy jump $+|\xi_{\varrho\varrho'}|$. Still in our model we shall either voluntary do so, i.e. we assume the surrounding *Hohlraum* to be perfectly black excited, or we assume it not to be excited at all. That makes in (58) in the former case $\sigma = 1$, in the latter case $\sigma = 0$. The latter case has already been dealt with in 5. 21 and we now take $\sigma = 1$.

We successively restore the original $\chi(r; \xi)$ for $\xi > 0$, beginning with the innermost thin in shell and then every next surrounding one. By a reasoning analogous to that in 5. 222 we find that the new virtual and direct up transitions have no effect on the succeding transitions. Therefore it is sufficient to consider only n_{in} down transitions in the ingoing phase and the latter contributes to the transition probability a factor 1, provided the surrounding *Hohlraum* is perfectly black excited (or not excited at all).

5. 224 *Central transition.* Whereas the contributions of the ingoing and outgoing phases to the probability reduce to 1, the contribution of the central transition remains given by the factor (42). In order to find the total probability of the transition $\varrho' \to \varrho$ during $t'_r t_r$ under given conditions we have still to multiply the factor (42) by the number of possibilities in which the combinations of the connected transitions can be realized. That is a matter of combinatorics.

5. 3 *Combinatorics.* As such conditions we take fixed values of the transition excesses n_{in} and n_{out}. Let us call the largest of the two n_{max}. According to 5. 222 and 5. 223 we need only consider apart from the central transition n_{in} down transitions in the ingoing phase and n_{out} up transitions in the outgoing phase. Then for the n_{max} pairs of down and up transitions there are altogether $n_{max}!^2$ terms of the type of (37). Some of these terms correspond to disconnected loops. If from the latter we omit those loose parts, which do not contain the central transition, we just retain all those combinations in which the central transition is connected with $n_{in} + n_{out}$ or less other transitions.

Now in the integrations over all thin shells each combination of n_{in} in shells already is met with $n_{in}!$ times and each combination of n_{out} out shells $n_{out}!$ times. There remains only a factor n_{max} still to be accounted for. That finally gives for the total probability of the central transition

$$\underset{in}{n_{out}}\, \gamma_{\varrho\varrho'} |t_r - t'_r| = (\underset{out}{n_{in}} + 1)\, \gamma_{\varrho\varrho'} |t_r - t'_r|, \tag{59}$$

the upper case if the central transition is downward, the lower case if it is upward. (59) is entirely symmetric in time.

The expessions (59) have to be equivalent to the transition probabilities of r from ϱ' to ϱ in the presence of n_{in} photons (in the appropriate state)

before or n_{out} of them after this transition as it is given by ordinary radiation theory [7]). Just as (54), (59) is $^3/_2$ times too large due to the use of the mutilated interaction function (14).

5. 4 *Grain in Hohlraum.* One of the oversimplifications of our model is the spheric symmetry, which makes that an observer at the centre could not see any structure at all. In order to see how things would work in nonsymmetric surroundings we bring inside the sphere r_0 (i.e. inside our *Hohlraum*) a small region $d\tau_k$ of matter (grain) as in 4. 13. Although still highly oversimplified this model can show some features of what can be observed of the surroundings by means of radiative interaction. For this purpose we ask how under given conditions the transition probability for the central atom is influenced by the introduction of the grain.

Just as in 5. 1 the various thin shells have been found to contribute a factor to the transition probability also the small region $d\tau_k$ contributes a certain factor. The latter can be found from the derivations of 4 by not performing the integration of the last $d\tau_k$ over a thin shell $r_k; dr_k$. This factor is

$$\frac{d\tau_k}{4\pi x_{rk}^2}\,\chi\,(x_k; -|\xi_{\varrho\varrho'}|) \tag{60}$$

for a downward transition,

$$\frac{d\tau_k}{4\pi x_{rk}^2}\,\chi\,(x_k; |\xi_{\varrho\varrho'}|) \tag{61}$$

for an upward transition and

$$\frac{d\tau_k}{4\pi x_{rk}^2}\,\{\chi\,(x_k; -|\xi_{\varrho\varrho'}|) + \chi\,(x_k; |\xi_{\varrho\varrho'}|)\} \tag{62}$$

for a virtual transition in the region.

In order to count the number of possible combinations we have to specify the conditions which determine the transition probability. As such we take initial conditions, leaving the given value of n_{in} of 5. 3 unaffected by the presence of the grain. The factors contributed by combinatorics to the transition probability for the various cases are given in the following scheme, in which n stands for n_{in}.

(63)

$\varrho' \to \varrho$	$\varkappa' \to \varkappa$	down	up	virtual
down	in	$(n+2)\,(n+1)-(n+1)$	n^2	$-n\,(n+1)$
up	in	$(n+1)^2-n$	$n\,(n-1)$	$-n^2$
down	out	$(n+2)\,(n+1)-(n+1)$	$(n+1)^2$	$-(n+1)^2$
up	out	n^2-n	$n\,(n-1)$	$-n\,(n-1)$

All contributions from the outgoing phase cancel. The contribution from the ingoing phase results in an additional transition probability

$$\frac{d\tau_k}{4\pi x_{rk}^2}\{(n_{in}+1)\,\chi(x_k;-|\xi_{\varrho\varrho'}|)-n_{in}\,\chi(x_k;|\xi_{\varrho\varrho'}|)\,\gamma_{\varrho\varrho'}. \tag{64}$$

Thus the probability of the central transition is not altered at all by future transitions in the grain. It is increased or decreased by past transitions if respectively

$$\frac{\chi(x_k;-|\xi_{\varrho\varrho'}|)}{\chi(x_k;|\xi_{\varrho\varrho'}|)} > \text{ or } < \frac{n_{in}}{(n_{in}+1)}. \tag{65}$$

One would need a more perfect measuring instrument than our simplified central atom in order to detect the direction (and distance) of the grain. Such an instrument would record the past grain as respectively a brighter or darker spot against the background of the surrounding *Hohlraum*. It would not record the future grain at all.

5. 5 *Transition equilibrium.* Along similar lines as in the foregoing derivations one finds that if at one side of a thin shell the transition excess is n, the probability of finding at the other side $n+1$ or $n-1$ is respectively proportional to

$$\begin{cases}(n+1)\,dr\,\chi(r;-|\xi_{\varrho\varrho'}|),\\ n\,dr\,\chi(r;|\xi_{\varrho\varrho'}|).\end{cases} \tag{66}$$

In order to obtain statistical equilibrium the average number of up and down transitions in a shell has to be equal and the average transition excess $\bar{n}$ has to be the same everywhere. That gives the condition

$$\frac{\chi(r;-|\xi_{\varrho\varrho'}|)}{\chi(r;|\xi_{\varrho\varrho'}|)} = \frac{\bar{n}}{\bar{n}+1} \tag{67}$$

independent of r. If also the grain is in equilibrium it is invisible against the background according to (65). If also the central atom is in equilibrium we have (because $f_{\varrho\varrho'}=f_{\varrho'\varrho}$)

$$\frac{\chi(r;-\xi_{\varrho\varrho'})}{\chi(r;\xi_{\varrho\varrho'})} = \frac{1-z_\varrho}{z_\varrho}\cdot\frac{z_{\varrho'}}{1-z_{\varrho'}}. \tag{68}$$

For thermal equilibrium at a temperature $T°$ we have

$$\frac{\chi(r;-\xi_{\varrho\varrho'})}{\chi(r;\xi_{\varrho\varrho'})} = \exp\left[-\frac{\xi_{\varrho\varrho'}}{kT}\right]. \tag{69}$$

That gives with (67)

$$\bar{n} = \frac{1}{\exp\left[\frac{|\xi_{\varrho\varrho'}|}{kT}\right]-1}. \tag{70}$$

This is formally a *B-E* distribution. For thermal equilibrium of the central electron we find of course a *F-D* distribution.

6. DAMPING.

6. 1 *Probability conservation.* Dealing with the transitions $\varrho' \to \varrho$ of the central electron we have up till now only considered the increase of the occupation probability of the final state ϱ, but not the decrease of the occupation probability of the initial state ϱ'. The latter has to be in balance with the former in order to conserve the total probability 1 of finding the electron in any state at all. We shall now consider the latter effect, which can be denoted as damping. It is described by virtual transitions of the central electron.

6. 2 *Low order transitions.* 6. 21 *Self damping.* The simplest of all closed loops is the one, which directly connects the electron with itself (fig. 1*a*). It describes a virtual transition, which can be regarded as leading to self screening. Just as the ordinary screening of 4. 24 its effect will be seen to diminish the probability of the same process with the screening left out, i.e. in this case the process in which there is no change. Being a part of self interaction, self screening actually belongs to problem Q_3. Anyhow we shall here evaluate the probability of the process by a bit of juggling, the justification of which has to be left to Q_3.

For the total probabillity of virtual transitions out of and into the state ϱ' we would have to sum over all intermediate states. As we want to compare the balance with the transition from ϱ' to ϱ we only consider the term arising from the one intermediate state ϱ. The matrix element for this virtual transition is found from (19) by identifying the electron k with r and the states $\varkappa$ with ϱ' and $\varkappa'$ with ϱ.

First we identify the states. Because in (19) the downward transition occurs before the upward one and because in the virtual transition $\varrho' \to \varrho$ occurs before $\varrho \to \varrho'$, self damping is only effective for virtual down transitions ($\xi_{\varrho\varrho'} < 0$), not for up ones ($\xi_{\varrho\varrho'} > 0$). For the former transitions we get from (19) for $\lim (\xi_{\varrho\varrho'} + \xi_{\varkappa\varkappa'}) \to 0$

$$(71) \qquad \frac{ie^2}{\hbar} f_{\varrho\varrho'} \frac{\exp\left[\frac{i}{\hbar c} |\xi_{\varrho\varrho'}| x_{rk}\right]}{x_{rk}} |t_r - t_r'|.$$

This term interferes with the 0 order term $\delta_{\varrho\varrho'}$. That gives in the probability of the transition $\varrho' \to \varrho'$, which in 0 order is $z_{\varrho'}$, the 2nd order term in e

$$(72) \qquad -z_{\varrho'}(1-z_{\varrho}) \frac{e^2}{\hbar} f_{\varrho\varrho'} \frac{\exp\left[\frac{i}{\hbar c} |\xi_{\varrho\varrho'}| x_{rk}\right] - \exp\left[-\frac{i}{\hbar c} |\xi_{\varrho\varrho'}| x_{rk}\right]}{ix_{rk}} |t_r - t_r'|,$$

accounting for the occupation number according to 3. 3. The higher order terms are of higher order in $t_r - t_r'$ and can be neglected.

Then we identify the electrons altogether. For (72) we get for $\lim x_{rk} \to 0$

$$(73) \qquad -\gamma_{\varrho\varrho'} |t_r - t_r'|.$$

By no more than a juggling trick with the limiting processes we have obtained a finite result for the self damping. This is analogous to the finite result which a classical theory of a point electron gives for radiation damping [8]). For our present model the self interaction term in the instantaneous COULOMB interaction would be infinite just as in the classical theory.

Without a consistent picture of self interaction it has no sense to try to represent the various obscure manipulations in a more acceptable form.

6. 22 *Mutual screening.* Because the nearest neighbours of the central atom r are supposed to be at a finite distance, there can in a 4-dimensional transition diagram be no closed loop connecting the virtual transitions of r and the other electrons. Therefore after summing over extended surroundings effects of screening only will be cancelled by interference.

6. 23 *Exchange damping.* Finally there are the closed loops which link together down and up transitions of other electrons and virtual transitions of r and possibly also of other electrons. The corresponding terms of the matrix elements interfere with those corresponding to the loops in which the virtual transition of r has been left out. From the standpoint of the other electrons the virtual transitions of r act as screening, which diminishes their transition probability. From the standpoint of r the paired transitions of other electrons give rise to exchange, possibly screened by virtual transitions of still other electrons, which enlarges the effect of self damping.

For a 3-electron process with a virtual transition of r and a paired transition of k and m the matrix element is given by (29) in which m has to be interchanged with r after which ϱ has to be replaced by ϱ' and ϱ'' by ϱ. The summing over small regions and thin shells proceeds in the same way as in 4.

6. 3 *High order processes.* The extension to more electron processes and the summing over the universe proceeds in the same way as in 5. The contributions to the transition probability due to a thin shell are again given by (43), (44) and (45), the contribution due to the central virtual transition by minus (42).

For a virtual transition of r we have $n_{in} = n_{out}$. If this transition excess has a given value n, we find in a similar way as in 5. 3 that the total probability of a virtual transition from ϱ' to ϱ and back again during $t_r' t_r$ adds to the probability $z_{\varrho'}$ of remaining in the initial state ϱ' a term

$$-\left\{\begin{matrix}(n+1)\\ n\end{matrix}\right\} \gamma_{\varrho\varrho'} |t_r - t_r'| \text{ for } \xi_{\varrho\varrho'} \lesseqgtr 0. \tag{74}$$

The term 1 is due to the self damping (73), the terms n to exchange damping.

The probability terms (74) for remaining in the state ϱ' are just in balance with the probability terms (59) for jumping to the state ϱ.

6. 4 *Blackness conditions.* In unexcited surroundings ($n_{in} = 0$) the derivation of (74) is independent of the oscillator distribution in the universe, but (59) only holds for perfectly black surroundings. Therefore we are forced to assume perfect blackness in order to establish probability conservation without which the description cannot be consistent.

The terms n_{in} in (59) and (74) are derived under the same assumptions about the excited universe. They would even be in balance for not perfectly black excited surroundings.

7. THE TWO ASPECTS.

Even if the transition equations (6) as far as they can be accepted at all had not been prepared in 2 by elimination of the photons from the dualistic theory, the derivations in the present 3 might have challenged at almost every stage the (re)introduction of an electromagnetic field (photons). Yet we have to resist this temptation just for another while in order to confront the two pictures of the radiative processes R_1 and R_2 of 1. 2.

7. 1 *Radiative processes.* 7. 11 *Emission and absorption.* The processes which in dualistic radiation theory are emission and absorption of photons are in the unitary picture described by the transition processes which just have been discussed at lenght. More complicated radiative processes as e.g. the emission of more than 1 photons in a single electron transition [9]) can also be treated in the unitary way. The two descriptions give the same electron transitions with the same probabilities, i.e. the same observable results. Only the interpretation looks quite different.

7. 111 *Sponataneous and induced processes.* Dualistic theory refers to initial conditions. The terms n_{in} in (59) and (74) are ascribed to emission and absorption processes induced by the n_{in} photons present just before. The term 1 is ascribed to spontaneous emission processes (occasionally interpreted as induced by the vacuum fluctuations of the photon field). In the unitary description one cannot stick to initial conditions. Every central transition is connected with other earlier or later transitions. They induce each other and none of them appears as spontaneous. The terms n_{in} in (59) and (74) describe central transitions which are connected with other earlier as well as later transitions. They arise from exchange. The term 1 in (59) describes central transitions which are connected with no earlier and 1 later transition. They arise from elementary paired transitions. The balance term in (74) is not connected with any other transitions at all. It arises from self damping. From the unitary standpoint the dualistic theory in looking to initial conditions only is liable to miss part of the connections and that is how it is lead to consider those transitions which are not connected with earlier ones as spontaneous.

7. 112 *Statistics.* In dualistic theory the factors n and $n + 1$ in (59) and (74) arise from the *B-E* properties of the photons in the same way as the factors z and $1 - z$ in 3. 3 arise from the *F-D* properties of the electrons. In the unitary description they appear as a result of combinatorics counting the various exchange connections of earlier down transitions and later up transitions into closed loops. This number is equal to the transition excess. Just as the indiscernible photons in dualistic theory the transition excess in the unitary picture follows in thermal equilibrium a BOSE-EINSTEIN distribution (70).

It should be noted that the statistical properties of radiation interaction are independent of the *F-D* properties of the electrons. They would hold equally well for radiative interaction between *B-E* charges.

7. 12 *Interference and diffraction.* The interaction matrix (14) shows after the integration over ξ_0 in (15) a periodic wave character, which can be attributed to beating of the initial and final electron waves. This wave character is retained in all matrix elements of the type (19), (29) and (36) and leads for connections along chains with different path lengths to interference effects such as we have met with in 4. 2 and 4. 3. In principle all interference and diffraction effects could (perhaps extremely prolixly) be described in the unitary way.

The screening and exchange effects in the unitary picture correspond to coherent scattering and coherent induced emission and absorption of photons in the dualistic radiation theory.

7. 2 *Conditions.* The relativistic invariance of the unitary discription has already been considered in 1. 31. As further conditions we have imposed in 21. 4 and 23. 3 the opaqueness condition and the asymmetry condition.

7. 21 *Opaqueness condition.* When in dualistic radiation theory we have in our model a single photon in an unexcited universe the opaqueness condition, which requires that it must with certainty be absorbed by any other electron at a later time, amounts to just the blackness condition of the unitary picture as it has been imposed in 5. 21 and found necessary in 6. 4. The condition that a photon must also have been emitted by a electron at an earlier time can be fulfilled either by voluntary assuming the universe to be perfectly black excited as in 5. 223 or by explicitly excluding incoming photon fields.

7. 22 *Asymmetry condition.* All our equations of motion are symmetric in time. They remain invariant if we reverse the order of time and at the same time interchange initial and final conditions and up and down transitions. There is only a peculiar asymmetry in the combination of the sense of two paired energy jumps and the order of time in which they occur in the + interaction of 2. 4 (fig. 3, where the higher energy states are represented by thicker world lines). This links together on the

one hand the asymmetry of up and down transitions, which arises from the asymmetry in the distribution of the electrons over the energy states, which have a ground level but no top level, and on the other hand the asymmetry in the temporal boundary conditions, which refer (as far as possible) to initial and not to final conditions. In this way the asymmetry in the energy conditions is transfered to the time conditions in such a way as to produce the asymmetric course of observed processes [10]).

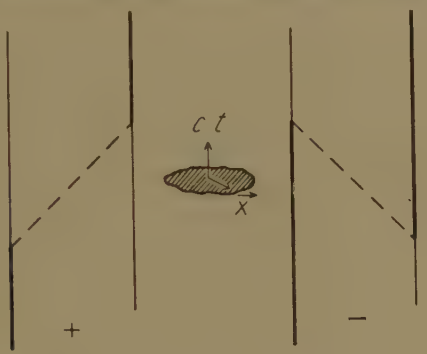

Fig. 3. + and — interaction.

In 6. 4 we have found that the surrounding universe has to be assumed perfectly black, but that the excitation may be arbitrary. We have shown in 5. 4 that an observer, though being in radiative interaction with past as well as future events, only observes past and not future ones. (A metaphoric central electron wanting to jump can trust that later down transitions will occur and does not need to know where and when, but it cannot be sure about earlier transitions without knowing about them).

The typical asymmetry of the radiative + interaction (fig. 3) corresponds to the dualistic feature that photons only occur with positive energy. This is a consequence of the *Hd* revision of 22. 151 by which creation and annihilation of photons in negative energy states is replaced by annihilation and creation in positive energy states. Another symbolic explanation of this situation is that the revision makes the negative energy photons travel backward in time instead of forward like positive energy photons are kept doing [11]).

7. 23 *Time conditions.* The unpleasant conditions of infinite time intervals introduced in 23. 1 have turned out to be a bit less worse than expected, because the form (15) of the interaction reduces them to intervals of the same order as the finite interval $t'_r t_r$ of the selected electron. The initial and final conditions (occupation numbers) can therefore be taken at the beginning and at the end of these finite intervals instead of at $-\infty$ and $+\infty$. This is very pleasant because during the infinite time intervals they might have entirely changed owing to other processes. For more general interaction functions these pleasant features would more or less be lost.

The results derived for infinitesimal short time intervals $t'_\tau t_\tau$ can readily by integrated for finite intervals of time.

7. 3 *Photons as radiative carriers.* 7. 31 *Dovetailing photons.* It would be tedious but instructive to repeat the unitary description of the radiative processes in our model in dualistic language. One would work with photons emitted during down transitions and absorbed during up transitions of the electrons. At any stage the integration over all intermediate photon states would by the dovetailing process of 22. 183 lead to the unitary description again. In 24. 32 slight objections have been raised against dovetailing the effective dualistic equations of motion. Because in our model the COULOMB interaction is treated in excactly the same way in the two descriptions these objections are not in force and the dovetailing can be restricted to the (mutilated) radiative part.

No matter at which stage the integration is performed the treatment of any particular electron starts just before and stops just after it performs a transition in one of the loops connected with the central transition. The description makes it easy by not bothering about the earlier and later adventures of the electron. But still it has to treat all electrons in the universe. The usual dualistic description, which is not the true translation of the unitary one, makes it still easier. In describing the central transition it only deals with the central electron and the photons and it starts just before and stops just after this central transition without bothering about the adventures of the electron and the photons at earlier and later times and about the adventures of the other electrons anyhow. All this insouciance can be tolerated because the disregarded processes have no influence on the probability of the central transition. In agreement with the greater insouciance in the dualistic description it turns out that also in the unitary description the transitions of the other electrons in the ingoing and outgoing phase contribute a factor 1 to the central transition probability, but that can only be seen after a laborious calculation.

7. 32 *Cutting links.* From the standpoint of the dualistic description the consequential procedure of the unitary description appears as a cheap trick giving simple matters a sciolistic intricateness. Therefore becoming entangled in the unitary transition loops it seems advisable to cut the links and to describe them by photons created and annihilated during the electron transitions. More precisely each link, which originates in the beating of an initial and final electron wave, corresponds to a single photon wave. The $2n$ links of a loop correspond to the quadratic expression in n photon waves for the transition probability. The number of ways in which the cut links, which are unlabeled, can be connected (combinatorics) is accounted for by the factors $n^{1/2}$ in the photon creation and annihilation operators according to S_d in 12. 34 (statistics). The factorization of the radiative interaction which is thus brought about by the

introduction of photons and which is the opposite of their eliminating by the dovetailing process of 7. 31 entails a tremendous simplification indeed (boundary conditions, conservation relations, practical applications, etc.).

7. 4 *Generalizations.* 7.41 *Transverse interaction.* In case the relation between the dualistic and unitary descriptions is clear for the mutilated interaction, it is also clear that the same thing can in principle be done for the supposedly correct transverse interaction. Only the more intricate angular dependence of the latter interaction leads to complications in such a degree that it seemed better to restrict the demonstrations to the academic model of mutilated interaction.

7. 42 *Other carrier particles.* Similar arguments apply to interaction by means of other carrier particles, e.g. mesons, provided they obey *B-E* statistics. The latter restriction is required by the relation which has been found to exist between statistics in dualistic description and combinatorics in unitary description. Mesonic interaction e.g. can in principle be treated almost along the same lines as the electrodynamic interaction we have been dealing with and which as has been seen in 23. 42 is a degenerate special case.

7. 5 *Balance.* Having returned after a long course of laborious derivations to the original starting point of the intimate dualistic theory, it seems justified to ask whether all our toiling has put any more weight into the balance of 21. 23 than the warning never to intermeddle with unitary source theory.

7. 51 *Other outlook.* Maybe the foregoing derivations have thrown a light on some aspects of radiative processes, which are not always seen by everyone in common dualistic radiation theory. Even if the unitary description could ultimately not be accepted at all and so the light had been false, the revealed aspect might still have some significance.

7. 52 *Untranslatable developments.* The importance of the possibility of further developments in the unitary description which cannot be translated into the dualistic description is mainly a matter of problem Q_3.

7. 6 *Other work.* The main content of the present paper 3 had been written before our papers 1 and 2. In the meantime there has appeared various other work, which is concerned with the same subject and which in general has not been duly digested even in the present version.

The classical theory has extensively been discussed by WHEELER and FEYNMAN [10]) [12]). Their work has in particular influenced our discussion of irreversibility. A description of radiative processes in a *Hohlraum* has been given by PLASS [13]), of which from our point of view the classical part counting the degrees of freedom is of interest.

There is much more similarity between our derivations and part of the recent work of FEYNMAN, than merely the reversed order in which papers have been written. In fact FEYNMAN's contributions to the recent developments in quantum electrodynamics [11] [14]) already essentially contain the ideas and equations of motion, which we have worked out in 3 for problem Q_2. FEYNMAN's applications mainly deal with problem Q_3. His general formalism corresponding to our (tentative) derivations in 2 has not yet been published.

Summary.

The radiative processes which in dualistic radiation theory are emission and absorption of photons are in a unitary charge picture described by all-electron processes composed out of elementary paired transitions. In the latter the down transition of the one electron occurs before the up transition of the other one. For composed more-electron processes also the effects of screening, which keeps the expressions convergent, and exchange, which replaces the part of multiple occupied photon states, appear. The statistical properties of the photons are accounted for by combinatorics of various linkings between down and up electron transitions. The wave character of photons (interference and diffraction) appears as beating of the electron waves before and after the transitions. The surrounding universe has to be assumed perfectly black, so that no photons could escape. Incoming photons are not admitted. The irreversible aspects of radiative processes are accounted for by a typical linking between asymmetrical energy and time conditions.

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REMARQUES SUR LES MÉTRIQUES NON-ARCHIMÉDIENNES. II

PAR

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Théorème 11. *Un espace uniforme muni d'une structure uniforme non-archimédienne et de la topologie déduite de cette structure est 0-dimensionnel.*

Démonstration. Soit $\mathcal{F}$ le système des entourages définiant la structure uniforme de l'espace E; par hypothèse ces ensembles sont symétriques et ouverts (dans $E \times E$). Soit $a \in E$ et considérons un entourage $U \in \mathcal{F}$. Alors $U(a)$ est un voisinage ouvert de a; $b \in U(a)$ si $(a, b) \in U$. Nous allons montrer que $U(a)$ est aussi fermé. Soit donc $x \neq a$ un point d'accumulation de $U(a)$. En vertu des suppositions concernant E il existe un voisinage ω de x ne contenant pas a; ω contient des points de $U(a)$. Puisque le système des voisinages, déduit de $\mathcal{F}$ est complet, il existe un $V \in \mathcal{F}$ tel que

$$V(x) \subset \omega.$$

$V(x)$ contient des points de $U(a)$ et a n'appartient pas à $V(x)$. En vertu de la deuxième condition imposée à $\mathcal{F}$ il existe un $W \in \mathcal{F}$ tel que

$$W \subset U \cap V.$$

Donc $W \subset V$ et

$$W(x) \subset V(x).$$

Montrons qu'on a

$$W(x) \subset U(a).$$

$W(x)$ contient des points de $U(a)$; soit $b \in E$ un point commun de ces voisinages. Donc

$$(x, b) \in W, \ (a, b) \in U.$$

Par suite et puisque les entourages sont symétriques

$$(x, a) \in U\ W.$$

Soit alors $y \in W(x)$ donc $(x, y) \in W$. Il s'ensuit

$$(a, y) \in U\ W\ W = U\ \overset{2}{W} \subset U\ W \subset \overset{2}{U} \subset U,$$

donc

$$y \in U(a)$$

On a donc

$$W(x) \subset U(a)$$

de sorte que x est un point intérieur de $U(a)$ et appartient donc à $U(a)$. Par suite $U(a)$ est fermé. Chaque point de E a donc des voisinages arbitrairement petit ouvert et fermé de sorte que E est 0-dimensionnel.

On sait que chaque espace uniforme est complètement régulier. Cette propriété est presque triviale si la structure uniforme est non-archimédienne, ce qui implique, d'après le théorème précédent, que l'espace est 0-dimensionnel. Soit en effet x_0 un point et $U(x_0)$ un voisinage de x_0, que nous pouvons supposer ouvert et fermé puisque l'espace est 0-dimensionnel. Le complément $C\,U(x_0)$ de $U(x_0)$ par rapport à l'espace est alors aussi ouvert et fermé. La fonction $f(x)$, étant 0 si $x \in U(x_0)$ et 1 si $x \in C\,U(x_0)$ est évidemment continue, d'où il résulte que l'espace est complètement régulier. Remarquons qu'on n'a pas utilisé l'uniformité de l'espace; le raisonnement s'applique dans chaque espace 0-dimensionnel. Faisons encore la remarque suivante, essentiel pour tout ce qui suit. La fonction continue, indiquée ci-dessus, ne prend que les valeurs 0 et 1. Elle peut donc être considérée comme une fonction continue, prenant ses valeurs dans un corps K muni d'une valuation non-archimédienne, donc dans un espace muni d'une métrique non-archimédienne. On peut donc dire: sur chaque espace 0-dimensionnel il existent des fonctions continues, non partout constante, prenant ses valeurs dans un corps K muni d'une valuation non-archimédienne. C'est une propriété remarquable: puisque l'image continue d'un espace connexe est connexe, elle n'est pas vraie par exemple dans les espaces connexes, tout corps valué de façon non-archimédienne étant totalement discontinu. [13])

En ce qui concerne la structure des espaces uniformes munis d'une structure non-archimédienne on a le théorème suivant, modification d'un théorème connu. [14])

Théorème 12. *Soit E un espace uniforme et supposons que la structure uniforme, déterminée par le système des entourages F, est non-archimédienne. Alors E est isomorphe (au sens de la structure uniforme) et donc homéomorphe à un sous-ensemble d'un produit d'espaces métriques, la métrique dans chacuns de ces espaces étant non-archimédienne.*

Démonstration. La démonstration est analogue à celle de M. A. Weil, qui, cependant, ne suppose pas que la structure est non-archimédienne et qui par suite ne peut tirer aucune conclusion concernant le caractère de la métrique des espaces métriques. Cependant il est utile de reprendre la démonstration.

D'abord il faut modifier la définition de la fonction continue sur E, ne prenant que les valeurs 0 et 1, indiquée ci-dessus. Soit U un entourage de la diagonale D dans E^2. Définions comme il suit une fonction $F(x, y)$ sur E^2

$$\left.\begin{aligned} F(x,y) &= 0 \text{ si } (x,y) \in U \\ F(x,y) &= 1 \text{ si } (x,y) \in C\,U \end{aligned}\right\}$$

[13]) Ils existent des espaces totalement discontinu avec dimension > 0. La question s'il est permis de remplacer ici les mots „espaces connexes" par: espaces avec dimension > 0, reste ouverte.

[14]) Voir Weil l.c. 10, p. 15.

Cette fonction est une fonction uniformément continue de (x, y) dans E^2. Soit pour cela (x, y) un point de E^2 et supposons $(x, y) \in U$. Considérons un point $(z, u) \in E^2$ tel que $(x, z) \in U$, $(y, u) \in U$, de sorte que z appartient au voisinage $U(x)$ de x et u au voisinage $U(y)$ de y. D'après la définition de $\overset{\circ}{U}$, les entourages étant supposés symétriques, les relations $(x, y) \in U$ et $(x, z) \in U$ entraînent $(y, z) \in \overset{\circ}{U} \subset U$. Ensuite les relations $(z, y) \in U$ et $(y, u) \in U$ entraînent $(z, u) \in U$. On en tire $F(z, u) = 0$, tandis que aussi $F(x, y) = 0$. Un raisonnement analogue s'applique si (x, y) n'appartient pas à U. Il en résulte la continuité désirée.

Nous considérons $F(x, y)$ comme une fonction prenant ses valeurs dans un corps K muni d'une valuation non-archimédienne.

Soit alors U_λ un système d'entourages de D dans E^2 tel que chaque entourage de D de la structure F contient un entourage appartenant au système des U_λ. En général, ce système ne sera pas dénombrable; le cas où l'on peut choisir les U_λ tels que le système des U_λ est dénombrable, sera considéré plus tard. Pour chaque U_λ construisons une fonction continue $F_\lambda(x, y)$ de la façon indiquée ci-dessus. Posons avec M. Weil

$$F_\lambda(x, y) = \varphi_x^{(\lambda)}(y).$$

Soit, pour chaque valeur de λ, $\mathfrak{E}_\lambda$ l'espace dont les points sont les fonctions continues bornées sur E, prenant ses valeurs dans le corps K. Ainsi que nous avons vu, ils existent de telles fonctions non partout constantes. Métrisons cet espace comme il suit: f et g appartenant à $\mathfrak{E}_\lambda$, la distance $\delta_\lambda(f, g)$ de f et g sera la borne supérieure de $|f(x) - g(x)|$ si x parcourt E, en désignant par $|f(x)|$ la valuation de l'élément $f(x)$ de K. En formule

$$\delta_\lambda(f, g) = \operatorname*{fin\,sup}_{x \in E} |f(x) - g(x)|.$$

Puisque la valuation de K est non-archimédienne, cette métrique est aussi non-archimédienne comme on voit aisément. Soit $\mathfrak{E}$ le produit topologique des espaces $\mathfrak{E}_\lambda$:

$$\mathfrak{E} = \prod_\lambda \mathfrak{E}_\lambda.$$

Considérons l'application suivante T de E dans $\mathfrak{E}$. A tout point x de E faisons correspondre le point uniquement déterminé Tx de $\mathfrak{E}$ dont la projection sur $\mathfrak{E}_\lambda$ est égale à la fonction continue $\varphi_x^{(\lambda)}(y)$, quel que soit λ.

La transformation T est biunivoque. Soit, pour cela, $z \neq x$. La structure uniforme étant supposée séparée (condition 1), l'espace satisfait à l'axiome de Hausdorff, de sorte qu'il existe un λ tel que z n'appartient pas à $U_\lambda(x)$, donc tel que (x, z) n'est pas dans U_λ. Il en résulte

$$\varphi_x^{(\lambda)}(z) = F_\lambda(x, z) = 1.$$

Puisque $D \subset U_\lambda$, on a de plus $\varphi_z^{(\lambda)}(z) = 0$. On a donc $\varphi_x^{(\lambda)} \neq \varphi_z^{(\lambda)}$, de sorte que les images de x et z dans $\mathfrak{E}$ sont distincts, d'où la biunivocité de la transformation.

La transformation est uniformément bicontinue. En premier lieu il faut donc montrer que, étant donné un entourage U de D dans E^2, il existe un entourage $\bar{U}$ de la diagonale dans $\mathfrak{E}^2$ tel que la relation $(\bar{x}, \bar{z}) \in \bar{U}$ ($\bar{x}$ et $\bar{z} \in \mathfrak{E}$) entraîne $(x, z) \in U$, en désignant par $\bar{x}$ et $\bar{z}$ les images de x et z. Pour cela, soit λ tel que $U_\lambda \subset U$. La relation

$$|f(y) - g(y)| < 1 \qquad (f \text{ et } g \in \mathfrak{E}_\lambda)$$

détermine un entourage dans $\mathfrak{E}$. Pour les images des points de E on a dans cet entourage

$$|\varphi_x^{(\lambda)} - \varphi_z^{(\lambda)}| < 1$$

et donc, puisque les fonctions ne prennent que les valeurs 0 et 1,

$$\varphi_x^{(\lambda)} - \varphi_z^{(\lambda)} = F_\lambda(x, y) - F_\lambda(z, y) = 0.$$

Donc pour $y = z$,

$$F_\lambda(x, z) = 0$$

et par suite, tenant compte de la définition de F_λ, $(x, z) \in U_\lambda \subset U$, ce qu'il fallait montrer. L'inverse, à savoir, étant donné un $\bar{U}$ il existe un U tel que $(x, z) \in U$ entraîne $(\bar{x}, \bar{z}) \in \bar{U}$, se démontre d'une façon analogue (voir Weil, l.c., p. 15). Le théorème est ainsi démontré.

Considérons maintenant le cas où l'on peut choisir les U_λ tels que le système est dénombrable. L'espace $\mathfrak{E}$ du théorème précédent est alors le produit topologique d'une infinité dénombrable d'espaces métriques, les métriques étant non-archimédiennes.

L'espace $\mathfrak{E}$ lui-même est alors un espace métrique et la métrique est non-archimédienne. En effet, en désignant par δ_i la métrique non-archimédienne dans $\mathfrak{E}_i$, l'espace $\mathfrak{E}$ est métrisé par

$$\delta(f, g) = \operatorname{fin\,sup}_i \frac{1}{2^i} \frac{\delta_i(f_i, g_i)}{1 + \delta_i(f_i, g_i)},$$

si f_i désigne la projection de f sur $\mathfrak{E}_i$. On voit sans peine que cette métrique est non-archimédienne.

Un système dénombrable d'entourages U_λ existe certainement si la structure est définie par une métrique. On a donc le théorème suivant:

Théorème 13. *Pour qu'un espace uniforme avec une structure non-archimédienne $\mathfrak{F}$ soit isomorphe (au sens de la structure uniforme) à un espace métrisé d'une façon non-archimédienne, il faut et il suffit que le système $\mathfrak{F}$ peut être défini au moyen d'une infinité dénombrable d'entourages U_λ.*

En ce qui concerne l'existence d'une structure uniforme non-archimédienne dans un espace topologique on a le théorème suivant. Faisons d'abord une remarque. On sait que dans chaque espace complètement régulier on peut définir une structure uniforme compatible avec la topologie et cette structure est même uniquement déterminée si l'on impose à la structure une certaine condition (concernant la continuité uniforme de certaines fonctions continues). D'après ce qui précède tout espace de HAUSDORFF 0-dimensionnel est complètement régulier (par rapport aux fonctions réelles et aussi par rapport aux fonctions prenant leurs valeurs dans un corps K muni d'une valuation non-archimédienne). Chaque espace DE HAUSDORFF 0-dimensionnel peut donc recevoir une structure uniforme, et c'est donc le problème de rechercher si cette structure est non-archimédienne ou, au moins, peut être choisie ainsi. La réponse est affirmative. [15])

Théorème 14. *Dans chaque espace de* HAUSDORFF 0*-dimensionnel on peut définir une structure uniforme non-archimédienne compatible avec la topologie donnée de l'espace.*

Démonstration. Considérons sur E^2 toutes les fonctions continues réelles $\Phi(x, y)$ satisfaisant aux conditions

1. $\Phi(x, x) = 0$,
2. $\Phi(x, y) = \Phi(y, x)$,
3. $\Phi(x, y) \leqq \max(\Phi(x, z), \Phi(y, z))$.

On ne suppose pas que $\Phi(x, y) \neq 0$ si $x \neq y$. Ils existent sur E^2 de telles fonctions. [16]) En effet, on en obtient une en posant

$$|f(x) - f(y)| = \Phi(x, y),$$

où $f(x)$ est une fonction continue sur E prenant ses valeurs dans un corps donné K valué d'une façon non-archimédienne. Remarquons que dans cet exemple, en supposant qu'une structure uniforme est définie sur E au moyen des entourages U_λ, la fonction $f(x)$ sera uniformément continue si, quel que soit $\varepsilon > 0$, il existe un λ tel que $(x, y) \in U_\lambda$ entraîne $\Phi(x, y) < \varepsilon$.

Définissons alors sur E, une structure uniforme comme il suit. On se donne un nombre fini de nombres réels $a_i > 0$ et un nombre fini de fonctions $\Phi_i(x, y)$. Soit alors V_a l'ensemble des points (x, y) de E^2 satisfaisant aux conditions

$$\Phi_i(x, y) < a_i.$$

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[15]) WEIL l.c., p. 16.

[16]) Une telle fonction est appelée un écart par N. BOURBAKI dans le livre Topologie générale, Act. Sci. et Ind. Paris, **1045** (1948). On y montre que chaque structure uniforme peut etre définie par une famille d'écarts. La démonstration présentée ici est analogue à celle de M. WEIL. L'inégalité, cependant, y est dans la forme habituelle (archimédienne). La fonction $\triangle_\varrho(x, y)$, introduite dans le § 1 de la partie I est un écart.

On voit qu'on a $V_a = \overset{-1}{V_a}$. De plus on a $\overset{2}{V_a} \subset V_a$. En effet, soit $(x, y) \in \overset{2}{V_a}$. Il existe donc z tel que $(x, z) \in V_a$, $(z, y) \in V_a$. Donc

$$\Phi_i(x, z) < \alpha_i \ , \ \Phi_i(z, y) < \alpha_i.$$

Il s'ensuit

$$\Phi_i(x, y) \leqq \max(\Phi_i(x, z), \ \Phi_i(z, y)) < \alpha_i,$$

donc $(x, y) \in V_a$, d'où il résulte $\overset{2}{V_a} \subset V_a$. Les ensembles V_a qu'on obtient en faisant varier les nombres α_i et les fonctions $\Phi_i(x, y)$, toujours en nombre fini pour un même V_a, constituent un système d'entourages d'une structure uniforme sur E. On voit que l'intersection de deux ensembles de ce système contient un ensemble du système, de sorte que les conditions 2, 3 et 4 sont vérifiées. Il ne reste donc que de montrer que la première condition est satisfaite. Soit, pour cela, $x \neq y$. Il existe une fonction continue f, prenant ses valeurs dans le corps K, telle que $f(x) = 0$, $f(y) = 1$. En posant.

$$\Phi(z, u) = |f(z) - f(u)|,$$

l'inégalité $\Phi(z, u) < 1$ détermine un V ne contenant pas le point (x, y) puisque

$$\Phi(x, y) = |f(x) - f(y)| = 1.$$

Il s'ensuit que la diagonale est l'intersection de tous les V_a. La structure ainsi définie est non-archimédienne. D'après la remarque précédente on voit que les fonctions continues f prenant ses valeurs dans K deviennent tous uniformément continues dans cette structure. Cette structure est même la moins fine rendant uniformément continues toutes ces fonctions. [17]) Il résulte de ceci que la topologie T_0, déduite de cette structure, est équivalente à la structure donnée T de E. En effet, d'abord on voit que T est plus fine que T_0, puisque les fonctions $f(x)$ sont tous continues dans T. Pour montrer que T est moins fine que T_0, soit $x_0 \in E$ et soit W un voicinage de x_0 dans la topologie T. Soit alors $f(x)$ tel que $f(x_0) = 0$ et $f(x) = 1$ si $x \in C\ W$. En posant $\Phi(x, y) = |f(x) - f(y)|$, l'inégalité $\Phi(x, y) < 1$ détermine un entourage V de la structure construite ci-dessus et les points y tels que $\Phi(x_0, y) < 1$ composent un voisinage de x_0 dans la topologie T_0. On a $y \in W$, puisque $\Phi(x_0, y) = 1$ si $y \in C\ W$. Les topologies sont donc bien équivalentes.

Les théorèmes 13 et 14 permettent de résoudre le problème 2, § 1, dans un sens affirmatif pour les espaces qui n'admettent qu'une seule structure uniforme compatible avec la topologie, en particulier donc pour les espaces métriques compacts 0-dimensionnels. Cependant, ceci n'a pas de valeur pour la résolution définitive du problème puisque, parmi les espaces métriques, les espaces compacts sont les seuls qui

[17]) Voir Bourbaki l.c. 16) p. 5.

admettent une structure uniforme unique. [18]) La résolution résulte alors du théorème 6.

Remarquons qu'il peut exister plusieurs structures uniformes distinctes non-archimédiennes compatibles avec une topologie donnée.

Exemple. Soit E un espace infini discret. La topologie discrète peut se déduire des structures uniformes suivantes.

a. la structure uniforme dont la diagonale est le seul entourage.

b. la structure uniforme des partitions finies de E. C'est la borne supérieure des structures uniformes qu'on obtient en considérant toutes les partitions finies de E et les structures uniformes définies par ces partitions. [19])

Ces structures sont non-archimédiennes; ils sont distinctes puisque E est supposé infini.

Est-il possible qu'une topologie donnée est compatible avec deux structures uniformes distinctes dont l'une est non-archimédienne et l'autre archimédienne? Un tel espace serait 0-dimensionnel et non-compact.

On sait que, pour qu'un espace topologique soit uniformisable et séparé il faut et il suffit que l'espace soit homéomorphe à un sous-espace d'un espace compact. Un espace uniforme peut donc être plongé topologiquement dans un espace compact. On peut encore dire qu'un tel espace peut être plongé dans un cube, en appelant un cube un espace topologique I^λ, produit topologique d'une famille d'espaces topologiques identiques à un intervalle I fermé de nombres réels. [20]) Si la structure de l'espace uniforme est non-archimédienne, on peut préciser ce résultat. Au lieu d'un cube, comme ci-dessus, on considère alors un produit topologique C^λ d'une famille d'espaces topologiques identiques à l'ensemble triadique C de Cantor. En remarquant que ce dernier ensemble est compact on montre de façon analogue *que tout espace uniforme à structure uniforme non-archimédienne, ou ce qui revient au meme d'après ce qui précède tout espace de* Hausdorff *0-dimensionnel, est homéomorphe à un sous-espace d'un tel "cube cantorien".* [21])

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[18]) R. Doss, On uniform spaces with a unique structure. Amer. J. Math. **71**, 19—23 (1949).

[19]) Bourbaki, Topologie générale, chap. II, Act. Sci. et Ind. **858**, 89 (1940).

[20]) Bourbaki, Topologie générale, chap. IX, Act. Sci. et Ind. **1045**, 11 (1948).

[21]) La démonstration modifiée n'est pas applicable si la structure uniforme n'est pas non-archimédienne, donc si la dimension est > 0. C'est puisque, alors, on n'est pas sûr qu'ils existent des applications continues, non partout égales à une constante, de l'espace en l'ensemble de Cantor, ce dernier étant totalement discontinu (comparer la note 13). Remarquons ici que les seuls sous-ensembles compacts et connexes de l'espace des nombres réels sont les intervalles fermés et bornés (Bourbaki, Topologie générale, chap. IV, Act. Sci. et Ind. **916**, 74 (1942)) et que tout ensemble compact, parfait, totalement discontinu dans cet espace est homéomorphe à l'ensemble de Cantor (Bourbaki, chap. IV, p. 140, et Bourbaki, chap. VI, Act. Sci. et Ind. **1029**, 33,). En considérant le cube I^λ et le cube cantorien C^λ on a donc épuisé les cas possibles.

Remarque. Pour les espaces métriques on peut simplement mettre en relation la définition 4 avec l'axiome d'Archimède dans les groupes ordonnés. Si G est un groupe ordonné (écrit dans la forme multiplicative) on sait que cet axiome exprime que, étant donnés a et b dans G, il existe un nombre entier n tel que $a^n > b$.

Dans un espace métrique E, ne vérifiant pas la définition 4 en ce sens qu'on a $\overset{2}{U} \supset U$ pour tout U, on a une pareille relation pour les entourages U. D'abord la métrique ϱ, définiant les U, ne peut être non-archimédienne comme le montre le théorème 10. U étant l'entourage défini par la relation

$$\varrho(x, y) < \alpha$$

le composé $\overset{n}{U}$ est défini, comme on voit aisément en utilisant l'inégalité triangulaire dans sa forme habituelle, par la relation

$$\varrho(x, y) < n\alpha$$

V étant un autre entourage $\varrho(x, y) < \beta$, on peut choisir n tel que $n\alpha > \beta$, d'où il résulte

$$\overset{n}{U} \supset V$$

On a donc le résultat: *étant donnés deux entourages U et V, il existe un entier n tel que*

$$\overset{n}{U} \supset V.$$

En prenant pour E un espace euclidien a une dimension on retrouve l'axiome d'Archimède dans forme géométrique bien connue.

Tout ceci n'est plus vrai si la structure uniforme est non-archimédienne au sens de la définition 4; on en trouve une justification de la dénomination "non-archimédien".

III. *Espaces topologiques.*

Dans chaque espace topologique dont la topologie peut se déduire d'une métrique non-archimédienne il existe, comme il résulte du théorème 2, un système S_E complet de voisinages U avec la propriété suivante: U_x et U_y étant deux voisinages appartenant à S_E on a une des relations suivantes qui s'excluent mutuellement

$$U_x . U_y = 0,$$
$$U_x \supset U_y,$$
$$U_x \subset U_y.$$

On peut prendre cette propriété comme point de départ de la définition d'une notion "non-archimédien" dans les espaces topologiques, non nécessairement métriques ou uniformes. Soit E un espace DE HAUSDORFF.

Définition 5. *La topologie T de E s'appelle non-archimédienne s'il existe une base S_E de T avec la propriété suivante: U_x et U_y étant deux voisinages de S_E on a une des relations*

$$U_x . U_y = 0,$$
$$U_x \supset U_y,$$
$$U_x \subset U_y.$$

Cette notion est bien topologiquement invariante. Si la topologie peut se déduire d'une métrique non-archimédienne, elle est non-archimédienne. Cependant une topologie non-archimédienne peut fort bien être compatible avec une métrique archimédienne; pour cela comparer la remarque précédent le théorème 11.

Théorème. *Tout espace* de Hausdorff *dont la topologie est non-archimédienne est 0-dimensionnel.*

Démonstration. On peut supposer que les voisinages constituant la base S_E sont tous ouverts. Soit $U_a \in S_E$ un voisinage de $a \in E$ et soit x un point d'accumulation de U_a: $x \neq a$. Chaque voisinage de x contient des points de U_a. Soit ω un voisinage de x ne contenant pas a. S_E étant une base, ω contient un voisinage $U_x \in S_E$; U_x contient des points de U_a et a n'appartient pas à U_x. Il suit donc de la définition 5 qu'on a $U_x \subset U_a$. Il s'ensuit $x \in U_a$ de sorte que U_a est fermé. Chaque point de E a donc des voisinages ouverts et fermés arbitrairement petits et l'espace est donc 0-dimensionnel

Théorème 16. *La topologie de chaque espace 0-dimensionnel séparable est non-archimédienne.*

Démonstration. C'est une conséquence du théorème 6 et de la propriété qu'une topologie, déduite d'une métrique non-archimédienne est non-archimédienne.

D'après ce qui précède la famille des espaces 0-dimensionnels est identique à celle des espaces dans lesquels on peut définir une structure uniforme non-archimédienne compatible avec la topologie. La chose est différente en partant de la définition 5: ils existent des espaces 0-dimensionnels dont la topologie n'est pas non-archimédienne au sens de cette définition de sorte qu'un espace susceptible de recevoir une structure uniforme non-archimédienne peut avoir une topologie qui n'est pas non-archimédienne. La définition 5 définit donc une vraie sous-famille de la famille des espaces 0-dimensionnels. Cette sous-famille ne contient que des espaces non-séparables.

Exemple. [22]) Considérons un espace euclidien R_2 à deux dimensions (x, y). Soit dans le demi-plan $x > 0$ un ensemble C de points isolés tel que tous les points de l'intervalle $y = 0,\ 0 \leqq x \leqq 1$ soient des points

[22]) Je dois cet exemple à M. J. de Groot, qui a bien voulu m'autoriser de le publier ici.

d'accumulations de C. Soit F l'espace qui se compose des points de C et des points $y = 0,\ 0 < x \leqq 1$. A partir de F nous construisons un nouvel espace E par la méthode d'identification. Les points de E seront a) les points de C et b) un point P obtenu par identification des points $y = 0,\ 0 < x \leqq 1$ en un seul point (à savoir le point P). La topologie de E est fixée de la façon habituelle. Pour P on procède comme il suit. Pour chaque point $y = 0,\ 0 < x \leqq 1$ on construit dans R_2 (ou F) un voisinage. Les points de C qui appartiennent à un quelconque de ces voisinages constituent par définition un voisinage de P dans E. L'espace E ainsi défini est dénombrable et donc 0-dimensionnel. Nous allons montrer d'abord que E n'a pas une base dénombrable dans P; il en résulte que E n'est pas métrisable.

Supposons qu'il y a une base dénombrable dans P. D'après la construction, il correspond à cette base une suite dénombrable de voisinages $\triangle_i$ dans F dont chacun est somme de voisinages des points $y = 0$, $0 < x \leqq 1$; on peut supposer

$$\triangle_1 \supset \triangle_2 \supset \triangle_3 \supset \ldots .$$

Désignons par σ_{ii} le voisinage du point $y = 0,\ x = x_i$ contribuant à la construction de $\triangle_i$. Puisque les $\triangle_i$ constituent une base, on peut construire une suite $x_1 > x_2 > \ldots \to 0$ telle que

$$\sigma_{ii} . \sigma_{jj} = 0 \quad (j \neq i),$$

au besoin en éliminant certains $\triangle_i$ (ce qui est permis). Choisissons pour $i = 1, 2, \ldots$, un point $P_i\,(x_i, y_i)$ appartenant à σ_{ii} et à C tel que

$$0 < y_i < y_{i-1}\ ,\ y_i \to 0.$$

On a $P_i \in \triangle_i$ et la suite $\{P_i\}$ a 0 comme seul point d'accumulation, considéré dans F. Considérons alors la suite $\{P_i\}$ dans E: on verra que P n'est pas point d'accumulation de cette suite (dont les points appartiennent à E). Il suffit de montrer qu'il existe un voisinage de P ne contenant aucun point P_i. Soient

ω_1 un voisinage de $x = x_1\ ,\ y = 0$ ne contenant pas P_1; $\omega_1 \subset \sigma_{11}$
ω_2 un voisinage de $x = x_2\ ,\ y = 0$ ne contenant pas P_2; $\omega_2 \subset \sigma_{22}$
. . . . . .

L'ensemble $\triangle^* = \Sigma\omega_i$ ne contient aucun point P_i. En ajoutant, au besoin, de voisinages convenablement choisis des points $x, y = 0$ qui ne sont pas points intérieurs de $\triangle^*$, on obtient dans F un voisinage $\triangle$, auquel correspond un voisinage de P dans E ne contenant aucun P_i.

En désignant par Σ la suite des points P_i, on voit que l'ensemble $U = \triangle_1 - \Sigma$ est un voisinage de P (dans l'espace E). Il résulte de la construction de Σ que U ne contient aucun $\triangle_i$, en contradiction avec l'hypothese que $\{\triangle_i\}$ soit une base. On en tire que qu'il n'y a pas de base dénombrable dans P.

Supposons maintenant qu'il existe dans E une base non-archimédienne S_E, donc une base telle que pour chaque couple de voisinages appartenant à cette base on sait que ou bien l'intersection de ces voisinages est vide ou bien un de ces voisinages contient l'autre. Puisque tous les voisinages de P ont ce point en commun, il existe donc en P une base monotone non-dénombrable

$$\triangle_1 \supset \triangle_2 \supset \ldots \supset \triangle_\omega \supset \ldots \supset \triangle_\mu \supset \ldots$$

Choisissons de nouveau, comme ci-dessus, une suite Σ ayant $x = 0$, $y = 0$ comme seul point d'accumulation. $\triangle_1 - \Sigma$ est un voisinage de P et on arrive à une contradiction tout comme ci-dessus.

IV. *Existence de triples non-archimédiens dans les espaces métriques.*

Appelons non-archimédien un triple x, y, z de points d'un espace métrique si l'on a

$$\varrho(x, y) \leqq \max(\varrho(x, z), \varrho(y, z))$$
$$\varrho(y, z) \leqq \max(\varrho(x, y), \varrho(x, z))$$
$$\varrho(x, z) \leqq \max(\varrho(x, y), \varrho(y, z)).$$

La définition 1 exprime que la métrique ϱ est non-archimédienne si l'espace, muni de cette métrique, ne contient que des triples non-archimédiens.

Etant donné un espace métrique E à métrique ϱ, la question se pose ce qu'on peut dire quant à l'existence de triples non-archimédiens (par rapport à ϱ) dans cet espace, autrement dit de triangles ayant deux côtés égaux et la troisième côté pas plus grande.

Théorème 17. *Dans chaque espace métrique séparable de dimension $n \geqq 2$ il existe une infinité de triples non-archimédiens.*

Démonstration. Choisissons x_0 et $a > 0$ tel que la frontière F du voisinage U: $\varrho(x_0, x) \leqq a$ soit de dimension $n - 1$, donc $\geqq 1$. La frontière F étant fermé, d'après le théorème de Cantor-Bendixson cet ensemble est la somme d'un ensemble parfait et d'un ensemble dénombrable. Cet ensemble parfait ne peut être vide, puisqu'alors F était dénombrable et donc de dimension 0. Soit y un point de cet ensemble parfait. y est point d'accumulation de F et il existe donc un point z de F tel que $\varrho(y, z) < a$. Le triple x_0, y, z est alors non-archimédien. En effet, on a $\varrho(x_0, y) = \varrho(x_0, z) = a$, $\varrho(y, z) < a$. Le même raisonnement s'applique pour toute valeur $b < a$. Puisque la puissance de E ne dépasse pas celle du continu, il s'ensuit que la puissance de l'ensemble des triples non-archimédiens est égale à celle du continu.

Toute autre est la situation dans les espaces séparables de dimension 0. Il résulte de la démonstration du théorème 6 qu'on peut métriser ces espaces tel qu'il n'y a aucun triple non-archimédien, mais aussi de façon

que tous les triples sont non-archimédiens. Le second cas n'est possible que dans les espaces de dimension 0; le premier cas est impossible si la dimension est $\geqq 2$.

Ils existent des espaces de dimension 1 qui puissent être métrisés de façon qu'il n'y a pas de triples non-archimédiens par rapport à cette métrique, comme on voit des exemples suivantes.

1. Un espace euclidien à une dimension munie de la métrique habituelle $\varrho(x, y) = |x - y|$. Il est cependant aussi possible de métriser cet espace topologiquement équivalent tel qu'il y a des triples non-archimédiens. Par exemple par la métrique $\bar{\varrho}(x, y) = |x - y|$ si $|x - y| \leqq 1$ et $\bar{\varrho}(x, y) = 1$ si $|x - y| > 1$.

2. Toutes les courbes suffisamment régulières, à savoir ceux sur les quelles on peut définir la longueur des arcs. Parmi ces espaces mentionnons en particulier ceux qui se composent d'un nombre fini de courbes de ce genre n'ayant en commun deux à deux qu'un point fixe O. Remarquons que, afin d'obtenir la distance de deux points A et B de deux courbes différentes, il faut passer par O.

3. Mentionnons enfin les espaces semicompact généralisés de Moore-Kline, introduit par M. J. de Groot [23]). Ces espaces sont topologiquement équivalent aux sous-ensembles de R_1. Ils puissent donc être métrisés tels qu'il n'y a pas de triples non-archimédiens.

Dans tous ces exemples la métrique ϱ, par rapport à laquelle il n'y a pas de triples non-archimédiens, satisfait à la relation

$$\varrho(x, y) = \varrho(x, z) + \varrho(y, z)$$

si $\varrho(x, y)$ est la plus grande des trois distances mutuelles. Cette relation est plus forte que celle que doit vérifier une métrique pour qu'il n'y a pas de triples non-archimédiens. En effet, il suffit qu'on a la relation moins restrictive

$$\max(\varrho(x, z), \varrho(y, z)) < \varrho(x, y) \leqq \varrho(x, z) + \varrho(y, z).$$

Cependant ils existent des espaces métriques de dimension 1 tels que, en considérant toutes les métriques équivalentes sur un tel espace, il y a des triples non-archimédiens par rapport à chaque métrique.

Exemple. Considérons dans l'espace euclidien R_2 les droites

$$l_n \ldots y = \frac{1}{n} x \ (n = 1, 2, \ldots),$$

$$l \ldots y = 0,$$

et soit E l'ensemble des points de ces droites vérifiant les relations

$$x \geqq 0 \text{ et } x^2 + y^2 \leqq 1.$$

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[23]) J. de Groot, Topological characterization of all subsets of the real number system. Proc. Kon. Nederl. Akad. v. Wetensch. Amsterdam, **50**, 876–884 (1947).

Soient ϱ une métrique arbitraire et U_ε un voisinage suffisamment petit de $O\,(0, 0)$. La frontière de U_ε contient un point P_n de chaque l_n et un point P de l. Choisissons un P_i tel que

$$\varrho\,(P_i, P) < \varepsilon.$$

Puisque

$$\varrho\,(P_i, O) = \varrho\,(P, O) = \varepsilon,$$

le triple (P_i, P, O) est non-archimédien. [24])

On est donc conduit au problème suivant.

Problème 4. *On demande à caractériser les espaces métriques de dimension* 1 *ayant la propriété que, parmi le système de toutes les métriques équivalentes, il existe au moins une métrique, telle qu'il n'y a pas de triples non-archimédiens par rapport à cette métrique.*

Ce problème fait partie d'un programme, introduit par M. L. M. Blumenthal, à savoir la recherche des propriétés topologiques qu'un espace doit posséder pourqu'il soit homéomorphe à un espace ayant des propriétés métriques données. Citons par exemple la propriété suivante de Blumenthal [25]):

Un espace compact convexe métrique contenant au moins deux points est congruent à un segment de droite si et seulement si l'espace ne possède aucun triple équilatéral.

Je veux remercier ici Prof. J. de Groot de quelques remarques qui m'ont été très utiles.

La Haye, le 8 mars 1950.

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[24]) Remarquons que la métrique, induite dans E par la métrique euclidienne dans R_2, n'est pas équivalente à la métrique dans E qu'on obtient en posant la distance du point P_i de l_i au point P_j de l_j égale à la somme des distances euclidiennes de P_i et de P_j à O, situation qui diffère donc de celle de l'exemple 2 précédent ou on ne considère qu'un nombre fini de courbes. Si on définit quand même sur E la métrique par la seconde méthode, on ne peut plus plonger l'ensemble E dans R_2 puisqu'alors on n'a pas $l_n \to l$ si $n \to \infty$.

[25]) L. M. Blumenthal. New characterizations of segments and arcs. Proc. Nat. Acad. Sci. U.S.A. 29, 107–109 (1943).

SUR L'ORDRE DE GRANDEUR DES FONCTIONS SOMMABLES

PAR

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§ 1. *Introduction.*

1. 1. Concernant l'ordre de grandeur d'une suite de fonctions sommables, on a un théorème bien connu:

Supposons que les fonctions $F_N(x)$ $N = 1, 2, \ldots$ *appartiennent à la classe* L^p *dans l'intervalle* $(0, 1)$ *et que*

$$\int_0^1 |F_N(x)|^p \, dx = O(\Phi(N)). \tag{1}$$

Si la fonction $\varphi(N) > 0$ *est telle que*

$$\sum_{N=1}^{\infty} \varphi(N)^{-1} < \infty,$$

alors

$$F_N(x) = o(\Phi(N)\,\varphi(N))^{1/p}, \tag{2}$$

presque partout dans $0 \leqq x \leqq 1$.

Nous allons démontrer ce théorème plus tard dans la forme d'un lemme (lemme 1), qui joue un certain rôle dans cet ordre d'idées. Montrons seulement que, dans un certain sens, on ne peut pas améliorer ce théorème.

Soient $\Phi(N) > 0$ et $\varphi(N) > 0$ deux fonctions croissantes quelconques, la deuxième choisie de telle façon que la série $\sum_1^{\infty} \varphi(N)^{-1}$ diverge vers $+\infty$. Il existe alors une suite $\{F_N(x)\}$ telle que (1) est valable, mais (2) est en défaut pour chaque valeur de x.

Pour construire une suite $\{F_N(x)\}$ jouissant la propriété énoncée, considérons les nombres

$$a_1 = 0 \quad , \quad b_1 = \varphi(1)^{-1} - [\varphi(1)^{-1}],$$

$$a_N = \sum_1^{N-1} \varphi(n)^{-1} \pmod 1 \quad \text{et} \quad b_N = \sum_1^{N} \varphi(n)^{-1} \pmod 1 \qquad (0 \leqq a_N < 1,\ 0 \leqq b_N < 1).$$

Si $a_N < b_N$, soit I_N l'intervalle ouvert (a_N, b_N). Si au contraire $a_N > b_N$, désignons par I_N la réunion des intervalles ouverts $(0, b_N)$ et $(a_N, 1)$. La mesure de l'ensemble I_N est en tout cas $\varphi(N)^{-1}$. Donc si nous posons

$$F_N(x) = \begin{cases} (\Phi(N)\,\varphi(N))^{1/p} & \text{dans } I_N, \\ 0 & \text{pour les autres valeurs de } x, \end{cases} \tag{3}$$

la relation (1) est valable. Mais nous avons supposé que $\sum_1^\infty \varphi(N)^{-1}$ diverge, donc d'après la définition de I_N, un point x quelconque appartient à une infinité d'ensembles I_N, c'est à dire $F_N(x) = (\Phi(N)\,\varphi(N))^{1/p}$ pour une infinité des valeurs de N. Donc la relation (2) n'est pas valable pour la fonction (3).

D'autre part il existe des suites $\{F_N(x)\}$ satisfaisantes (1) dont l'ordre de grandeur est en réalité beaucoup plus petit que l'ordre donné par la formule (2) si l'on suppose que $\varphi(N) > 0$ est croissante, telle que $\sum_{N=1}^\infty \varphi(N)^{-1}$ converge. Posons par example $F_N(x) = N^{1/p}$ dans (0, 1), nous pouvons choisir $\Phi(N) = N$. Le théorème précédent ne donne que des résultats moins forts que $O(N^{2/p})$, quoique d'après la définition $F_N(x) = O(N^{1/p})$ pour tout x dans (0, 1).

1. 2. Les remarques précédentes nous amènent de poser la question suivante: Quelles sont les hypothèses naturelles, plus restrictives que (1), qui permettent de déterminer plus précisement l'ordre de grandeur de la suite $\{F_N(x)\}$?

On peut supposer par example, que l'estimation

$$\int_0^1 |F_{M+N}(x) - F_M(x)|^p\,dx = O(\Phi(M, N)) \tag{4}$$

$M = 0, 1, \ldots;\ N = 1, 2, \ldots$ *est valable, et on cherche l'ordre de grandeur de* $F_N(x)$.

Dans cette direction pour le cas $p = 2$, $\Phi(M, N) = N$, M. H. Rademacher a démontré un résultat, qu'on peut énoncer comme suit [1]:

Si la suite de fonctions $F_N(x) - F_{N-1}(x)$ *est une suite presque-orthogonale dans* (0, 1), *alors*

$$F_N(x) = o(N \log^{3+\varepsilon} N)^{1/2} \tag{5}$$

presque partout dans (0, 1).

Parce que nous n'avons pas besoin de la notion d'une suite presque-orthogonale, nous renvoyons le lecteur pour les détails au mémoire de M. Rademacher [2].

M. Rademacher a basé la démonstration de (5) sur un théorème concernant les séries orthogonales (théorème de H. Rademacher [3]) et

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[1]) H. Rademacher, Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen, Math. Ann. **87**, 112—138 (1922).

Ici on trouve aussi des renseignements sur la littérature antérieure. L'idée fondamentale de Rademacher avait été utilisée déjà dans des recherches de Hobson et Plancherel; il prend son origine dans un travail de Weyl.

[2]) Voir [1]). D'après le mémoire de M. Rademacher, la notion de la suite presque-orthogonale a été introduite probablement par I. Schur. Voir aussi le mémoire de MM. M. Kac, R. Salem et A. Zygmund, A gap theorem. Trans. Amer. Math. Soc. **63**, 235—243 (1948).

[3]) Voir [1]).

D. MENCHOFF [4])). On ne peut pas utiliser ce procédé pour le cas plus général, c'est à dire pour $p > 2$, $\Phi(M, N) \neq N$. En utilisant certaines modifications nous allons démontrer un théorème général (§ 2, théorème 1), qui renferme comme cas particulier le résultat de M. RADEMACHER (sans la condition que la suite soit presque-orthogonale) et quelques résultats précédents des auteurs [5]) [6]).

Après la démonstration de notre théorème, nous discuterons les cas spéciaux qui ont une importance au point de vue des applications (§ 3, théorèmes 2—7).

§ 2. *Le théorème général.*

2. 1. Tout d'abord nous allons donner quelques définitions. Il y a lieu de remarquer, en passant, que la fonction $F(M, N; x) \geqq 0$ de laquelle nous parlerons, n'est pas autre chose, que la généralisation de la fonction

$$F(M, N; x) = |F_{M+N}(x) - F_M(x)|.$$

L'inégalité (6) ci dessous, de la même façon, signifie la généralisation de l'inégalité du triangle

$$|F_{M+N}(x) - F_M(x)| \leqq |F_{M+N'}(x) - F_M(x)| + |F_{M+N}(x) - F_{M+N'}(x)|.$$

En général, dans les applications la fonction $F_N(x)$ n'est autre chose que la somme partielle d'une série

$$f_1(x) + f_2(x) + \ldots + f_N(x) + \ldots$$

non nécessairement convergente.

Définition. *Soit S une partie mesurable d'un espace euclidien lieu de points x. Supposons que les fonctions $F(M, N; x) \geqq 0$; $x \in S$; $M = 0, 1, \ldots$; $N = 0, 1, \ldots$ définies sur S, appartiennent à la classe L^p, où $p \geqq 1$. Supposons enfin que $F(M, 0; x) = 0$ pour $M = 0, 1, \ldots$ et que*

$$F(M, N; x) \leqq F(M, N'; x) + F(M + N', N - N'; x) \tag{6}$$

pour tout $M = 0, 1, \ldots$ et pour tout couple (N, N') satisfaisant $0 \leqq N' \leqq N$.

A partir de cette définition on peut démontrer le théorème général suivant:

Théorème 1.

A. Supposons que pour tout couple $M = 0$, $N = 2^n$ $(n = 1, 2, \ldots)$ et

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[4]) MENCHOFF, D., Sur les séries de fonctions orthogonales. Fund. Math. **4**, 82—105 (1923).

[5]) GÁL, I. S. et J. F. KOKSMA, Sur l'ordre de grandeur des fonctions sommables. C. R. Acad. Sci. Paris **227**, 1321—1323 (1948).

[6]) GÁL, I. S., Sur l'ordre de grandeur des fonctions sommables. C. R. Acad. Sci. Paris **228**, 636—638 (1949).

pour tout couple M, N avec $M \geqq 1$, $0 \leqq N \leqq M-1$, M et N étant entiers non-negatifs,

$$\int_S F(M, N; x)^p \, dx \leqq C \Phi(M, N) \tag{7}$$

où $\Phi(M, N) > 0$ et où C désigne une constante positive.

B. Soit $\varkappa(n, \lambda) \geqq 1$; $n = 1, 2, \ldots$; $\lambda = 1, 2, \ldots, n$ et soit $\varphi(N) > 0$ une fonction non-décroissante, telle que

$$\sum_{n=1}^{\infty} \left(\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1})\right) \varphi(2^n)^{-1} < \infty.$$

C. Soit finalement

$$K(N) = \begin{cases} 1 & \text{si } p = 1; \\ \underset{(2^\nu \leqq N)}{\text{maximum}} \left(1 + \sum_{\lambda=1}^{\nu} \varkappa(\nu, \lambda)^{-\frac{p}{p-1}}\right)^{p-1}, & \text{si } p > 1, \end{cases} \tag{8}$$

où le maximum est pris par rapport à tous les $\nu = 1, 2, \ldots, [\log N / \log 2]$; $N \geqq 2$.

Conclusion.

$$F(0, N; x) = o\left((K(N) \varphi(N))^{1/p}\right)$$

pour presque toutes les valeurs $x \in S$.

Si, par example, $F(M, N; x) = |f_{M+1}(x) + \ldots + f_{M+N}(x)|$ où les fonctions $f_N(x)$ sont à p-ième puissance sommables, notre théorème donne une estimation pour la somme

$$f_1(x) + f_2(x) + \ldots + f_N(x)$$

qui est valable presque partout.

2. 2. Pour la démonstration de ce théorème nous avons besoin de deux lemmes. L'un de ces lemmes n'est autre chose que le théorème du § 1.

Lemme 1. *Si les fonctions $a_n(x) \geqq 0$; $n = 1, 2, \ldots$ sont intégrables au sens de Lebesgue dans S et si*

$$\sum_{n=1}^{\infty} \int_S a_n(x) \, dx < \infty,$$

alors $a_n(x) = o(1)$ presque partout dans S.

Démonstration. D'après un théorème connu de Abel [7]), il existe une suite de nombres croissants positifs, telle que

$$\alpha(n) \to \infty \quad \text{si} \quad n \to \infty$$

et que

$$\sum_{n=1}^{\infty} \int_S \alpha(n) \, a_n(x) \, dx < \infty. \tag{9}$$

Soit E l'ensemble des points $x \in S$ pour lesquels $\limsup \alpha(n) \, a_n(x) \geqq 1$,

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[7]) Voir K. Knopp, Theorie und Anwendung der unendlichen Reihen, Berlin, 302 (1931).

si $n \to \infty$. Désignons par $\mu(E)$ la mesure de l'ensemble E. Définissons E_n comme l'ensemble des points x pour lesquels $2\alpha(n)\, a_n(x) > 1$.

Donnons nous maintenant un nombre $\varepsilon > 0$ en avance. D'après (9) nous pouvons choisir $n_0 = n_0(\varepsilon)$ de telle sorte que

$$\sum_{n=n_0}^{\infty} \int_S \alpha(n)\, a_n(x)\, dx < \frac{\varepsilon}{2}. \tag{10}$$

Or, selon la définition des ensembles E et E_n, nous avons $E \subset E_{n_0} + E_{n_0+1} + \ldots$ quelque soit $n_0 \geqq 1$ donc

$$\mu(E) \leqq \sum_{n=n_0}^{\infty} \mu(E_n) \leqq 2 \sum_{n=n_0}^{\infty} \int_{E_n} \alpha(n)\, a_n(x)\, dx \leqq 2 \sum_{n=n_0}^{\infty} \int_S \alpha(n)\, a_n(x)\, dx.$$

Par conséquent, vu (10), $\mu(E) < \varepsilon$ quelque soit $\varepsilon > 0$. Cela veut dire que $\mu(E) = 0$. Ainsi, nous avons $\limsup \alpha(n)\, a_n(x) < 1$ pour presque toutes les valeurs $x \in S$. Mais, $\alpha(n) \to \infty$, donc en effet, l'on a $a_n(x) = o(1)$.

Pour compléter l'exposé vérifions en passant le théorème de § 1. Choisissons $S \equiv (0, 1)$ et $a_n(x) = |F_n(x)|^p\, (\Phi(n)\, \varphi(n))^{-1}$. Selon (1) nous avons

$$\int_S a_n(x)\, dx = O\,(\varphi(n)^{-1})$$

alors, d'après l'hypothèse $\sum_1^{\infty} \varphi(n)^{-1} < \infty$, on a

$$\sum_{n=1}^{\infty} \int_S a_n(x)\, dx < \infty.$$

Or, nous pouvons utiliser le lemme 1, qui donne

$$a_n(x) = |F_n(x)|^p\, (\Phi(n)\, \varphi(n))^{-1} = o(1),$$

c'est à dire que $F_n(x) = o\,(\Phi(n)\, \varphi(n))^{1/p}$ pour presque tout x, $0 \leqq x \leqq 1$.

2. 3. Considérons maintenant le second lemme, qui est nécessaire pour la démonstration du théorème 1. Ce lemme contient la généralisation d'une inégalité fondamentale de M. RADEMACHER. [8])

Lemme 2. *Supposons que* $p \geqq 1$ *et* $\varkappa(n, \lambda) \geqq 1$ *pour* $n = 1, 2, \ldots$; $\lambda = 1, 2, \ldots, n$. *Désignons par* $F(M, N; x) \geqq 0$; $F(M, 0; x) = 0$; $M, N = 0, 1, \ldots$ *une suite de fonctions de* $x \in S$ *pour lesquelles l'inégalité* (6) *est valable. Désignons de plus par* $K(N)$ $(N \geqq 2)$ *la fonction définie plus haut par* (8). *Posons pour* $n \geqq 1$:

$$H(n, p; x) = \left(F(0, 2^n; x)^p + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} F(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}; x)^p \right)^{1/p}.$$

Alors nous avons

$$F(0, N; x) \leqq K(N)^{1/p}\, H(n(N), p; x)$$

pour tout $x \in S$ $(N \geqq 2)$, *où l'entier* $n = n(N)$ *est défini par* $2^{n(N)} \leqq N < 2^{n(N)+1}$.

[8]) Voir [1]).

Démonstration. Soit $N \geqq 2$ donné en avance. On peut écrire

$$N = 2^n + \varepsilon_{n-1}\, 2^{n-1} + \ldots + \varepsilon_1 \cdot 2 + \varepsilon_0 \tag{11}$$

où $n = n(N) \geqq 1$ et $\varepsilon_i = 0, 1$ pour $i = 0, 1, \ldots, (n-1)$. D'après (6), nous obtenons de proche en proche, que

$$\begin{aligned} F(0, N; x) &\leqq F(0, 2^n; x) + F(2^n, \varepsilon_{n-1}\, 2^{n-1} + \ldots + \varepsilon_0; x) \leqq \\ &\leqq F(0, 2^n; x) + F(2^n, \varepsilon_{n-1}\, 2^{n-1}; x) + \\ &\qquad + F(2^n + \varepsilon_{n-1}\, 2^{n-1}, \varepsilon_{n-2}\, 2^{n-2} + \ldots + \varepsilon_0; x) \leqq \\ &\leqq F(0, 2^n; x) + F(2^n, \varepsilon_{n-1}\, 2^{n-1}; x) + \\ &\qquad + F(2^n + \varepsilon_{n-1}\, 2^{n-1}, \varepsilon_{n-2}\, 2^{n-2}; x) + \\ &\qquad + F(2^n + \varepsilon_{n-1}\, 2^{n-1} + \varepsilon_{n-2}\, 2^{n-2}, \varepsilon_{n-3}\, 2^{n-3} + \ldots + \varepsilon_0; x) \leqq \\ &\leqq F(0, 2^n; x) + F(2^n, \varepsilon_{n-1}\, 2^{n-1}; x) + \\ &\qquad + \sum_{\lambda=1}^{n-1} F(2^n + \varepsilon_{n-1}\, 2^{n-1} + \ldots + \varepsilon_\lambda\, 2^\lambda, \varepsilon_{\lambda-1}\, 2^{\lambda-1}; x). \end{aligned}$$

Soient maintenant

$$\mu_n = 0 \text{ et } \mu_\lambda = \varepsilon_\lambda + \varepsilon_{\lambda+1}\, 2 + \ldots + \varepsilon_{n-1}\, 2^{n-\lambda-1} \text{ pour } \lambda = 1, 2, \ldots, (n-1). \tag{12}$$

D'après l'inégalité précédente, nous obtenons

$$F(0, N; x) \leqq F(0, 2^n; x) + \sum_{\lambda=1}^{n} F(2^n + \mu_\lambda\, 2^\lambda, \varepsilon_{\lambda-1}\, 2^{\lambda-1}; x). \tag{13}$$

Parce que nous avons supposé que $F \geqq 0$ et que $F(M, 0; x) = 0$, nous avons

$$F(2^n + \mu_\lambda\, 2^\lambda; \varepsilon_{\lambda-1}\, 2^{\lambda-1}; x) \leqq F(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}; x).$$

En vertu de (13) on a donc

$$F(0, N; x) \leqq F(0, 2^n; x) + \sum_{\lambda=1}^{n} F(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}; x) \tag{14a}$$

pour tout $N \geqq 2$, où les valeurs

$$n \geqq 1 \quad \text{et} \quad 0 \leqq \mu_\lambda \leqq 2^{n-\lambda} - 1 \tag{14b}$$

sont définies par (11) et (12).

Supposons maintenant que $p = 1$, par conséquent $K(N) = 1$. Vu (14 a) et $\varkappa(n, \lambda) \geqq 1$, nous avons

$$F(0, N; x) \leqq F(0, 2^n; x) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)\, F(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}; x),$$

alors d'après (14 b)

$$\begin{aligned} F(0, N; x) &\leqq F(0, 2^n; x) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda) \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} F(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}; x) = \\ &= H(n, 1; x) = K(N)^{1/p}\, H(n(N), 1; x). \end{aligned}$$

Le lemme est ainsi démontré pour $p = 1$.

Si $p > 1$, nous allons utiliser l'inégalité de HÖLDER [9]) en forme

$$\sum A_i B_i \leqq \left(\sum A_i^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} (\sum B_i^p)^{1/p} \qquad (A_i \geqq 0,\ B_i \geqq 0).$$

D'après l'inégalité (14 a), nous trouvons que

$$F(0, N; x) \leqq F(0, 2^n; x) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^{-1} (\varkappa(n, \lambda) F(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}; x)) \leqq$$

$$\leqq \left(1 + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^{-\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\cdot \left(F(0, 2^n; x)^p + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p F(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}; x)^p\right)^{1/p}.$$

Mais on a

$$\left(1 + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^{-\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \leqq \left(\underset{(\nu \leqq n)}{\text{maximum}} \left(1 + \sum_{\lambda=1}^{\nu} \varkappa(\nu, \lambda)^{-\frac{p}{p-1}}\right)^{p-1}\right)^{1/p}$$

$$\text{et donc} \qquad \leqq K(N)^{1/p}.$$

En effet, $n = n(N)$ est telle que $2^n \leqq N < 2^{n+1}$, donc le maximum est pris par rapport à tous les $\nu = 1, 2, \ldots, [\log N/\log 2]$, c'est à dire pour les mêmes valeurs comme dans (8).

D'autre part en utilisant (14b), on obtient que

$$\left(F(0, 2^n; x)^p + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p F(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}; x)^p\right)^{1/p} \leqq$$

$$\leqq \left(F(0, 2^n; x)^p + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} F(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}; x)^p\right)^{1/p} = H(n, p; x).$$

Enfin, en combinant les trois dernières inégalités, nous obtenons

$$F(0, N; x) \leqq K(N)^{1/p} H(n(N), p; x).$$

Ainsi le lemme est aussi démontré pour $p > 1$.

2. 4. Nous allons démontrer maintenant le théorème 1. D'après la définition de $H(n, p; x)$ (Lemme 2), en utilisant l'hypothèse A nous obtenons que

$$\int_S H(n, p; x)^p \, dx \leqq C \left\{\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1})\right\}.$$

Donc, si nous posons

$$a_n(x) = H(n, p; x)^p \cdot \varphi(2^n)^{-1},$$

nous avons

$$\int_S a_n(x) \, dx \leqq C \left\{\left(\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1})\right) \varphi(2^n)^{-1}\right\}.$$

[9]) Voir G. H. HARDY, J. E. LITTLEWOOD et G. PÓLYA, Inequalities, Cambridge, 24 (1934).

Ainsi, en utilisant l'hypothèse B l'on a

$$\sum_1^\infty \int_S a_n(x)\,dx < \infty.$$

Or, d'après le lemme 1, on obtient

$$a_n(x) = \frac{H(n,p;x)^p}{\varphi(2^n)} = o(1)$$

presque partout dans S.
Posons $n = n(N)$, tel que $2^n \leqq N < 2^{n+1}$, alors il résulte que

$$H(n(N), p; x) = o(\varphi(N))^{1/p}$$

pour presque tout $x \in S$. Donc, en utilisant le résultat du lemme 2, nous avons en effet

$$F(0, N; x) = o(K(N)\,\varphi(N))^{1/p}$$

presque partout dans S. Ainsi le théorème 1 est complètement démontré.

§ 3. *Les cas spéciaux.*

3. 1. Dans ce chapitre nous allons considérer quelques cas spéciaux du théorème 1. Nous avons seulement supposé que $\Phi(M,N)$ est une fonction positive. Maintenant supposons que $\Phi(M,N) = \Psi(N)\,\Phi^*(M,N)$, où $\Psi(N)/N^{1-\gamma}$; $\gamma \geqq 0$, est une fonction non-décroissante, et $\Phi^*(M,N) > 0$. C'est principalement le cas $\Phi^*(M,N) \geqq 1$ qui nous interesse [10]). Tout d'abord, pour le cas $\gamma = 0$, on peut montrer le théorème suivant:

Théorème 2.
Supposons que (C *désignant une constante positive*)

$$\int_S F(M,N;x)^p\,dx \leqq C\,\Psi(N)\,\Phi^*(M,N) \qquad (M \geqq 0,\ N \geqq 1)$$

où $\Psi(N)/N$ *est non-décroissante et* $\Phi^*(M,N) \geqq 1$. *Soit* $\psi(N) > 0$ *une fonction non-décroissante et telle que*

$$\psi(2^n) \geqq \Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} 2^{\lambda-n}\,\Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}),$$

alors, si $\chi(N)$ *désigne une fonction non-décroissante et positive telle que* $\sum_{N=1}^{\infty} (N\chi(N))^{-1} < \infty$, *on a*

$$F(0, N; x) = o(\Psi(N)\,\psi(N)\,\chi(N)\,(\log N)^{p-1})^{1/p}$$

presque partout dans S.

Remarque. Parce que $\chi(N)$ est non-décroissante les assertions

$$\sum_{(N)} (N\chi(N))^{-1} < \infty \quad \text{et} \quad \sum_{(n)} \chi(2^n)^{-1} < \infty$$

sont équivalentes. Notons encore une fois qu'il suffit avoir l'estimation $O(\Psi(N)\,\Phi^*(M,N))$ seulement pour les couples $(0, 2^n)$ et (M,N) $(M \geqq 1,\ 0 \leqq N \leqq M-1)$.

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[10]) Dans les théorèmes 2—5 la Définition de § 2. 1 est sousentendue.

Démonstration. Posons dans le théorème 1

$$\varphi(N) = 2\,\Psi(N)\,\psi(N)\,\chi(N) \quad \text{et} \quad \varkappa(n, \lambda) \equiv 1.$$

Si $p > 1$, nous avons

$$K(N) = \underset{(2^\nu \leqq N)}{\text{maximum}} \left(1 + \sum_{\lambda=1}^{\nu} 1\right)^{p-1} = O\,(\log N)^{p-}, \tag{15}$$

par conséquent (15) est valable pour $p \geqq 1$.

Montrons que l'hypothèse B du théorème 1 est vérifiée. D'après la supposition on a

$$\Psi(2^\lambda) \leqq 2^{\lambda-n}\,\Psi(2^n)$$

pour $\lambda = 1, 2, \ldots n$. Donc vu $\varkappa(n, \lambda) = 1$ on obtient

$$\sum_{n=1}^{\infty} (\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}))\, \varphi(2^n)^{-1} \leqq$$

$$\leqq \sum_{n=1}^{\infty} (\Psi(2^n)\, \Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Psi(2^\lambda)\, \Phi^*(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}))$$

$$(\Psi(2^n)\, \psi(2^n)\, \chi(2^n))^{-1} \leqq$$

$$\leqq \sum_{n=1}^{\infty} (\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} 2^{\lambda-n}\, \Phi^*(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}))\, (\psi(2^n)\, \chi(2^n))^{-1}$$

$$\leqq \sum_{n=1}^{\infty} (\chi(2^n))^{-1} < \infty.$$

Maintenant toutes les hypothèses A, B, et C du théorème 1 sont satisfaites. On peut donc utiliser le théorème 1, et par (15):

$$F(0, N; x) = o\Big((\log N)^{p-1}\, \Psi(N)\, \psi(N)\, \chi(N)\Big)^{1/p}$$

presque partout.

3. 2. Maintenant le théorème 2 étant établi, nous l'allons spécialiser pour le cas $\Phi^*(M, N) \equiv 1$. Ainsi nous avons

$$\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda\ 0}^{2^{n-\lambda}-1} 2^{\lambda-n}\, \Phi^*(2^n + \mu_\lambda\, 2^\lambda, 2^{\lambda-1}) =$$

$$= 1 + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} 2^{\lambda-n} = n + 1 < 4 \log 2^n.$$

Par conséquent on peut choisir $\psi(N) = 4 \log N$, $\chi(N) = \log^{1+\varepsilon} N$ et l'on obtient le résultat suivant:

Théorème 3.
Si

$$\int_S F(M, N; x)^p\, dx = O\,(\Psi(N))$$

uniformement dans $M = 0, 1, \ldots$, *où* $\Psi(N)/N$ *est une fonction non-décroissante, alors pour tout* $\varepsilon > 0$ *on a*

$$F(0, N; x) = o\,(\Psi(N)\, (\log N)^{p+1+\varepsilon})^{1/p}$$

presque partout dans S.

Finalement notons que dans le cas $\Psi(N) = N$, $p = 2$, ce théorème donne le résultat (5), c'est à dire l'estimation de M. RADEMACHER. [11])

3. 3. Considérons maintenant l'autre cas, c'est à dire le cas où $\Psi(N)/N^{1+\gamma}$ est non-décroissante et $\gamma > 0$.

Pour ce cas on peut démontrer le théorème suivant:

Théorème 4:
Supposons que

$$\int_S F(M, N; x)^p \, dx \leqq C \, \Psi(N) \, \Phi^*(M, N) \qquad (M \geqq 0; \; N \geqq 1),$$

où C désigne une constante positive et où $\Psi(N)/N^{1+\gamma}$ est non-décroissante, γ désignant une constante positive et $\Phi^*(M, N) \geqq 1$. Soit $\psi(N) > 0$ une fonction non-décroissante et telle que

$$(16) \qquad \psi(2^n) \geqq \Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)(1+\alpha)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}),$$

avec une valeur constante $a > 0$ et $< \gamma$, alors, si $\chi(N)$ désigne une fonction positive, non-décroissante de N, telle que $\sum (N \chi(N))^{-1} < \infty$, on a

$$(17) \qquad F(0, N; x) = o\left(\Psi(N) \, \psi(N) \, \chi(N)\right)^{1/p}$$

presque partout dans S.

Démonstration. Pour obtenir ce résultat, nous allons prouver, ainsi que nous l'avons déjà fait, que l'hypothèse B du théorème 1 est satisfaite. Choissisons maintenant

$$(18) \qquad \varphi(N) = \Psi(N) \, \psi(N) \, \chi(N)$$

et

$$(19) \qquad \varkappa(n, \lambda) = 2^{\frac{n-\lambda}{p}(\gamma-\alpha)}.$$

Par conséquent nous avons

$$K(N) = \underset{(2^\nu \leqq N)}{\text{maximum}} \left(1 + \sum_{\lambda=1}^{n} 2^{\frac{\lambda-\nu}{p-1}(\gamma-\alpha)}\right)^{p-1} < \left(1 + \sum_{\Lambda=0}^{\infty} \left(2^{-\frac{\gamma-\alpha}{p-1}}\right)^{\Lambda}\right)^{p-1} = O(1),$$

pour $p > 1$. Ainsi $K(N) = O(1)$ pour $p \geqq 1$.

Or, considérons la série

$$\sum_{n=1}^{\infty} \left(\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda)^p \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1})\right) \varphi(2^n)^{-1},$$

et montrons sa convergence. D'après (18) et (19) on obtient

$$\sum_{n=1}^{\infty} \left(\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(n-\lambda)(\gamma-\alpha)} \frac{\Psi(2^{\lambda-1})}{\Psi(2^n)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1})\right)$$
$$(\psi(2^n) \, \chi(2^n))^{-1}.$$

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[11]) Voir aussi I. S. GÁL, A Theorem concerning diophantine approximations, Nw. Arch. Wisk. 23, 13—38 (1949).

Ici se trouve une démonstration d'un cas spécial du théorème 3 avec $\Psi(N) = N(\log\log N)^2$, qui utilise les mêmes idées que ceux du théorème 1.

Mais d'après l'hypothèse du théorème 4, on a

$$\frac{\Psi(2^{\lambda-1})}{\Psi(2^n)} \leqq 2^{(\lambda-n)(1+\gamma)}$$

pour $\lambda = 1, 2, \ldots, n$. Il suffit donc de montrer que la série

$$\sum_{n=1}^{\infty} (\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)(1+\alpha)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}))\,(\psi(2^n)\chi(2^n))^{-1}$$

converge. En effet

$$(\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)(1+\alpha)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}))\,\psi(2^n)^{-1} \leqq 1,$$

par conséquent l'hypothèse B est vérifiée. Or, en utilisant le théorème 1 nous obtenons, d'après (18) et $K(N) = O(1)$, le résultat du théorème 4.

3. 4. Considérons encore le cas particulier du théorème 4 pour lequel $\Phi^*(M, N) \equiv 1$. Il résulte que

$$\Phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} 2^{(\lambda-n)(1+\alpha)}\,\Phi^*(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}) =$$

$$= 1 + \sum_{\lambda=1}^{n} 2^{(\lambda-n)\alpha} < 1 + \sum_{\Lambda=0}^{\infty} (2^{-\alpha})^\Lambda = c(\alpha),$$

où $c(\alpha)$ est indépendant de n. Ainsi nous pouvons choisir $\psi(N) = c(\alpha) = O(1)$, $\chi(N) = (\log N)^{1+\varepsilon}$ $(\varepsilon > 0)$, et nous obtenons le théorème suivant, qui a été annoncé dans le cas spécial $\Psi(N) = N^k$ $(k > 1)$ par l'un des auteurs [12]).

Théorème 5.
Supposons que

$$\int_S F(M, N; x)^p\,dx = O(\Psi(N))$$

uniformement dans $M = 0, 1, \ldots$, *où* $\Psi(N)/N^{1+\gamma}$ $(\gamma > 0)$ *est une fonction non-décroissante, alors*

$$F(0, N; x) = o(\Psi(N)(\log N)^{1+\varepsilon})^{1/p}$$

presque partout dans S, *et pour tout* $\varepsilon > 0$.

3. 5. Nous voulons noter encore deux cas spéciaux, qui jouent un certain rôle dans les applications. Au point de vue de ces applications nous allons considérer seulement le cas

$$F(M, N; x) = |f_{M+1}(x) + f_{M+2}(x) + \ldots + f_{M+N}(x)|$$

où les fonctions $f_\nu(x)$ appartiennent à la classe L^2.

Tout d'abord un résultat, que M. Erdös et l'un des auteurs ont employé dans une autre publication. [13])

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[12]) Voir [6]).

[13]) Erdös, P. et J. F. Koksma, On the uniform distribution modulo 1 of sequences $(f(n, \vartheta))$. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam **52**, 299–302 (1949).

Théorème 6.
Supposons que

$$\int_a^b |f_{M+1}(x)+f_{M+2}(x)+\ldots+f_{M+N}(x)|^2\,dx = O\,(N\,(\log N)^\sigma) \qquad (\sigma \geqq 0)$$

uniformement dans M, alors pour tout $\varepsilon > 0$:

$$|f_1(x)+f_2(x)+\ldots+f_N(x)| = o\,(N\,(\log N)^{3+\sigma+\varepsilon})^{1/2},$$

presque partout dans $a \leqslant x \leqslant b$. En effet, la fonction $\psi(N)/N = (\log N)^\sigma$ est non décroissante, donc d'après le théorème 3 on a

$$|f_1+f_2+\ldots+f_N| = o\,(N \log^\sigma N \log^{3+\varepsilon} N)^{1/2} = o\,(N \log^{\sigma+3+\varepsilon} N)^{1/2}.$$

Finalement voici un théorème, que M. Salem et l'un des auteurs ont utilisé dans un autre travail [14]).

Théorème 7. *Soit $f_n(x) \in L^p\,(0, 1)$ $(n = 1, 2, \ldots)$ $(p > 1)$ une suite de fonctions pour laquelle*

$$\int_0^1 \Big|\sum_{n=M+1}^{M+N} f_n(x)\Big|^p\,dx \leqslant C\,(M+N)^{p-\sigma}\,N^\sigma\,\eta(N) \tag{20}$$

où $C > 0$ et $\sigma > 1$ sont des constantes. La fonction $\eta(N) > 0$ est non-croissante et $\sum \eta(N)/N < \infty$. Alors on a

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(x) = 0$$

presque partout dans $(0, 1)$.

Remarque: Nous pouvons supposer naturellement que $(1 <)\ \sigma \leqslant p$, parce que pour $\sigma > p$ nous obtenons $C(M+N)^{p-\sigma}\,N^\sigma\,\eta(N) \leqslant CN^p\,\eta(N)$, donc on peut remplacer dans (20) σ par la valeur p. De la même façon, on peut supposer que $\eta(N)\log^2 N$ est non-décroissante.

Démonstration. Nous allons utiliser le théorème 4 en posant

$$F\,(M, N; x) = \Big|\sum_{n=M+1}^{M+N} f_n(x)\Big|\ , \quad \Psi(N) = N^{\frac{\sigma+1}{2}}$$

$$\Phi^*\,(M, N) = (M+N)^{p-\sigma}\,N^{\frac{\sigma-1}{2}}\,\eta(N)\ , \quad \gamma = \frac{\sigma-1}{4}.$$

Dans ce cas pour tout $\alpha\left(0 < \alpha < \dfrac{\sigma-1}{4}\right)$ nous avons

$$\left.\begin{aligned} &\Phi^*\,(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)\,(1+\alpha)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi^*\,(2^n+\mu_\lambda\,2^\lambda,\ 2^{\lambda-1}) \leqslant \\ &\leqslant 2^{n\left(p-\frac{\sigma+1}{2}\right)}\,\eta(2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)\,(1+\alpha)} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} (2^n+(\mu_\lambda+1)\,2^\lambda)^{p-\sigma}\,2^{\frac{(\sigma-1)\,(\lambda-1)}{2}}\,\eta(2^{\lambda-1}). \end{aligned}\right\} \tag{21}$$

[14]) Koksma, J. F. et R. Salem, Uniform distribution and Lebesgue integration, Acta. Sci. Math. Szeged, 12B, 87—96 (1950).

La fonction $\eta(N)\log^2 N$ est non-décroissante, par conséquant

$$2^{\frac{(\sigma-1)(\lambda-1)}{2}}\eta(2^{\lambda-1}) \leqq 2^{\frac{(\sigma-1)n}{2}}\eta(2^n)$$

pour tout $\lambda \leqslant n$. D'après cette inégalité nous obtenons que le membre à droit de (21)

$$\leqq 2^{n\left(p-\frac{\sigma+1}{2}\right)}\eta(2^n) + 2^{p-\sigma}\cdot 2^{n(p-\sigma)}\,2^{\frac{(\sigma-1)n}{2}}\eta(2^n)\sum_{\lambda=1}^{n} 2^{(\lambda-n)\alpha} <$$

$$< 2^{p-\sigma}\cdot 2^{n\left(p-\frac{\sigma+1}{2}\right)}\eta(2^n)\left\{1+\sum_{\lambda=0}^{\infty} 2^{-\alpha\lambda}\right\} < c\,2^{n\left(p-\frac{\sigma+1}{2}\right)}\eta(2^n),$$

où la constante $c > 0$ ne dépend que de p, σ et α. Ainsi (16) est valable avec $\psi(N) = cN^{p-\frac{\sigma+1}{2}}\eta(N)$.

Posons maintenant $\chi(N) = \eta(N)^{-1}$ dans le résultat (17) et nous obtenons ainsi, que

$$F(0,N;x) = o(\Psi(N)\psi(N)\chi(N))^{1/p} = o\left(N^{\frac{\sigma+1}{2}}\cdot N^{p-\frac{\sigma+1}{2}}\eta(N)\eta(N)^{-1}\right)^{1/p} = o(N)$$

presque partout dans (0, 1).

§ 4. *Quelques remarques sur le théorème* 1.

4. 1. Dans ce dernier chapitre nous examinerons avec quel succès on peut appliquer le théorème 1 à des problèmes concrets. Finalement nous allons mentionner une possibilité naturelle de généralisation à laquelle nous reviendrons ailleurs.

Nous avons remarqué dans (1. 1) que l'ordre de grandeur obtenu par le théorème qui y figure et l'ordre de grandeur effectif de $F(0, N; x)$ peuvent différer beaucoup, à savoir par un facteur d'ordre $N^{1/p}$. Il est facile de construire un example où ce facteur est supérieur à une fonction $\Phi(N)$ arbitraire, donnée à l'avance. De ce point de vue notre théorème n'est pas meilleur. Soit en effet

$$f_n(x) = \begin{cases} (2^n\Phi(n)^p)^{1/p} & \text{si } 0 \leqq x \leqq 2^{-n}, \\ 0 & \text{si } 2^{-n} < x \leqq 1, \end{cases}$$

où $\Phi(N)$ est une fonction croissante tendant vers l'infini. Il est évident que $f_1(x) + f_2(x) + \ldots + f_N(x) = O(1)$; $N \to \infty$ pour $0 < x \leqslant 1$. D'autre part en vertu de la definition

$$\int_0^1 |f_{M+1} + \ldots + f_{M+N}|^p\,dx \geqslant \int_0^{2^{-(M+N)}} 2^{M+N}\Phi(M+N)^p\,dx \geqq \Phi(N)^p,$$

donc le théorème 1 ne peut donner qu'un résultat moins fort que $o[\Phi(N)(\log N)^{1/p}]$.

Mais en comparaison avec le théorème qui figure dans (1. 1), le théorème 1 est plus fort, car dans l'exemple précédent (1. 1) ne donne que l'estimation $O[\Phi(N)N^{1/p}]$. C'est d'ailleurs la situation générale: les résultats obtenus

à partir du théorème dans (1. 1) et ceux obtenus à partir du théorème 1 diffèrent par un facteur $N^{1/p}(\log N)^{-\delta}$ $(0 < \delta < 2)$.

4. 2. Pour que le résultat obtenu à partir du théorème 1 approche bien l'ordre de grandeur effectif de $F(0, N; x)$, il faut que cet ordre de grandeur soit atteint "assez uniformément" dans $x \in S$. Pour certains problèmes ceci est satisfait, on peut montrer cependant un exemple très simple où ce n'est pas le cas:

Soit

$$F(M, N; x) = \left| \sum_{n=M+1}^{M+N} e^{2\pi i h n x} \right|,$$

où h $(h = \pm 1, \pm 2, \ldots)$ est un nombre fixe. Nous obtenons

$$\int_0^1 F(M, N; x)^2 \, dx = \int_0^1 \left(\sum_{n_1=M+1}^{M+N} e^{2\pi i h n_1 x} \right) \left(\sum_{n_2=M+1}^{M+N} e^{-2\pi i h n_2 x} \right) dx = N.$$

En choisissant $p = 2$ dans le théorème 3, on obtient $F(0, N; x) = o(N \log^{3-\varepsilon} N)^{1/2}$ presque partout. D'autre part la sommation directe de $F(0, N; x)$, en tant que progression géométrique, fournit le résultat connu

$$F(0, N; x) \leqq \frac{2}{|\sin \pi h x|} = O(1)$$

pour tout nombre irrationel x.

4. 3. Nous voulons donner encore un exemple en connexion avec le théorème 1 concernant le cas $\mathfrak{C}(M, N) = 1$. Dans ce cas on pourrait penser que $F(0, N; x) = O(1)$ et que le théorème 1 fournirait au moins le résultat $F(0, N; x) = o(\log N)^2$. La situation est cependant complètement différente, comme le montre l'exemple suivant dû à M. P. Erdös:

Soit $p > 1$ et posons $s_n = 1 + \frac{1}{2} + \ldots + 1/n$ $(n \geqslant 1)$; $a_n = s_n - [s_n]$, $b_n = s_{n+1} - [s_{n+1}]$ pour $n = 1, 2, \ldots$. Si $a_n < b_n$, soit I_n l'intervalle $a_n \leqslant x < b_n$. Si au contraire $a_n > b_n$, désignons par I_n la réunion des intervalles $0 \leqslant x < b_n$, $a_n \leqslant x < 1$. Ainsi tout x de $(0, 1)$ appartient à une infinité d'ensembles I_n. Définissons dans $0 \leqslant x < 1$ les fonctions $f_n(x)$ de la manière suivante:

$$f_1(x) = 1$$

$$f_{2m+1}(x) = \begin{cases} (m+1)^{1/p} & \text{dans } I_m, \\ 0 & \text{pour les autres valeurs de } x; \end{cases}$$

et

$$f_{2m}(x) = -f_{2m-1}(x)$$

pour $m = 1, 2, 3, \ldots$.

Ainsi nous avons $\int_0^1 |f_n(x)|^p \, dx = 1$ $(n = 1, 2, \ldots)$ et

$$F(M, N; x) = \left| \sum_{n=M+1}^{M+N} f_n(x) \right| \leqslant |f_{M+1}(x)| + |f_{M+N}(x)|.$$

Par conséquent d'après l'inégalité de HÖLDER nous obtenons

$$F(M, N; x)^p \leqq 2^{p-1}\,(|f_{M+1}(x)|^p + |f_{M+N}(x)|^p)$$

donc

$$\int_0^1 F(M, N; x)^p\,dx \leqq 2^{p-1}\,\{\int_0^1 |f_{M+1}(x)|^p\,dx + \int_0^1 |f_{M+N}(x)|^p\,dx\} = 2^p$$

D'autre part, on a pour N impair $F(0, N; x) = f_N(x)$ c'est à dire que pour tout x il y a une infinité de valeurs de N telles que

$$F(0, N; x) > 2^{-p} N^{1/p}. \tag{22}$$

En utilisant le théorème 2 avec

$$\Psi(N) = N,\ \Phi^*(M, N) = c N^{-1},\ \psi(N) = O(1) \text{ et } \chi(N) = \log^{1+\varepsilon} N$$

on trouve que

$$F(0, N, x) = o\,(N \log^{p+\varepsilon} N)^{1/p}$$

presque partout dans (0, 1). Cette estimation sans connaître l'inégalité (22) pourrait sembler être un mauvais résultat.

4. 4. Il y a naturellement des cas où le théorème 1 donne des bornes de l'ordre $O(\log N)^\delta$ $(0 \leqslant \delta)$ pour la function $F(0, N; x)$. Tel est le cas classique de MENCHOFF-RADEMACHER, où

$$F(M, N; x) = \sum_{n=M+1}^{M+N} c_n \varphi_n(x),\ \sum c_n^2 \log^2 n < \infty,\ \int_0^1 \varphi_m \varphi_n = 0 \text{ et } \int_0^1 \varphi_n^2 = 1.$$

Pour mentionner un autre cas, soit par exemple

$$\int_0^1 F(M, N; x)^2\,dx \leqq \Phi(M, N)$$

où

$$\Phi(M, N) \leqq \begin{cases} c \text{ pour } M = 0,\ N \geqslant 1 \\ c/M \text{ pour } M \geqslant 1,\ 1 \leqslant N < M. \end{cases}$$

Dans ce cas on peut utiliser le théorème 1 avec

$$\varkappa(n, \lambda) = 1,\ K(N) = O(\log N) \quad \text{et} \quad \varphi(N) = (\log N)^{1+\varepsilon}.$$

Montrons que B est valable. En effet

$$\sum_{n=1}^{\infty} (\Phi(0, 2^n) + \sum_{\lambda=1}^{n} \varkappa(n, \lambda) \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} \Phi(2^n + \mu_\lambda 2^\lambda, 2^{\lambda-1}))\,\varphi(2^n)^{-1} \leqslant$$

$$\leqslant c \sum_{n=1}^{\infty} (\log 2^n)^{-1-\varepsilon} + c \sum_{n=1}^{\infty} (\log 2^n)^{-1-\varepsilon} \sum_{\lambda=1}^{n} \sum_{\mu_\lambda=0}^{2^{n-\lambda}-1} (2^n + \mu_\lambda 2^\lambda)^{-1} \leqslant$$

$$\leqslant \frac{c}{(\log 2)^{1+\varepsilon}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} + \frac{c}{(\log 2)^{1+\varepsilon}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \sum_{\lambda=1}^{n} 2^{-\lambda} < \infty.$$

Donc on a presque partout $F(0, N; x) = o\,(\log^{2+\varepsilon} N)$.

4. 5. Dans notre note [5]) nous avons annoncé le théorème 1 dans une forme plus générale mais plus compliquée. Au lieu d'un développement dyadique nous avons utilisé un développement analogue en remplaçant la fonction 2^n une fois par une fonction croissante $\Omega(n)$ et l'autre fois par une pareille fonction $\psi(n)$. Parce que nous n'avons pas rencontré un exemple où l'introduction des développements nommés est absolument nécessaire, nous avons préféré de ne pas compliquer l'exposé et nous avons décidé de publier ici le théorème dans la forme du théorème 1 ci dessus. [15])

Au lieu de cette généralisation il est beaucoup plus intéressant de considérer la forme à "plusieurs paramètres" du théorème: au lieu de la suite $f_n(x)$ on considère $f_{n_1,n_2}(x)$ $(n_1, n_2 = 1, 2, \ldots)$ et la somme

$$F(M_1, N_1, M_2, N_2; x) = \left| \sum_{n_1=M_1+1}^{M_1+N_1} \sum_{n_2=M_2+1}^{M_2+N_2} f_{n_1,n_2}(x) \right|.$$

L'étude même n'apporte aucune nouvelle difficulté, il faut seulement utiliser l'inégalité de HÖLDER deux fois de suite. On peut envisager plus généralement certaines fonctions $F(M_1, N_1, \ldots, M_k, N_k; x)$. Une classe de ces fonctions fournit des renseignements sur la discrépance de la répartition modulo 1. Nous reviendrons cependant à ce problème à un autre endroit.

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[15]) La dernière formule à page 1322 de la note [5]) il faut lire comme suit:

$$\Lambda_k(K) = \sum_{\substack{K_\lambda \in T_\lambda \\ K_\lambda \text{ corresp. à } K}} \left\{ \frac{\Phi(0, \Psi(k)+K_\lambda)^{1/p}}{\Phi(0, \Psi(k)+K)^{1/p}} \frac{\varphi(\Psi(k)+K_\lambda)}{\varphi(\Psi(k)+K)} \right\}^{\frac{p}{p-1}}.$$

ON INTEGRALS OCCURRING IN THE THEORY OF DIFFRACTION OF ELECTROMAGNETIC WAVES BY A CIRCULAR DISK

BY

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In an earlier paper [1] certain integrals have been evaluated occurring in the theory of acoustic radiation by a freely vibrating circular disk and related problems [2]. Similar integrals are encountered in the rigorous theory of diffraction of a plane-polarized electromagnetic wave by circular disks and apertures. The purpose of the present paper is to express the latter integrals in terms of known functions. Details of the physical problem will appear elsewhere.

The integral in question is

$$(1)\qquad \left\{\begin{array}{l} I(n,m,\mu;\varrho,\varphi)= \\ \displaystyle\int_0^1 \frac{P_{2n}^{2m}(\sqrt{1-\varrho'^2})}{\sqrt{1-\varrho'^2}}\,\varrho'\,d\varrho' \int_0^{2\pi} \cos 2m\varphi' [\varrho^2-2\varrho\varrho'\cos(\varphi-\varphi')+\varrho'^2]^{\frac{1}{2}(\mu-1)}\,d\varphi', \end{array}\right.$$

in which n, m, and μ are integers subject to $0 \leqslant m \leqslant n$, $\mu \geqslant 0$, while $0 \leqslant \varrho \leqslant 1$, $0 \leqslant \varphi < 2\pi$; P_{2n}^{2m} is Ferrers's associated Legendre function [3, p. 323].

Two different expressions will be derived for $I(n, m, \mu; \varrho, \varphi)$. The first of them is an expansion in Legendre functions, while the second is a hypergeometric series, namely,

$$(2)\qquad I(n,m,\mu;\varrho,\varphi)=\sum_\nu A_\nu(n,m,\mu)\,P_{2\nu}^{2m}(\sqrt{1-\varrho^2})\cos 2m\varphi,$$

$$(3)\qquad \left\{\begin{array}{l} I(n,m,\mu;\varrho,\varphi)= \\ B(n,m,\mu)\,\varrho^{2m}\,F(-\frac{1}{2}\mu+n+m+\frac{1}{2},-\frac{1}{2}\mu-n+m;2m+1;\varrho^2)\cos 2m\varphi, \end{array}\right.$$

in which the coefficients A_ν and B are expressible in terms of gamma functions as follows:

$$(4)\qquad \left\{\begin{array}{l} A_\nu(n,m,\mu)= \\ \dfrac{\pi(-1)^{n+\nu}(2\nu+\frac{1}{2})\,\Gamma(\mu+1)\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(n+m+\frac{1}{2})\,\Gamma(\nu-m+\frac{1}{2})}{\Gamma(n-m+1)\,\Gamma(\nu+m+1)\,\Gamma(\frac{1}{2}\mu-n-\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-\nu+1)\,\Gamma(\frac{1}{2}\mu-n+\nu+1)\,\Gamma(\frac{1}{2}\mu+n+\nu+\frac{3}{2})} \end{array}\right.$$

$$(5)\qquad \left\{\begin{array}{l} B(n,m,\mu)= \\ \dfrac{\pi(-1)^{n+m}\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(n+m+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\,\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(\frac{1}{2}\mu-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-m+1)}. \end{array}\right.$$

It is evident from these expressions that $I(n, m, \mu; \varrho, \varphi)$ is a polynomial in ϱ^2. In eq. (2) the summation over ν must be taken according to $\max(m, n - \frac{1}{2}\mu) \leqslant \nu \leqslant n + \frac{1}{2}\mu$ (μ even) or $m \leqslant \nu \leqslant \frac{1}{2}(\mu-1)-n$ (μ odd). In the latter case the integral is zero if $m + n > \frac{1}{2}(\mu - 1)$.

To prove eq. (2) we need several lemmas.

Lemma I. Let J denote the Bessel function of the first kind; then

$$(6) \quad J_{2m}(\varrho t) = 2^{-2m+\frac{1}{2}} t^{-\frac{1}{2}} \sum_{\nu=m}^{\infty} (2\nu+\tfrac{1}{2}) \frac{\Gamma(\nu-m+\frac{1}{2})}{\Gamma(\nu+m+1)} J_{2\nu+\frac{1}{2}}(t) P_{2\nu}^{2m}(\sqrt{1-\varrho^2}).$$

Lemma II.

$$(7) \quad \left\{ \begin{aligned} & P_{2\nu}^{2m}(\sqrt{1-\varrho^2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+m+\frac{1}{2})\,\Gamma(\nu+m+1)}{\Gamma(m+\frac{1}{2})\,\Gamma(m+1)\,\Gamma(\nu-m+\frac{1}{2})\,\Gamma(\nu-m+1)} \times \\ & \times \varrho^{2m} F(\nu+m+\tfrac{1}{2}, -\nu+m; 2m+1; \varrho^2). \end{aligned} \right.$$

Proof. Eq. (7) is an immediate consequence of relations known from the theory of spherical harmonics; see Hobson [4, pp. 205, 219]. Eq. (6) follows at once from a general formula given by Watson [5, p. 140] if use is made of Lemma II [1]). For an independent proof of eq. (6), see Bouwkamp [6, p. 268].

Lemma III. Let n and ν denote non-negative integers, $k \geqslant 0$, then

$$(8) \quad \left\{ \begin{aligned} & \int_0^{\infty} \frac{J_{2n+\frac{1}{2}}(t) J_{2\nu+\frac{1}{2}}(t)}{\sqrt{t^2-k^2}}\, dt = \\ & \sum_{\mu=0}^{\infty} \frac{\frac{1}{2}(-1)^{n+\nu}\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,(ik)^{\mu}}{\Gamma(\frac{1}{2}\mu+n-\nu+1)\,\Gamma(\frac{1}{2}\mu-n+\nu+1)\,\Gamma(\frac{1}{2}\mu-n-\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n+\nu+\frac{3}{2})} \end{aligned} \right.$$

($\sqrt{t^2-k^2}$ is defined as $-i\sqrt{k^2-t^2}$ when $0 \leqslant t \leqslant k$).

Proof. According to Watson [5, p. 436] one has, if $-\frac{1}{2} < \mathrm{Re}(\mu+\nu) < 0$,

$$(9) \quad J_{2\mu}(t) J_{2\nu}(t) = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\,\Gamma(2s+2\mu+2\nu+1)\,(\frac{1}{2}t)^{2\mu+2\nu+2s}}{\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)}\, ds,$$

in which the contour of integration coincides with the imaginary axis except for an indentation (toward the left) at the origin. Using some transformations of gamma-function theory, we get

$$J_{2\mu}(t) J_{2\nu}(t) =$$

$$-\frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\frac{1}{2})\,\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+1)\,t^{2s+2\mu+2\nu}}{\Gamma(s+1)\,\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)} \frac{ds}{\sin \pi s},$$

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[1]) There is a printer's error in eq. (19) of the previous paper [1]; for P_{2n+1} read P_{2n}.

and consequently

$$\int_0^\infty \frac{J_{2\mu}(t)\,J_{2\nu}(t)}{\sqrt{t^2-k^2}}\,dt =$$

$$-\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(\frac{1}{2})\,\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+1)}{\Gamma(s+1)\,\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)}\,\frac{ds}{\sin\pi s}\int_0^\infty \frac{t^{2s+2\mu+2\nu}}{\sqrt{t^2-k^2}}\,dt.$$

Now it is readily shown that

$$\int_0^\infty \frac{t^{2\lambda}}{\sqrt{t^2-k^2}}\,dt = -\tfrac{1}{2}\,k^{2\lambda}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}\,\frac{e^{-i\lambda\pi}}{\sin\pi\lambda} \qquad (-\tfrac{1}{2} < \operatorname{Re}\lambda < 0)$$

so that

$$\int_0^\infty \frac{J_{2\mu}(t)\,J_{2\nu}(t)}{\sqrt{t^2-k^2}}\,dt =$$

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+\frac{1}{2})}{\Gamma(s+1)\,\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)}\,\frac{\pi\,(k^2\,e^{-i\pi})^{s+\mu+\nu}\,ds}{2\sin\pi s\,\sin\pi(s+\mu+\nu)}.$$

The poles of the integrand to the right of the imaginary axis are located at $s = n$ and $s = n-\mu-\nu$ $(n = 0, 1, \ldots)$. Closing the contour toward the right and applying the residue theorem yields

$$2\sin\pi(\mu+\nu)\int_0^\infty \frac{J_{2\mu}(t)\,J_{2\nu}(t)}{\sqrt{t^2-k^2}}\,dt =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{1}{2})\,k^{2n}}{\Gamma(n-\mu-\nu+1)\,\Gamma(n+\mu-\nu+1)\,\Gamma(n-\mu+\nu+1)\,\Gamma(n+\mu+\nu+1)} -$$

$$-e^{-i(\mu+\nu)\pi}\,k^{2\mu+2\nu}\sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(n+\mu+\nu+\frac{1}{2})\,\Gamma(n+\mu+\nu+\frac{1}{2})\,k^{2n}}{\Gamma(n+1)\,\Gamma(n+2\mu+1)\,\Gamma(n+2\nu+1)\,\Gamma(n+2\mu+2\nu+1)}.$$

By analytical continuation, this formula may be shown to hold for $\operatorname{Re}(\mu+\nu) > -\frac{1}{2}$. In particular, the left-hand member of (8) is equal to

$$\sum_{n=0}^{\infty} \frac{\frac{1}{2}(-1)^{n+\nu}\,\Gamma(\mu+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})\,(ik)^{2\mu}}{\Gamma(\mu-n-\nu+\frac{1}{2})\,\Gamma(\mu+n-\nu+1)\,\Gamma(\mu-n+\nu+1)\,\Gamma(\mu+n+\nu+\frac{3}{2})} +$$

$$+\sum_{n=0}^{\infty} \frac{\frac{1}{2}(-1)^{n+\nu}\,\Gamma(\mu+n+\nu+1)\,\Gamma(\mu+n+\nu+1)\,(ik)^{2\mu+2n+2\nu+1}}{\Gamma(\mu+1)\,\Gamma(\mu+2n+\frac{3}{2})\,\Gamma(\mu+2\nu+\frac{3}{2})\,\Gamma(\mu+2n+2\nu+2)}.$$

In the last sum the terms corresponding to $-n-\nu \leqslant \mu \leqslant -1$ may be added since these terms are identically zero. The modified sum then contains all positive odd powers of ik, while the first sum consists of all

even powers of ik. The two series can be combined so as to lead to the right-hand member of eq. (8).

We shall now prove eqs (2) and (4). To this end we observe that the integrals $I(n, m, \mu; \varrho, \varphi)$ are generated by the function

$$(10)\qquad \begin{cases} G = \displaystyle\int_0^1 \frac{P_{2n}^{2m}(\sqrt{1-\varrho'^2})}{\sqrt{1-\varrho'^2}}\,\varrho'\,d\varrho' \int_0^{2\pi} \cos 2m\varphi' \,\frac{e^{iks}}{s}\,d\varphi' = \\ \quad = \displaystyle\sum_{\mu=0}^{\infty} I(n, m, \mu; \varrho, \varphi)\,\frac{(ik)^\mu}{\Gamma(\mu+1)}, \end{cases}$$

where s is an abbreviation for $[\varrho^2 - 2\varrho\varrho'\cos(\varphi-\varphi') + \varrho'^2]^{\frac{1}{2}}$. Using the known relation

$$\frac{e^{iks}}{s} = \int_0^\infty J_0(st)\,\frac{t\,dt}{\sqrt{t^2-k^2}},$$

in which the square root is defined as in Lemma III, and applying the addition theorem of Bessel functions, we find that

$$\int_0^{2\pi} \cos 2m\varphi' \,\frac{e^{iks}}{s}\,d\varphi' =$$

$$= \int_0^\infty \frac{t\,dt}{\sqrt{t^2-k^2}} \int_0^{2\pi} \cos 2m\varphi' \; J_0\left(t\sqrt{\varrho^2 - 2\varrho\varrho'\cos(\varphi-\varphi') + \varrho'^2}\right) d\varphi' =$$

$$= 2\pi \cos 2m\varphi \int_0^\infty J_{2m}(\varrho t)\, J_{2m}(\varrho' t)\,\frac{t\,dt}{\sqrt{t^2-k^2}}.$$

Consequently, in virtue of Lemma I and using the normalization properties of the Legendre functions, we obtain

$$(11)\qquad \begin{cases} G = 2\pi\cos 2m\varphi \displaystyle\int_0^\infty J_{2m}(\varrho t)\,\frac{t\,dt}{\sqrt{t^2-k^2}} \int_0^1 \frac{P_{2n}^{2m}(\sqrt{1-\varrho'^2})}{\sqrt{1-\varrho'^2}}\, J_{2m}(\varrho' t)\,\varrho'\,d\varrho' = \\ \quad = 2^{2m+\frac{1}{2}}\pi\cos 2m\varphi\,\dfrac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+1)} \displaystyle\int_0^\infty J_{2n+\frac{1}{2}}(t)\, J_{2m}(\varrho t)\,\frac{t^{\frac{1}{2}}\,dt}{\sqrt{t^2-k^2}}. \end{cases}$$

Once more applying Lemma I, we are led to

$$(12)\quad G = \pi\cos 2m\varphi \sum_{\nu=m}^{\infty} (4\nu+1)\, C_{n,\nu}(k)\,\frac{\Gamma(n+m+\frac{1}{2})\,\Gamma(\nu-m+\frac{1}{2})}{\Gamma(n-m+1)\,\Gamma(\nu+m+1)}\, P_{2\nu}^{2m}(\sqrt{1-\varrho^2}),$$

in which $C_{n,\nu}(k)$ is the integral considered in Lemma III. If the power series (8) is substituted in (12) and the resulting expansion compared with the right-hand member of (9), we at once obtain eqs (2) and (4).

Though it might be possible to prove eq. (3) directly from eq. (2), since both are essentially polynomials in ϱ^2, we prefer to establish a

different proof, based on eq. (11) [2]). To that end we start with the relation

$$(13)\quad \left\{ \begin{aligned} & J_{2n+\frac{1}{2}}(t)\, J_{2m}(\varrho t) = \\ & = -\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\varrho^{2m}(\frac{1}{2}t)^{2s+2n+2m+\frac{1}{2}}\, F(-s,-s-2n-\frac{1}{2};2m+1;\varrho^2)}{\Gamma(2m+1)\,\Gamma(s+1)\,\Gamma(s+2n+\frac{3}{2})}\,\frac{\pi}{\sin \pi s}\,ds, \end{aligned} \right.$$

in which the contour of integration is the same as in eq. (9), while for the present moment we confine ourselves to the case $-1 < \mathrm{Re}(n+m) < -\frac{1}{2}$. The truth of eq. (12) is evident, as may be seen by applying the residue theorem and comparing the resulting series with a known expression for the product of two Bessel functions, given by Watson [5, p. 148].

As a consequence

$$\int_0^\infty J_{2n+\frac{1}{2}}(t)\, J_{2m}(\varrho t)\,\frac{t^{\frac{1}{2}}\,dt}{\sqrt{t^2-k^2}} =$$

$$-\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\varrho^{2m}\,F(-s,-s-2n-\frac{1}{2};2m+1;\varrho^2)}{\Gamma(2m+1)\,\Gamma(s+1)\,\Gamma(s+2n+\frac{3}{2})\,2^{2s+2n+2m+\frac{1}{2}}}\,\frac{\pi\,ds}{\sin \pi s}\int_0^\infty \frac{t^{2s-2n-2m+1}}{\sqrt{t^2-k^2}}\,dt =$$

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\frac{1}{2})\,\Gamma(s+n+m+1)\,\varrho^{2m}\,F(-s,-s-2n-\frac{1}{2};2m+1;\varrho^2)}{2^{\frac{1}{2}}\,\Gamma(2m+1)\,\Gamma(s+1)\,\Gamma(s+2n+\frac{3}{2})\,\Gamma(s+n+m+\frac{3}{2})}\,\frac{\pi\,(\frac{1}{2}ke^{-i\pi/2})^{2s+2n+2m+1}}{\sin \pi s\,\sin \pi(s+n+m+\frac{1}{2})}\,ds.$$

Once more closing the contour toward the right, and taking the sum of the residues at the poles of the integrand to the right of the imaginary axis, we find for the value of the integral under consideration:

$$-\sum_{\tau=0}^{\infty}\frac{(-1)^\tau}{\sin \pi(\tau+n+m+\frac{1}{2})}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\tau+n+m+1)\,\varrho^{2m}\,F(-\tau,-\tau-2n-\frac{1}{2};2m+1;\varrho^2)}{2^{\frac{1}{2}}\,\Gamma(\tau+1)\,\Gamma(\tau+2n+\frac{3}{2})\,\Gamma(\tau+n+m+\frac{3}{2})\,\Gamma(2m+1)}\times$$

$$\times\,(\tfrac{1}{2}ke^{-i\pi/2})^{2\tau+2n+2m+1}$$

$$-\sum_{\tau=0}^{\infty}\frac{(-1)^\tau}{\sin \pi(\tau-n-m-\frac{1}{2})}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\tau+\frac{1}{2})\,\varrho^{2m}\,F(-\tau+n+m+\frac{1}{2},-\tau-n+m;2m+1;\varrho^2)}{2^{\frac{1}{2}}\,\Gamma(\tau-n-m+\frac{1}{2})\,\Gamma(\tau+n-m+1)\,\Gamma(\tau+1)\,\Gamma(2m+1)}\times$$

$$\times\,(\tfrac{1}{2}ke^{-i\pi/2})^{2\tau}.$$

By analytical continuation, this expression may be shown to represent the integral for $\mathrm{Re}\,(n+m) > -1$. In particular, for non-negative integers n and m, we have for the sum of the two series:

$$\sum_{\tau=0}^{\infty}\frac{(-1)^{n+m}\,\Gamma(\frac{1}{2})\,\Gamma(\tau+n+m+1)\,\varrho^{2m}\,F(-\tau,-\tau-2n-\frac{1}{2};2m+1;\varrho^2)}{2^{\frac{1}{2}}\,\Gamma(2m+1)\,\Gamma(\tau+1)\,\Gamma(\tau+2n+\frac{3}{2})\,\Gamma(\tau+n+m+\frac{3}{2})}\,(\tfrac{1}{2}ik)^{2\tau+2n+2m+1}+$$

$$\sum_{\tau=0}^{\infty}\frac{(-1)^{n+m}\,\Gamma(\frac{1}{2})\,\Gamma(\tau+\frac{1}{2})\,\varrho^{2m}\,F(-\tau+n+m+\frac{1}{2},-\tau-n+m;2m+1;\varrho^2)}{2^{\frac{1}{2}}\,\Gamma(2m+1)\,\Gamma(\tau-n-m+\frac{1}{2})\,\Gamma(\tau+n-m+1)\,\Gamma(\tau+1)}\,(\tfrac{1}{2}ik)^{2\tau}=$$

$$\sum_{\mu=0}^{\infty}\frac{(-1)^{n+m}\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})\,\varrho^{2m}\,F(-\frac{1}{2}\mu+n+m+\frac{1}{2},-\frac{1}{2}\mu-n+m;2m+1;\varrho^2)}{2^{\mu+\frac{1}{2}}\,\Gamma(2m+1)\,\Gamma(\frac{1}{2}\mu-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-m+1)\,\Gamma(\frac{1}{2}\mu+1)}\,(ik)^\mu.$$

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[2]) The author is indebted to Dr Bremmer for an independent proof of eq. (3) using operational calculus.

If this series is substituted in eq. (11), a power series for G is obtained which, when compared with the expansion of eq. (10), will lead to eq. (3).

In view of the physical applications referred to, it is important to know the effect of the two-dimensional Laplace operator of potential theory when applied to the function $I(n, m, \mu; \varrho, \varphi)$, considered as a function of the polar coordinates ϱ, φ on the circular disk of unit radius. The operator mentioned is given by

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2}{\partial \varphi^2},$$

while the formula in question is

$$\triangle I(n, m, \mu; \varrho, \varphi) = (\mu-1)^2 I(n, m, \mu-2; \varrho, \varphi). \tag{14}$$

The proof of eq. (14) is quite simple. If s is written in terms of rectangular coordinates instead of polar coordinates, $s = [(x-x')^2 + (y-y')^2]^{\frac{1}{2}}$, it is easy to see that

$$\triangle s^{\mu-1} = (\mu-1)^2 s^{\mu-3},$$

which in combination with eq. (1) leads at once to eq. (14). The proof of the corresponding relation in the previous paper [1, eq. (2)] is unnecessarily complicated.

Finally, if in the physical problem the direction of propagation of the incident electromagnetic wave is normal to the plane of the disk, we need explicit values of the functions $I(n, m, \mu; \varrho, \varphi)$ for $m = 0$ and $m = 1$. The first few of them may be taken from the list below, in which the function J is defined by

$$I(n, m, \mu; \varrho, \varphi) = \pi^2 \cos 2m\varphi \; J(n, m, \mu). \tag{15}$$

The argument, $\sqrt{1-\varrho^2}$, of the Legendre functions is omitted throughout.

$$J(0, 0, 0) = P_0^0 = 1$$

$$J(1, 0, 0) = \frac{1}{4} P_2^0 = \frac{1}{4}\left(1 - \frac{3}{2}\varrho^2\right)$$

$$J(2, 0, 0) = \frac{9}{64} P_4^0 = \frac{9}{64}\left(1 - 5\varrho^2 + \frac{35}{8}\varrho^4\right)$$

$$J(3, 0, 0) = \frac{25}{256} P_6^0 = \frac{25}{256}\left(1 - \frac{21}{2}\varrho^2 + \frac{189}{8}\varrho^4 - \frac{231}{16}\varrho^6\right)$$

$$J(4, 0, 0) = \frac{1225}{16384} P_8^0 = \frac{1225}{16384}\left(1 - 18\varrho^2 + \frac{297}{4}\varrho^4 - \frac{429}{4}\varrho^6 + \frac{6435}{128}\varrho^8\right)$$

$$J(1, 1, 0) = \frac{3}{8} P_2^2 = \frac{9}{8}\varrho^2$$

$$J(2, 1, 0) = \frac{5}{32} P_4^2 = \frac{225}{32}\varrho^2\left(1 - \frac{7}{6}\varrho^2\right)$$

$$J(3, 1, 0) = \frac{105}{1024} P_6^2 = \frac{11025}{512}\varrho^2\left(1 - 3\varrho^2 + \frac{33}{16}\varrho^4\right)$$

$$J(4, 1, 0) = \frac{315}{4096} P_8^2 = \frac{99225}{2048}\varrho^2\left(1 - \frac{11}{2}\varrho^2 + \frac{143}{16}\varrho^4 - \frac{143}{32}\varrho^6\right)$$

$$J(0,0,1) = \frac{2}{\pi} P_0^0 = \frac{2}{\pi}$$

$$J(n,m,1) = 0 \quad (n+m>0)$$

$$J(0,0,2) = \frac{2}{3} P_0^0 - \frac{1}{6} P_2^0 = \frac{1}{2}\left(1+\frac{1}{2}\varrho^2\right)$$

$$J(1,0,2) = -\frac{1}{30} P_0^0 - \frac{1}{42} P_2^0 - \frac{3}{560} P_4^0 = -\frac{1}{16}\left(1-\varrho^2+\frac{3}{8}\varrho^4\right)$$

$$J(2,0,2) = -\frac{1}{336} P_2^0 - \frac{9}{2464} P_4^0 - \frac{5}{4224} P_6^0 = -\frac{1}{128}\left(1-\frac{9}{2}\varrho^2+\frac{45}{8}\varrho^4-\frac{35}{16}\varrho^6\right)$$

$$J(3,0,2) = -\frac{15}{18304} P_4^0 - \frac{5}{4224} P_6^0 - \frac{35}{79872} P_8^0 =$$

$$= -\frac{5}{2048}\left(1-10\varrho^2+\frac{105}{4}\varrho^4-\frac{105}{4}\varrho^6+\frac{1155}{128}\varrho^8\right)$$

$$J(1,1,2) = -\frac{1}{28} P_2^2 - \frac{1}{560} P_4^2 = -\frac{3}{16}\varrho^2\left(1-\frac{1}{2}\varrho^2\right)$$

$$J(2,1,2) = -\frac{5}{336} P_2^2 - \frac{5}{1232} P_4^2 - \frac{5}{8448} P_6^2 = -\frac{45}{128}\varrho^2\left(1-\frac{5}{3}\varrho^2+\frac{35}{48}\varrho^4\right)$$

$$J(3,1,2) = -\frac{35}{18304} P_4^2 - \frac{7}{5632} P_6^2 - \frac{7}{26624} P_8^2 =$$

$$= -\frac{525}{1024}\varrho^2\left(1-\frac{7}{2}\varrho^2+\frac{63}{16}\varrho^4-\frac{231}{160}\varrho^6\right)$$

$$J(0,0,3) = \frac{8}{3\pi} P_0^0 - \frac{4}{3\pi} P_2^0 = \frac{4}{3\pi}\left(1+\frac{3}{2}\varrho^2\right)$$

$$J(1,0,3) = -\frac{4}{15\pi} P_0^0 = -\frac{4}{15\pi}$$

$$J(n,m,3) = 0 \quad (n+m>1)$$

$$J(0,0,4) = \frac{6}{5} P_0^0 - \frac{6}{7} P_2^0 + \frac{9}{280} P_4^0 = \frac{3}{8}\left(1+3\varrho^2+\frac{3}{8}\varrho^4\right)$$

$$J(1,0,4) = -\frac{6}{35} P_0^0 + \frac{1}{14} P_2^0 + \frac{9}{1540} P_4^0 + \frac{1}{2464} P_6^0 =$$

$$= -\frac{3}{32}\left(1+\frac{3}{2}\varrho^2-\frac{3}{8}\varrho^4+\frac{1}{16}\varrho^6\right)$$

$$J(2,0,4) = \frac{1}{280} P_0^0 + \frac{1}{308} P_2^0 + \frac{243}{160160} P_4^0 + \frac{1}{2464} P_6^0 + \frac{7}{146432} P_8^0 =$$

$$= \frac{9}{1024}\left(1-2\varrho^2+\frac{9}{4}\varrho^4-\frac{5}{4}\varrho^6+\frac{35}{128}\varrho^8\right)$$

$$J(1,1,4) = \frac{3}{28} P_2^2 + \frac{3}{1540} P_4^2 + \frac{3}{49280} P_6^2 = \frac{27}{64}\varrho^2\left(1-\frac{1}{3}\varrho^2+\frac{1}{16}\varrho^4\right)$$

$$J(2,1,4) = \frac{5}{308} P_2^2 + \frac{27}{16016} P_4^2 + \frac{1}{4928} P_6^2 + \frac{1}{73216} P_8^2 =$$

$$= \frac{45}{256}\varrho^2\left(1-\frac{3}{2}\varrho^2+\frac{15}{16}\varrho^4-\frac{7}{32}\varrho^6\right)$$

$$J(0,0,5)=\frac{256}{45\pi}P_0^0-\frac{320}{63\pi}P_2^0+\frac{16}{35\pi}P_4^0=\frac{16}{15\pi}\left(1+5\varrho^2+\frac{15}{8}\varrho^4\right)$$

$$J(1,0,5)=-\frac{64}{63\pi}P_0^0+\frac{32}{45\pi}P_2^0=-\frac{32}{105\pi}\left(1+\frac{7}{2}\varrho^2\right)$$

$$J(2,0,5)=\frac{16}{315\pi}P_0^0=\frac{16}{315\pi}$$

$$J(1,1,5)=\frac{16}{15\pi}P_2^2=\frac{16}{5\pi}\varrho^2$$

$$J(n,m,5)=0\quad(n+m>2)$$

$$J(0,0,6)=\frac{20}{7}P_0^0-\frac{125}{42}P_2^0+\frac{135}{308}P_4^0-\frac{25}{3696}P_6^0=\frac{5}{16}\left(1+\frac{15}{2}\varrho^2+\frac{45}{8}\varrho^4+\frac{5}{16}\varrho^6\right)$$

$$J(1,0,6)=-\frac{25}{42}P_0^0+\frac{125}{231}P_2^0-\frac{675}{16016}P_4^0-\frac{5}{3696}P_6^0-\frac{5}{109824}P_8^0=$$

$$=-\frac{25}{256}\left(1+6\varrho^2+\frac{9}{4}\varrho^4-\frac{1}{4}\varrho^6+\frac{3}{128}\varrho^8\right)$$

$$J(1,1,6)=\frac{125}{154}P_2^2-\frac{225}{16016}P_4^2-\frac{1}{4928}P_6^2-\frac{1}{256256}P_8^2=$$

$$=\frac{225}{128}\varrho^2\left(1+\frac{1}{2}\varrho^2-\frac{1}{16}\varrho^4+\frac{1}{160}\varrho^6\right)$$

$$J(0,0,7)=\frac{512}{35\pi}P_0^0-\frac{256}{15\pi}P_2^0+\frac{192}{55\pi}P_4^0-\frac{32}{231\pi}P_6^0=$$

$$=\frac{32}{35\pi}\left(1+\frac{21}{2}\varrho^2+\frac{105}{8}\varrho^4+\frac{35}{16}\varrho^6\right)$$

$$J(1,0,7)=-\frac{256}{75\pi}P_0^0+\frac{128}{35\pi}P_2^0-\frac{96}{175\pi}P_4^0=-\frac{32}{105\pi}\left(1+9\varrho^2+\frac{63}{8}\varrho^4\right)$$

$$J(1,1,7)=\frac{192}{35\pi}P_2^2-\frac{32}{175\pi}P_4^2=\frac{288}{35\pi}\varrho^2\left(1+\frac{7}{6}\varrho^2\right).$$

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Eindhoven, March, 1950.

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TECTONICS OF A PRETECTONIC ORTHOGNEISS MASSIF NEAR ST. JEAN DU GARD IN THE SOUTHEASTERN CEVENNES, FRANCE. II

BY

D. DE WAARD

(Communicated by Prof. F. A. VENING MEINESZ at the meeting of February 25, 1950)

13. *The shape of the gneiss massif.*

As a whole the gneiss body is lenticular-shaped with irregularly receding and projecting parts. The outcrop is drawn out in the direction of the cleavage. Though the type of contact may change quickly within a short distance, in sections the contact is generally seen to be about parallel to the cleavage as well. In tectonogram fig. 13 the pole axes of contacts

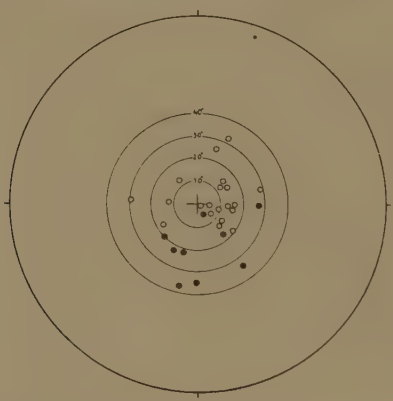

Fig. 13. Pole diagram with pole axes of northern (white) and southern contacts (black) after rotation with the average of cleavage to the centre of the diagram. Mainly a concordancy of contact planes to the general trend of cleavage. Somewhat steeper contact planes at the southern contact.

are shown in comparison to the average of cleavage of this area. All data are rotated, together with the rotation of the average N 84 E, 40 N to the centre of the diagram. Most contacts prove to be concentrated near this average. Exceptions, up to 40° are shown.

Not all contact planes could be measured direct. In many cases approximations according to the type of contact are used. The southern contact of the western part, for instance, appears to have a general tendency to be nestled. This is also illustrated by the structure map at the back. The cleavage of nestled parts, when approaching the contact, bend towards the direction of the contact plane. The measurements in the

material behaving most plastically, viz. the slate, closest to the contact, will thus give the approximate position of the contact plane in that place.

In the northern contact, concordant, nestled and crossed contacts are used in the diagram. As in the southern contacts, the cleavage directions of the slate closest to the contact are supposed to give approximations of the contact planes in places where nestling has been found. Concordant

Fig. 14. Block diagram of the gneiss massif of St. Jean du Gard (slates sunken and posttectonic granite eliminated).

contacts give direct data of the position of the contact plane. As far as crossed contacts are observed, the contact planes cross the cleavage generally at a low-angle (photo 4). Besides strike differences, steeper

as well as flatter (photo 4) dips in comparison with the cleavage have been found.

According to the diagram, the southern contacts show stronger differences from the general cleavage in the area than the northern contacts. Most of their pole axes are situated between 10° and 40° south of the centre. This indicates a generally steeper contact plane than the average of the cleavage at the southern contact of the gneiss body.

The only regular divergence from the average of cleavage in the northern contact is the concentration SE and E of the centre in the diagram. This concentration of mainly smaller strikes than the average of cleavage coincides with the northwestern direction of the contact of the eastern part of the gneiss body. Apart from this, the divergences are small and they do not show regularities. The contact planes are sometimes more, sometimes less inclined than the average of cleavage.

Summarizing these results, the gneiss body proves to be sheetlike, dipping to the north about parallel to the general cleavage of the area (fig. 14). The massif has a tendency to broaden towards the depth. The southern contact is about 10° to 40° steeper than the northern contact.

14. *The shape of the intrusion.*

In view of the intrusive origin of the massif, the reconstruction of the original shape of the granitic body will be discussed.

In the present outcrop the dimensions of the massif are roughly: 9.8 km long and 1.9 km wide. As the contact planes are generally inclined in the direction of the cleavage, the actual width is 1.9 sin 40 km or 1.2 km, which gives a ratio length/width of 8.1. As for the long western strip of gneiss this ratio is considerably larger, viz. about 10. As a whole the massif has a composite sheet-like shape. Though the amount of deformation during the orogenic phase is difficult to determine, an original roundish batholith strongly flattened in epizonal conditions is not very likely.

The lenticular body in the present outcrop resembles a laccolith. If a moderate deformation is supposed an original laccolithic magma chamber is highly probable. In that case magma intruded mainly in the bedding planes of the sediments, which would coincide largely with the general secondary foliation in this area. There are some indications supporting this hypothesis.

In several localities in the Mt. Aigoual area — 20 km WNW of this area — the secondary cleavage is found to be parallel to the primary bedding (DE WAARD, 1949*b*). Though well-separated beds of quartzite are not observed in this area, primary facies alterations in the slates proved to be about parallel with the cleavage. In general, important and wide-spread differences between bedding and cleavage do not seem likely in this area.

In one of the clearest examples of a crossed contact in the northeastern

wedge of gneiss, two systems of planes are visible (fig. 15). The most distinct are the cleavage planes crossing the contact at 25° into the hardend

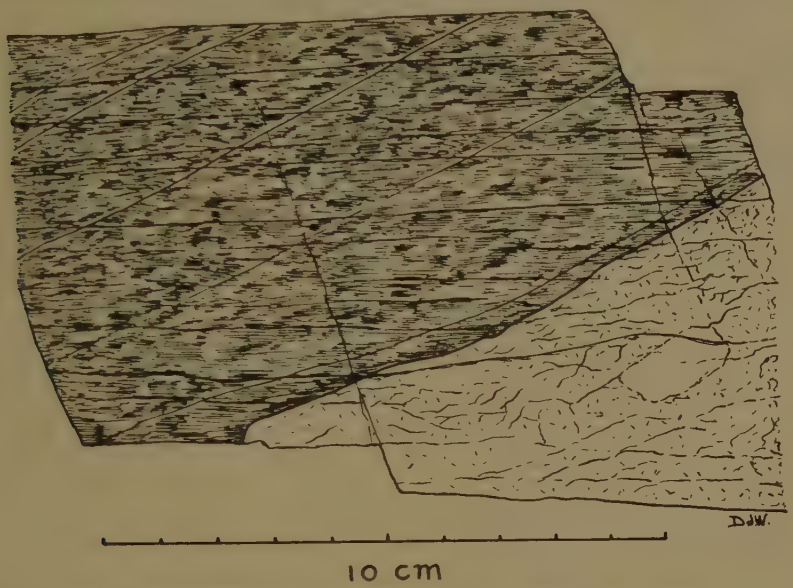

Fig. 15. Detail of a crossed contact with cleavage planes (horizontally) developed in both rocks. Stratification in the spotted country rock (left) and a crushed feldspar in the gneiss (right): probably ghosts of primary structures parallel to the contact plane.

contact rock. The second set of planes — crossing the cleavage — is parallel to the contact, but only visible in the country rock. They are probably ghosts of a primary bedding parallel to the contact plane.

At the end of the eastward pointing strip of slate near the centre of the massif, the contact is complicated by a strongly frayed contour. At least eight fingers of gneiss — several metres long — point downward and to the north, in the slate (fig. 16). In detail the contacts are con-

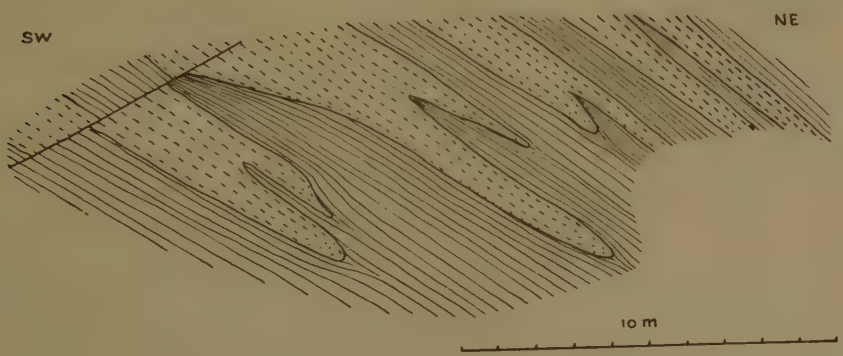

Fig. 16. Part of the frayed outline with dike-like fingers of gneiss (stippled) at the end of the eastward pointing strip of slate near the centre of the massif. Section at right angles to the fold axis. Plane table mapping by Mr. J. W. C. M. VAN DER SLIJP.

cordant at the sides and nestled at the end of the fingers (photo 3). In other places too, dike-like offshoots mostly parallel to the cleavage have been observed. These dikes or lenses of gneiss, connected with and parallel to the massif have a similar appearance as the whole massif on a small scale.

The gneiss body itself has a complicated outline. As far as still present it is composed of a number of large offshoots in the direction of the cleavage. From a central part two or three offshoots go to the west (fig. 14). The continuation of the structure to the south and east has been removed by the posttectonic intrusion. A wedge points from the north-eastern contact to the north. Especially the western part gives strongly an impression of a compound laccolith, with perhaps three sheet-like chambers above each other, sprouting from a central part of the laccolith.

A parallelism generally of bedding planes and cleavage except in crossed contacts where parallelism between contact and stratification is mentioned, together with a not exceptionally deformed lenticular shape of the massif in that direction suggest a laccolith as the original intrusive form. This suggestion is favoured by the composite shape of the body in large parallel sheets and by a number of smaller sheets of similar appearance parallel to the main parts.

15. *The origin of the cleavage pattern.*

The compaction of the clay after sedimentation and the development of cleavage planes during the orogeny may be considered as the essential effects in the mode of origin of the slates. The development of cleavage realized the structural pattern, which is represented by the intersecting lines of the cleavage planes with map or section.

The cleavage is usually considered as a predominant development of one of the shear planes in the strain ellipsoid. The material gives way to the stress by differential movements along this set of shear planes. Though it is not possible to define the exact direction of the stress, it is supposed to have been more or less parallel to the developed shear plane. Only one of the components of a divergent stress results in differential movements. The other component — acting perpendicular to the shear plane — causes compression of the material.

Without any other influence the cleavage pattern would show a set of parallel lines on the map. Every structure pattern in a crystalline area however is determined by a number of influences of which some — regionally as well as locally — may dominate the others.

In the area in question many more or less important influences can be traced, which affected the ideal set of parallel cleavage lines of the map. In a relatively homogeneous material a laccolith of granite intruded, attended with a doming up of strata. The body, with properties different from the intruded rock, has been the main cause of deformation of the ideal cleavage pattern. Both rocks reacted differently upon compaction,

flattening and shearing during the orogeny. Besides other — usually uncontrollable — inhomogeneities as a result of differences in material or a local domination of one of the mentioned influences may affect the structure pattern at certain points.

The influence of the primary bedding upon the cleavage pattern may be neglected in this case. The doming of strata by the intrusion of the laccolith would result in a perfectly enveloping structure. The material however is very homogeneous here. The development of a cleavage pattern may be supposed to have easily destroyed the primary doming structure and to have crossed the bedding planes only dependent on the direction of stress.

As the most important influences in the deformation of the cleavage pattern, will be considered here the effect of continued compaction during the folding, the lateral broadening and simultaneous thinning effect upon the tectonic unit by tension of the fold axis and the development of shear or schistosity planes in differently behaving materials as granite and slate upon stress.

Compaction started directly after sedimentation by gradual increase of overburden and is continued by orogenic compression. During the compaction the material undergoes a decrease of porosity and an increase of specific gravity by loss of free and adsorbed water and by recrystallization to denser mineral forms (Hedberg, 1936). The first stages of compaction have been calculated by Athy (1930). The porosity of clay decreases from 50 % at 100′ to 18 % at 3000′ and to 8 % at 8000′ depth of overburden.

The continuation of this process by recrystallization of the constituents into denser minerals probably takes place mainly during orogenic activities. The sequence, clay-shale-slate (phyllite), is characterized by a considerable increase of density from about 2.0 to 2.75. Especially the latter half of the sequence from shales to slates may be ascribed to combined orogenic activities. The specific gravities of slates and phyllites of the Cevennes area have been determined for an average of about 2.75. Compared with those, shales of a less transformed area in the Ardennes proved to have an average of about 2.5.

According to the preceding paragraph the emplacement of the laccolith occurred before the main orogenic phase. This orogenic phase is supposed to be the main cause of the cleavage, hence of the genesis of slates. Thus, during the time of intrusion the condition of the country rock must have been somewhere between that of clay and slate. A condition like a shale will be most likely.

If shale with a s.g. of 2.5 is transformed to slate with a s.g. of 2.75 without transport of components except of water, it has been compressed to $\frac{2.5}{2.75}$ or about 90 % of its volume during its genesis into slate. The amount of compression is determined by the orogenic stress and the increasing load of folded strata on top. The development of cleavage

at the same time means relief to the stress in the direction of the shear plane. The effect of the compression will thus be found at right angles to the cleavage plane as a result of the second component of the stress. During this process the laminae of the slaty rock thus flattened about 10 % of the original thickness.

In order to get a general idea of the structural consequences near the contacts of a rigid body by differential compaction, these rough calculations will be used in the area in question. The compaction of the gneiss may be neglected with respect to that of the country rock. An original thickness of 1000 metres shales decreased by compression to 900 metres of slaty country rock next to the gneiss body parallel to the cleavage. The effect of the missing 100 metres will be expressed by a bending of the cleavage round the corners of the rigid body.

Some theoretical examples of the structural behaviour near the contacts are constructed in fig. 17 *a*, *b*, *c* by a differential compaction of 10 %. The general enveloping tendency around such a rigid body is illustrated in fig. 17*c*. The origin of nestled contacts may be elucidated

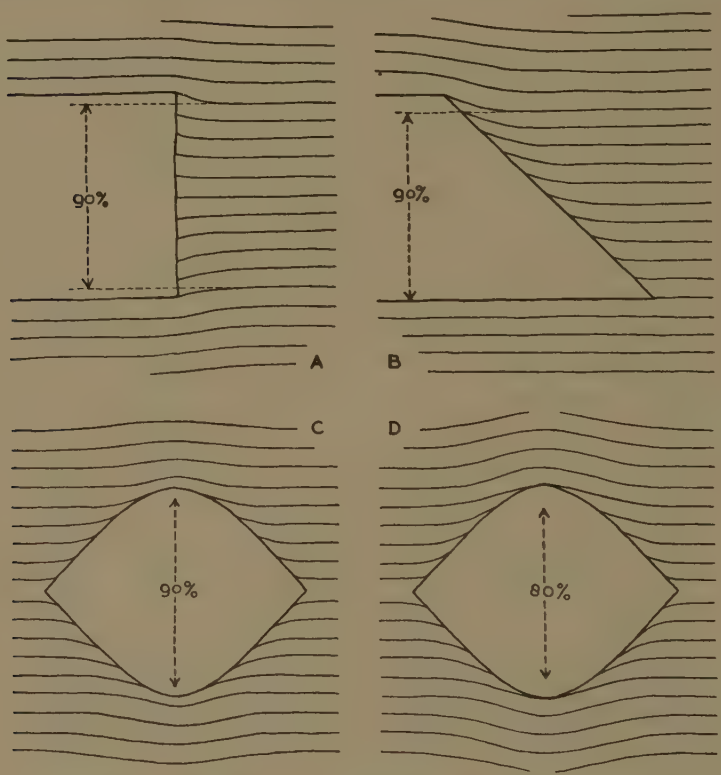

Fig. 17. Structures developed by differential compaction or flattening of an inhomogenious series. Compression is only supposed in the country rock surrounding a rigid body. An originally, parallel structure pattern in the country rock shows a deformation at the contacts where locally concordant, crossed and nestled structures are found. In *a*, *b* and *c*, a compression of 10 % and in *d* of 20 % has been constructed.

by fig. 17*b*. Along such contacts the connection between wall rock and body may easily dislocate to a faulted contact. Concordant contacts will be found along contacts parallel to the cleavage. Crossed contacts will develop especially in places where the displacement as caused by compaction is minimal. Neutral zones near the central line of a block will be favourable spots (fig. 17*a*).

Flattening of a series of a tectonic unit caused by extension may be of importance to interprete a structure pattern. This effect is mainly dependent to the extent of plastic behaviour of the material. Tension in brittle material develops tension joints. In plastic condition the material reacts upon tension with flow.

Extension is doubtless a common feature in tectonic units. Many examples e.g. torn out and extended fossils and crystals, prove its effect in those localities. In the upper parts of a series folded to an anticline for instance tension takes place (parallel to the *a* axis). Moreover, every bending, horizontal as well as vertical, of the fold axis of a freely folding unit causes lateral extension (parallel to the *b* axis). Usually it will be very difficult to determine the amount of the extension and the attended flattening of the series. A certain amount may be expected in many cases. But the exact rôle of the folding and that of the distortion of the fold axis, as well as the extent of freedom of folding in the unit and the actual plasticity of behaviour of the material are variable factors which are difficult to calculate.

In the area in question tension joints perpendicular to the fold axis have been observed. Still, a certain plasticity of behaviour must be supposed according to the nicely bended structures in the slates. Actual proofs of extension as well as plastic behaviour are shown by boudinage structures of torn out and fractured pretectonic dikes and veins.

The result of the extension of the series either parallel to the *b* axis, to the *a* axis or to both, is expansion in the *s* plane and flattening of the series perpendicular to it. The effect of flattening of the series is about similar to that of compaction. Flattening of the country rock will have been much easier than that of the gneiss body. Enveloping structures and nestled contacts may, to an unknown extent, have also been due to this effect. The result of 20 % compression of an inhomogenious series by the combined action of compaction and flattening is constructed in fig. 17*d*. The comparison of this structure with the boudinage of fig. 11 suggests even a larger compression by combined actions than 20 %.

Differential movements are the cause of the alteration of the granitic material to gneiss. This alteration is mainly a mechanical deformation. A perfect orientation of feldspars as caused by flow in many augen gneisses does not exist here. The schistosity is mainly visible by sericitic layers and light coloured bands and lenses of cataclastically deformed minerals. Irregular cleavage planes, covered with sericite and provided with drag ribs or wrinkles demonstrate the restriction in zones of the

rock where motion mainly occurred. These planes are caused by shear, not by flow, and the cleavage pattern of the gneiss is in this way similar in origin to the shear cleavage in the slate.

Differences may appear in the time of origin and the direction of the cleavage planes in and outside the gneiss body. As the mechanical properties of granitic and shaly rock are unequal, a difference in time and direction of failure of both rocks is most likely. Difference in direction of the cleavage at both sides of the contact would show the same effect as refraction at the contact of media of different density. The refraction effect may be found at various points along the contact in the structure map. A further analysis of the importance of this effect upon the structure pattern is not possible however, because of the activity of other deformational influences.

Difference in time of the origin of cleavage in both media may be of added influence. Predisposition of the cleavage direction is supposed by orientation of particles during sedimentation, by the early stages of compaction and by preceding differential movements in an earlier phase of the orogeny before the emplacement of the laccolith. A general parallelism of the cleavage pattern in and outside the gneiss body points however to a relative small influence of the mentioned effects, except locally.

Schematically, the origin of the cleavage pattern may have been as follows. In the beginning of the orogenic phase the cleavage developed as a nearly parallel system of shear planes through both rocks. The continued compaction and the flattening of the series — affecting mainly the slaty rock — acted during the whole orogenic phase and differential movements took place along gradually deforming cleavage planes.

The general enveloping tendency of the pattern around the body and nestled contacts are the results of these deformations. In the gneiss, the nestling tendency is usually very small and in a narrow zone at the contact. This tendency is the result of the relative motion of the thinning country rock at the contact which changed the stress direction locally (drag).

Crossed contacts are locally promoted by the existence of the contact aureole. Though the contact is always very sharp, in some parts the mechanical properties of the rock gradually pass from the gneiss, via the hardend contact rock into the slates. In those places nestled contacts are not found. The cleavage pattern shows preference for crossed contacts and a diffuse enveloping structure throughout and outside the contact zone. Examples will be found in the northeastern contact on the structure map.

16. *The history of the gneiss massif.*

According to the preceding paragraphs the history of the massif may be summarized as follows.

Argillaceous sediments of probably Cambrian or Lower Ordovician age are intruded in the diagenetically altered but unmetamorphosed condition of about shale. The form of the body was defined by the direction of the stratification. The granite intruded largely in the bedding planes of the shaly rock on various levels to a compound laccolith.

On emplacement, the granitic material has been in magmatic condition. The intruded rock must have been a porphyritic granite or a granite porphyry of orthoclase-albite/oligoclase-quartz-biotite-apatite composition. In the contact aureole developed a feldspar-quartz-biotite-muscovite-garnet-cordierite-andalusite rock from the slates. Veins of aplitic, pegmatitic, porphyritic and lamprophyric material in and around the massif point to differentiation in the magmatic body.

The granitic mass consolidated in a period of relative orogenic calmness. As far as preserved, feldspars have been oriented at random, or slightly, before the secondary orientation. Pre-existence of a certain primary parallelism by intrusive flow is observed e.g. near crossed contacts (fig. 15). A pre-existence of a perfect orientation as developed in a magmatic mass in active surroundings (syntectonic) did not exist, however (photo 1).

After the consolidation of the laccolith the main phase of the orogeny deformed and blurred the primary structures in this area. The secondary orientation of the gneiss with broken feldspars and mortar structure, and the tortuosity of veins and dikes indicate a deformation in solid state. Even the youngest traces of magmatic activity, viz. the veins of the differentiation series are strongly deformed. The facies of metamorphism and tectonics, during the development of the cleavage in the slate and of the orbicular texture in the gneiss is epitype.

The relative rigidity of the massif against the more plastic behaviour of the country rock effected a distorted structure pattern. As a whole a certain agreement of the cleavage with the primary structure on the area has been mentioned. The difference in the reaction upon deformational forces is thought to be expressed by co-operation or dominant operation of three effects of the orogenic movements, viz. further differential compaction of the series after the intrusion, differential flattening in the tectonic unit by lateral extension and possible difference in the time of origin and in the direction of cleavage in both media. The direction of the original contacts, the irregularities in the contact plane and the effect of the hardend contact rock caused the local difference in the structure of the contact zone, in which four types are distinguished, viz. concordant, nestled, crossed and faulted contacts.

After the orogeny the area is affected mainly by fault movements. The intrusion of the posttectonic Mt. du Liron granite was accompanied by diapiric structures. The southern part of the gneiss massif disappeared and faulting and warping phenomena by diapiric action have been noticed in the remaining part. Some large faults with slickensiding and quartz-fillings of probably alpine age cross the area.

17. *Conclusions.*

The characteristics in structure and history can be shortly defined by using the granite tectonic terminology.

The gneiss massif of St. Jean is a small, lenticular, nuclear pluton, pretectonic intruded in epitype conditions as a compound laccolith of porphyritic granite or granite porphyry which is deformed in epitype conditions to a planar, orbicular gneiss, harmoniously infolded, partly concordantly and conformably, in the Orthocevennes Complex of the variscan orogen.

18. *Correlation with augen gneiss massifs in the Orthocevennes Complex.*

The mapped area is part of the Orthocevennes Complex (Demay, 1931*a*), which is comparable — as to tectonics and facies — with the penninic part of the Alpes (von Gaertner, 1937*a*, *b*, *c*).

The nearest gneiss massif is situated 25 km NE of the mapped area. This elongated massif — 40 km in NS direction — is marked on the geological map with a signature indicating an augen gneiss of regional or contact metamorphic origin (Thiéry, 1923). According to Demay (1931*a*, *b*, 1932), the gneiss — originated in katametamorphic conditions — is the core of a recumbent fold. The rock type has been described as an augen gneiss in which mechanical deformations as slip planes have been observed and with feldspar augen of locally variable size (Demay, 1931*a*). Recently, the origin of this massif has been mentioned as: "The hypothesis of a syntectonic magmatic injection into laccolithic shape seems to be the most probable" (Demay, 1948).

In the western part of the orthocevennes — the Rouergue — augen gneiss massifs are found, which partly have been interpreted as syntectonic laccoliths. The augen gneiss as well as the contact rock of spotted andalusite schist (Bergeron, 1889), shows often a cataclastic texture, which is caused by younger mylonitization (Durand and Raguin, 1932, Angel, 1937, von Gaertner, 1937*a*). According to Roques (1941), the mechanical deformation occurred during the consolidation of the laccolith. The intrusion is younger than the regional metamorphism of the area, which retrograded during the orogenic phase (Roques, 1941).

Of probably older origin than the syntectonic laccoliths are the massif of Rieupeyroux and similar gneiss bodies in the same area. According to Roques (1941), the augen gneiss has a microcline-albite-sericite-muscovite-quartz-chlorite composition. A dominantly cataclastic texture is characteristic for this rock. The feldspar augen are broken and dislocated. Shear planes of sericite are found throughout the rock. The biotite is destructed and quartz and muscovite are recrystallized. The structure pattern of the laccolithic massif is parallel to that of the country rock of sericite schist. As the St. Jean massif, this gneiss body is emplaced before the orogenic phase, which caused the epizonal deformation of the rock.

1°26'
1°28'
1°30'
1°32'
44°8'
44°7'
44°5'
STRUCTURE MAP
OF THE GNEISS MASSIF W OF ST JEAN DU GARD
IN GARD DEPARTMENT, FRANCE
CONSTRUCTED BY DIRK DE WAARD, JULY-AUGUST 1948
slaty country rock
augen gneiss
contact aureole
veins in country rock
strike and dip of cleavage
ribs and fold axes
fault
fault with downthrow of S
1°32' E. OF PARIS MERIDIAN

Like in most crystalline areas, the granitic bodies in the Orthocevennes Complex can be distinguished in pre-, syn- and posttectonically intruded ones, with respect to the local orogenic phase. Each of these intrusive types has its own structural character. Syntectonic and pretectonic massifs usually have a lenticular outline, parallel to the general schistosity. The shape and the structure pattern of pretectonic laccoliths are effected by the intrusive conditions and by the deformation during the orogenic phase.

The conditions in the zone of the orogen in which intrusion and deformation took place have great influence upon the structure pattern of the massif. As to pretectonic bodies, the conditions of the intruded zone and the intermixed structures of this zone during the orogenic phase compose the ultimate pattern. Differences may be expected, e.g. between a mesozonal intrusion which is epitype deformed and an epizonal intrusion which is deformed in a deeper zone.

Though lack of detailed mapping of gneiss massifs hampers a structural comparison, the Rieupeyroux gneiss seems to be emplaced in a deeper zone of the orogen and epitype deformed. This in contrast with the St. Jean massif, which is intruded as well as deformed in epitype conditions.

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DIE ELEKTRONENOPTISCHE UNTERSUCHUNG VON GEWEBESCHNITTEN

DIE TECHNIK DER UNTERSUCHUNG ERLÄUTERT AM BEISPIEL DES MAMMA-CARCINOMS DER MAUS

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1. *Einleitung.*

Seit unserer ersten Mitteilung über die elektronen-mikroskopische Untersuchung von Gewebeschnitten (L. H. BRETSCHNEIDER, 1949) war es uns möglich die dementsprechende Technik in mehreren Punkten zu verbessern. In dieser vorliegenden Mitteilung beschränken wir uns lediglich auf die Beschreibung der technischen Seite, während in einer nächsten Mitteilung die eigentlichen wissenschaftlichen Ergebnisse folgen werden. Es war zunächst zu untersuchen welchen Einfluss die Fixation, die Schnittdicke und die Strahlspannung auf die Qualität des Bildes haben und wo die derzeitige Grenze der elektronen-optischen Schnittuntersuchung liegt.

Die elektronen-mikroskopische Untersuchung wurde mit den beiden elektromagnetischen Elektronen-Mikroskopen für 100 kV und 400 kV in Delft vorgenommen (LE POOLE, 1947, VAN DORSTEN, OOSTERKAMP & LE POOLE, 1947). Für die bereitwillige Unterstützung danken wir den Herren Ing. J. B. LE POOLE, J. KRAMER und J. VAN HENGEL, Delft. Da die Untersuchung erst durch die "Niederländische Organisation für fundamental-wissenschaftliche Forschung" (Z.W.O.) finanziell ermöglicht wurde, danken wir dieser Organisation an dieser Stelle.

2. *Die histologische Technik.*

Als Untersuchungsmaterial diente das spontane Mammacarcinom der Maus. Das Material wurde von Herrn Dr O. MÜHLBOCK aus dem "A. v. Leeuwenhoek Huis" in Amsterdam zur Verfügung gestellt. Auf die Einzelheiten des untersuchten Tiermaterials werden wir in der nächsten Mitteilung eingehend zurückkommen. Kleine Stücke der Geschwulste wurden in der Fixationsflüssigkeit von CHAMPY 24 Stunden fixiert, hernach gewässert und durch die Alkoholreihe bis Benzen gebracht. Hieraus wurden die Objekte in das von H. F. STEEDMAN (1947)

zusammengestellte "Ester Wax" eingebettet.[1]) Dieses Ester Wax hat gegenüber dem von uns anfangs gebrauchten Bienenwachs-Paraffingemenge (BRETSCHNEIDER, 1949) verschiedene Vorzüge bei der gleichguten Schneidbarkeit.

Es hat einen niedrigen Schmelzpunkt (48° C.) und dennoch den gleichen Härtegrad als das Wachs-Paraffin, es löst sich leichter in Kohlenwasserstoffen und erlaubt eine Färbung oder überhaupt Behandlung mit in Wasser gelösten Agentien ohne vorherige Auflösung des Ester Waxes. So kann man b.w. die auf dem Objektträgerfilm aufgezogenen Schnitte zur lichtmikroskopischen Kontrolle mit einer wässerigen Eosin oder Safraninlösung anfärben und auf diese Weise die geeignetsten Schnitte heraussuchen oder eventuel vor der elektronen-mikroskopischen Untersuchung zur Orientierung skizzieren.

In enger Zusammenarbeit mit der "Cambridge Instrument Co. Ltd." wurde durch diese Firma ein modernes "Cambridge Rocking Microtome for fine sections" entwickelt, mit welchem ein Teil der hier abgebildeten Schnitte angefertigt wurde, Abb. 1. Mit diesem Mikrotom kann eine minimale Schnittdicke von 0,2 Mikron erzielt werden. Ein besonderer Messerhalter ermöglicht das Schneiden mit Rasierklingen. Voraussetzung für diese sehr dünnen Schnitte ist allerdings eine Temperatur die nicht tiefer oder höher ist als $\pm 0°$ bis $+ 6°$ C. In diesem Temperaturgebiet gelingt es ohneweiters völlig unbeschädigte Schnittbänder von 0,2 und 0,4 Mikron Dicke zu erhalten. Es wurde von uns entweder im Winter bei dieser Temperatur geschnitten oder mit dem von uns beschriebenen Eiskasten (BRETSCHNEIDER, 1949). Für das weitere Hantieren und Montieren der Schnitte schien es am besten wenn man bloss kurze Bänder von 4 Schnitten schneidet, wobei der erste und abschliessende vierte etwas dicker gewählt wird. Diese Begrenzung der zwei dünnen Schnitte durch zwei dickere hat den grossen Vorteil, dass man beim Abnehmen, Übertragen und Wiederaufnehmen der Schnittbänder die mittleren, für die Untersuchung bestimmten Schnitte, nicht zu berühren braucht. Wir schneiden das ganze Objekt fortlaufend und legen die Bänder in eisgekühlten Petrischalen ab, die danach in einem mit Chlorcalcium beschickten Exsiccator im Eisschrank bis zur weiteren Verarbeitung aufbewahrt werden. Bei einer event. Erwärmung der Petrischalen auf Zimmertemperatur kleben die Bänder nl. am Glasboden fest und sind dann nicht mehr zu gebrauchen. Um Misserfolgen vorzubeugen sei noch darauf hingewiesen, dass das Aufnehmen der Schnitte nur mit besonders hierzu präparierten Aquarellpinseln zu geschehen hat. Die einzelnen Haare selbst der besten Pinsel durchlöchern diese zarten Schnitte und spiessen sie gewissermaassen auf. Man durchtränkt darum den Aquarellpinsel mit einer 4 % Aether-Celloidinlösung, lässt trocknen und schneidet mit einer Rasierklinge soviel Haare weg bis nur

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[1]) Es ist zu beziehen durch die Firma "The British Drug Houses Ltd., London".

eine mittlere Schichte übrig bleibt. Vor und auch während der Arbeit benetzt man ab und zu den Pinsel mit der Celloidinlösung und zieht ihn durch zwei Fingerspitzen hindurch. Nach dem Verdampfen des Aethers müssen die Haare des Pinsels durch das trockene Celloidin schaufelartig zusammengehalten bleiben, aber doch noch etwas elastisch sein. Man nimmt die Schnitte wie mit einer Schaufel von unten auf und transportiert sie so weiter. Für die in Ester Wax eingebetteten Objekte erwies sich das Strecken der Schnitte auf einem Warmwasserbad nicht geeignet, da in diesem Medium Schnitte unterhalb 2 Mikron durch die Oberflächenspannung auseinander gezerrt werden und sich über die Wasseroberfläche verbreiten. Wir haben daher das Strecken auf die folgende Weise abgeändert. Man nimmt das mit dem Trägerfilm versehene Kodaknetz (siehe weiterunten) in die Pinzette und fixiert es indem man einen Gummiring über die Flanken schiebt. Mann nimmt die Pinzette in die linke Hand und legt das Netz auf einen Objektträger, während man mit der rechten Hand mit einer Pipette 4 bis 5 Tropfen aqua bidestillata (auf Zimmertemperatur) auf das Netz tröpfelt. Man hält inzwischen das Netz noch immer mit der Pinzette unter Wasser fest, da es sonst wegen der stark hydrophoben Eigenschaft des Trägerfilmes auf dem Wassertropfen schwimmen würde. Dann nimmt man mit einem Pinsel mit der rechten Hand einer der Schnittbändchen und legt es auf die Wasseroberfläche oberhalb des Netzes. Die Schnitte strecken sich unmittelbar und indem man das Netz aus dem Wassertropfen hebt, fängt man die mittleren zwei dünnen Schnitte auf.

Im Laufe der Untersuchungen hat sich überdies ergeben, dass das Montieren des Trägerfilms auf das Kodaknetz weit einfacher und sicherer geschehen kann als wir in unserer ersten Mitteilung (Bretschneider, 1949) beschrieben. Man lässt einen Tropfen einer 2 % Lösung von „Parlodion" (Kollodion) in chemisch reinem Amylacetat (sic!) auf aqua bidest. fallen. Der Durchmesser der betreffenden Petrischalen darf nicht kleiner sein als 9 cm, da sich sonst der Tropfen nicht völlig ausbreiten kann. Während des Verdunstens des Amylacetates wandern über den Film lebhafte Interferenzfarben. Erst wenn diese verschwunden sind, ist der Trägerfilm gebrauchsfertig. Dann legt man in die Mitte des Filmes ein Kodaknetz. Man nimmt einen runden Stab von der Grösse und Dicke eines Bleistiftes dessen Ende leicht gewölbt ist und drückt mit dieser Wölbung das Netz und den darunterliegenden Film ins Wasser. Durch rollende Bewegung des Stabes in einer Richtung, verbunden mit der Annäherung an den Rand der Petrischale rollt man den Film um das Stabende herum. Hebt man es dann aus dem Wasser, dann zeigt sich dass der Film glatt über das Netz gespannen ist. Ritzt man sodann mit einer feinen Nadel den Film am Rande des Netzes durch, dann kann man es mit der Pinzette abheben. Nach dem Trocknen bei ca. 50° C. ist der Trägerfilm fertig. Das Ester Wax wird vor der Untersuchung mit Xylen gelöst.

3. *Die Notwendigkeit von Übersichtsbildern in der el. opt. Schnittuntersuchung.*

Mit der elektronen-optischen Schnittuntersuchung wird ein Gebiet erschlossen, das der Elektronen-mikroskopie eigentlich wesensfremd ist, da sie berechnet ist für viel kleinere Dimensionen als diejenigen welche Zellverbänden eigen sind. Der Dünnschnitt bringt nl. Objekte die an sich weit über der Durchdringungsgrenze der Elektronenstrahlen liegen unterhalb diese und ermöglicht damit eine Analyse des räumlichen Aufbaues dieser Objekte. Erschwert wird diese Analyse aber durch die grosse Inhomogenität der Zelle und noch mehr der Gewebe. Daher erweist sich bei der Schnittuntersuchung zur topographischen Bestimmung der zu untersuchenden Strukturen eine jeweilige elektronen-optisch schwächere Übersichtsvergrösserung als unumgänglich. Scharf eingestellte elektronenoptische Aufnahmen können überdies bei den derzeitigen sehr feinkörnigen Filmen stark nachvergrössert werden, wobei sich die elektronenoptische Auflösung für das menschliche Auge mit seinem viel schlechteren Auflösungsvermögen, noch weiterhin entfaltet. Wir wollen dies an der Abb. 2. illustrieren. Eine bestimmte Stelle des Präparates wurde bei einer elektronen-optischen Vergrösserung von 2000 : 1 aufgenommen und auf eine Bildvergrösserung von 4000 : 1 gebracht. In diesem Areal sehen wir ungefähr 17 Zellen in ihrem natürlichen Zusammenhange beisammen. Die zwei dunkleren schmalen Zellen rechts unterhalb der Bildmitte sind nekrotisierende, weggedrückte Carcinomzellen, während man links oben und in der Mitte unten die Reste bereits zerfallener Zellen sieht. In den meisten anderen Zellen fällt der relativ grosse Kern auf, der von einer dichten, fädigen Struktur ziemlich homogen erfüllt wird. Sehr deutlich zeichnen sich elektronen-optisch die Zellgrenzen ab vielleicht als Ausdruck ihres Lipoidreichtums der eine stärkere Osmiumreduktion bewirkt und dadurch so kontrastreich wird. Licht-optisch fallen die Zellgrenzen überhaupt nicht auf. Innerhalb eines solchen im Übersichtsbild festgehaltenen Bezirkes kann man nun bestimmte Stellen bei elektronen-optisch weit stärkeren Vergrösserungen untersuchen ohne hinterher in Zweifel zu geraten in welchem Zusammenhange diese standen.

4. *Der Einfluss der Strahlspannung auf das elektronen-mikroskopische Bild.*

Bekanntlich steigt mit der Strahlspannung das Durchdringungsvermögen der Elektronenstrahlen und sinkt die Elektronenstreuung an dem Objekt. Bei steigender Voltgeschwindigkeit wird das Präparat gewissermassen „aufgehellt". Um bei unserem Objekt im Zusammenhang mit der Schnittdicke und der OsO_4 Fixation die günstigste Strahlspannung zu ermitteln, untersuchten wir bei verschiedenen zwischen 50 kV und 350 kV liegenden Spannungen. Die erzielten Bilder illustrieren überdies auch ganz allgemein den Einfluss der Voltgeschwindigkeit auf

die Auflösung der Plasma-Kernstrukturen in dünnen Gewebeschnitten. Im ersten Versuch wurde die selbe Präparatstelle bei einer elektronen-optischen Vergrösserung von 6000 : 1 bei einer Strahlspannung von 50 kV aufgenommen, Abb. 3*a*. Sie kennzeichnet sich durch den fast völligen Mangel an Durchdringungsvermögen und darum durch eine überwiegende Silhouettewirkung. Es überwiegen die schroffen schwarz-weiss Kontraste ohne etwaiger Differenzierung. Die Abb. 3*b* zeigt die gleiche Stelle bei einer Strahlspannung von 80 kV. Obwohl die Silhouette-wirkung zu weichen beginnt, lässt doch die Durchstrahlung viel zu wünschen übrig. Im Gegensatz zum vorigen Bild füllen sich die schein-baren Lücken zwischen den schwarzen Teilchen nun mit Grautönen feinster fibrillärer Plasmastrukturen.

Die nächste Abbildung 3*c* stammt von einer dritten Aufnahme bei einer Strahlspannung von 110 kV. An ihr bemerkt man nun, dass die in der vorigen Abb. 3*b* noch dunklen Bildpartien nun durchsichtig geworden sind, ohne aber an der an sich gewünschten Kontrastwirkung eingebüsst zu haben. Für diese Schnittdicke von 0,4 Mikron erwies sich im Allgemeinen eine Strahlspannung von 110 kV völlig hinreichend, weshalb wir die meisten Untersuchungen mit dieser Spannung ver-richten. An einem anderen etwas dickeren Schnitt (ca 0,6 Mikron) des gleichen Objektes untersuchten wir mit dem E.M. 400 kV ein und dieselbe Stelle des Präparates mit zwei höheren Strahlspannungen bei einer elektronen-optischen Vergrösserung von 7000 : 1. Mit der nun höheren Massendicke nimmt auch die Elektronenstreuung zu, wodurch die Bildqualität trotz der höheren Spannung von 200 kV, Abb. 4*a* gleich zu setzen ist mit der vorhergehenden Aufnahme bei 110 kV der Abb. 3*c*. Erst bei der Erhöhung der Spannung auf 350 kV, Abb. 4*b*, wird das Bild durchsichtiger und die Einzelheiten klarer und deutlicher. Man vergleiche z.B. den etwa dreieckigen Nukleolus dieser Abbildung mit dem vorhergehenden der Abb. 4*a* um zu sehen, dass bei dieser Spannung nun erst einige massendichtere runde Körper in ihm zu sehen sind, oder die rechte untere Ecke des Gesichtsfeldes, die gegenüber der Abb. 4*a* nun viel gleichmässiger durchstrahlt ist. Die Inhomogenität der Cellen stellt an ihre elektronen-optische Darstellung hohe Anforderungen. Man will nl. einerseits die extrem dichteren Partien oder Strukturen so viel wie möglich durchstrahlen, anderseits die extrem dünnen doch auch hinreichend kontrastreich darstellen. Für eine gewisse Spannung schliesst das Eine das Andere aber aus, denn dickere Objektpartien kommen erst mit schnelleren Elektronen zu einer befriedigenden Darstellung, dünnere aber vorallem mit langsameren Elektronen. Zweifelsohne ist das Plasma in der Aufnahme bei 200 kV kontrastreicher als in der von 350 kV. Dem steht aber gegenüber, dass die Auflösung mit langsameren Elektronen abnimmt. Obwohl also die Schwarz-Weiszgraduierung in der Abb. 4*b* innerhalb geringerer Grenzen verläuft als in Abb. 4*a* gewinnt doch durch die bessere Auflösung das Bild an Klarheit. Für manche Untersuchungen

wird es wohl der Mühe wert sein, die gleiche Präparatstelle mit verschiedenen Spannungen aufzunehmen. Bemerkt sei noch, dass die bessere „Aufhellung" der Abb. 4*b* nicht etwa zu setzen ist auf eine längere Durchstrahlungszeit im Laufe der Untersuchung. Diese führt nl. oftmals bei organischen Präparaten wahrscheinlich durch die Ionisation zu einer gewissen Aufhellung. Die Untersuchung geschah nl. in der umgekehrten Richtung, wobei von der höchsten Spannung treppenweise nach der niedrigeren gegangen wurde. Eine Reihe dazwischen liegender Aufnahmen wurden hier nicht reproduziert, weil ihre Unterschiede zu gering sind. Man kann nun aus diesen beiden Versuchsreihen den Schluss ziehen, dass man bei einer Schnittdicke von ca. 0,5 Mikron mit Strahlspannungen non ca. 110 kV auskommt, bei Schnitten oberhalb 0,5 Mikron mit bedeutend höheren Spannungen von ca. 300 kV. Wir sehen weiterhin, vass das Durchdringungsvermögen als Funktion der Objektstreuung dicht etwa linear verläuft, sondern dass am Beginne der Spannungserhöhung z.B. von 80 auf 100 kV eine geringere Erhöhung nötig ist um die Bildqualität wesentlich zu verbessern als bei den höheren Spannungen von z.B. 200 auf 350 kV.

5. *Der Einfluss der Fixation auf das elektronen-optische Bild.*

Bei der Beurteilung der Güte eines Fixationsmittels wird sich in der Zukunft das elektronen-optische Bild als maaszgebend erweisen. Es hat sich herausgestellt, dass Fixationsmittel die lichtoptisch als sehr gut gelten, sich elektronen-optisch als untauglich erweisen. Der Maaszstab den man bei der Beurteilung anlegt, bezieht sich auf die Teilchengrösse — also auf den Grad der Feinheit — mit welcher das Fixationsmittel das Plasma ausfällt. Je feiner das Plasma als Gefüge linearer und sphärischer Elementarteilchen, die durch Wasserhüllen umgeben sind, niedergeschlagen wird, desto besser nennen wir den Grad der Fixation. Im lebenden Plasma erreicht die Teilchendispersion wahrscheinlich ihr Maximum und jede Fixation ist bereits eine Vergröberung der Grundstruktur. Je schwächer diese ist, desto näher steht sie dem lebenden Zustande. Für die elektronenoptische Untersuchung des Carcinomgewebes ergab die Champy'sche Flüssigkeit die feinste Ausflockung die wir bisher sahen. Dies betrifft sowohl den Kern als auch das Cytoplasma. Dahingegen war die lichtoptisch so gute Zenker'sche Flüssigkeit für die elektronenmikroskopische Kernanalyse gänzlich untauglich, Abb. 5, da sich der Kerninhalt auf einige grobe Balken zusammenklumpte und optisch leer wurde. Auch die etwas bessere Cytoplasmafixation erreichte bei weitem nicht das Bild der Champy'schen Fixation. Die Teilchengrösse im Kern nach der Champy'schen Fixation beträgt im Mittel 20 mμ, nach der Zenker'schen Fixation 200 mμ, im Plasma nach Champy 20 mμ, nach Zenker 120 mμ. Neben der schlechten Fixierung bemerkt man auch, dass das Gewebe geschrumpft ist und deutliche Zellgrenzen fehlen. Die weiteren Untersuchungen wurden daher ausschliesslich mit der

Champy'schen Fixation ausgeführt. Die Abb. 5 dokumentiert also den Fixationszustand der Zellen nach einer der üblichstenFixationstechniken womit wir in der licht-optischen Histologie arbeiten. Die üble Strukturvergröberung gegenüber der nach Champy fixierten gleichartigen Zellen fällt bei den wesentlich schwächeren lichtoptischen Vergrösserungen nicht so störend auf, wie bei dieser elektronenoptischen Untersuchung. Wir geben zum Vergleich in der linken unteren Ecke (Abb. 5) die elektronenoptische Aufnahme der gleichen Präparatstelle bei einer Vergrösserung von 1700 : 1 wider, um zu zeigen wie sich dieses Präparat im lichtoptischen Vergrösserungsbereich ausnehmen würde. Es bietet dem Histologen das ihm vertraute Bild der bläschenförmigen, nahezu optisch leeren Kerne mit zusammen geklumpten Chromatin und einem retikulären Cytoplasma. Die oft geäusserte Behauptung dass die Osmiumsäure vorallem das Kernplasma auf unnatürliche Weise „homogenisierte", während Sublimatgemenge mehr den natürlichen Zustand des Kernes fixierten, lässt sich also nach den elektronen-optischen Untersuchungen nicht aufrecht erhalten, da sich gerade diese licht-optische „Homogenisierung" als eine Annäherung an den natürlichen Dispersionsgrad erweist, während das Sublimat eine sehr starke Vergröberung herbeiführt. Einer allgemeineren Anwendung osmiumhältiger Fixationsflüssigkeiten in der Histologie steht leider die nur sehr beschränkte Färbungsmöglichkeit der Gewebe nach Überosmiumsäure im Wege, sodass man eben zu anderen Fixationsflüssigkeiten greifen musste. Da für die elektronenoptische Untersuchung dieses Hindernisz der Färbung wegfällt, wird man nach dem Gesagten den osmiumhältigen Fixationsmitteln in der Elektronenmikroskopie eine Vorzugsstellung einräumen können.

6. *Die derzeitige, durch die Fixation bedingte Grenze der elektronenoptischen Untersuchung.*

Die Untersuchung eines Gewebes hat gegenüber der eines Einzelteilchens den Nachteil der erforderlichen Fixation und Nachbehandlung. Diese bestimmen darum den jeweiligen Grad der Strukturvergröberung. Wir haben an dünnen Schnitten durch das Carcinomgewebe bei höheren elektronenoptischen Vergrösserungen von 90000 : 1 versucht, wie weit diese Vergröberung nach der Champy'schen Fixation, Alkoholreihe und Einbettung in Ester-wax reicht. Wir kommen zu dem Schlusse, dass das Resultat ziemlich befriedigend ist und zu weiteren elektronenmikroskopischen Schnittuntersuchungen ermutigt. Die mittlere Dicke der in Abb. 6 wiedergegebenen filaren Kernplasmastruktur, die überwiegend aus Eiweissen bestehen, beträgt 160 Å, die dünnsten noch zur Abbildung gelangten 80 Å. Nimmt man den Radius eines Linearproteines mit ungefähr 20 Å an, dan bedeutet dies, dass man Strukturen von höchstens 4 bis 8 nebeneinander liegenden Eiweissketten vor sich hat. Die Länge der einzelnen Glieder des dreidimensionalen Maschen-

werkes dieser Abbildung schwanken zwischen 80 mμ und 400 mμ. Selbst wenn man sich den Kerninhalt hauptsächlich aus Histonen vorstellt, denen man 86 Aminosäurereste mit einem Abstand von 3,5 Å in der Hauptvalenzrichtung zukennt, das heisst einer Kettenlänge von 300 Å, dann können bloss drei Histonmoleküle in diesen kleinsten Netzkanten hintereinander liegen. Legt man den Eiweissmolekülen im Kern eine Eiweissart zugrunde, mit dem niedrigsten Svedberg'schen Molekulargewicht von 17.600, und einer Zusammensetzung aus ca. 158 Aminosäureresten, dann beträgt die Kettenlänge ca. 525 Å. In diesem Falle würden nur $1\frac{1}{2}$ Eiweissketten in der Kantenlänge hintereinander liegen. Die dünnsten und kürzesten zur Abbildung gelangten Glieder des Maschenwerkes in unserer Abb. 6 bestehen demnach nur noch aus $1\frac{1}{2}$ hintereinander liegenden und 4 nebeneinander liegenden Eiweissmolekülen niedriger Eiweisse, eine Feststellung die für die weitere elektronenoptische Gewebsuntersuchung nicht ungünstig ist. In diesem 0,2 μ dicken Schnitt (Abb. 6) durch einen kleinen Teil des Kerns, sehen wir in diesem Vergrösserungsgebiet von 90.000 : 1 ein deutliches dreidimensionales Maschenwerk von ziemlich regelmässiger Anordnung. Nimmt man an, dass die Kette der Makromoleküle im lebenden Zustande regelmässiger, also in fast gleichen Abständen von einander verliefen, dann ist die Strukturvergröberung durch die Fixation, allen Erwartungen zuwider, dennoch relativ gering. Aus dem periodischen Aufbau der Maschenfaseren aus abwechselnd dickeren und dünneren Orten-Chromidien und Interchromidien, im Sinne Monné's (1946) könnte man anderseits den Schluss ziehen, dass dieser Effect nur dann zustande kommen kan, wenn schon im lebenden Zustande die Molekülketten sehr nahe nebeneinander lagen oder wie Monné anführt auf autoreproduktivem Wege zustande kämen. Wahrscheinlich beruhen auch die Knotenpunkte des Maschenwerkes auf die im lebenden Zustand bereits vorhandenen Haftpunkte, wie sie Frey-Wyssling (1938) aus theoretischen Gründen voraussagte.

7. *Zusammenfassung.*

1. Diese Mitteilung bezweckt die Beschreibung der elektronenmikroskopischen Technik welche für die Untersuchung der Mammacarcinoms der Maus angewandt wurde. Sie gilt im Allgemeinen als Basis für die elektronen-mikroskopische Schnitt-Untersuchung auch anderer Gewebe.

2. Die von uns beschriebene Technik der Schnittuntersuchung (Bretschneider, 1949) wird durch einige Verbesserungen ergänzt.

a. Als Einbettungsmittel wurde nun das sogenannte „Ester Wax" von Steedman (1947) vorteilhaft gebraucht.

b. Geschnitten wurde bereits mit dem modernen „Cambridge Rocking Microtome for fine sections" auf 0,2, 0,4 und 0,6 Mikrons.

3. Zur Orientierung in so inhomogenen Systemen wie Gewebe es

sind, wird der Gebrauch von elektronen-optisch schwachen Vergrösserungen (2000 : 1) als Übersichtsaufnahmen empfohlen.

4. Im Zusammenhang mit der Art des Fixationsmittels und der Schnittdicke wird in zwei Versuchsreihen der Einfluss der Strahlspannung auf die Qualität des elektronenoptischen Bildes besprochen. Es wurden Aufnahmen bei 50, 80, 110, 200 und 350 kV reproduziert.

5. Untersucht wird die Wirkung zweier Fixationsflüssigkeiten (ZENKER und CHAMPY) auf das Kern- und Zellplasma. Es wird festgestellt, dass die OsO_4haltige Flüssigkeit nach CHAMPY sehr befriedigende Resultate zeitigte. An der Hand einer elektronen-optisch stärkeren Vergrösserung bei 90.000 : 1 wird festgestellt, dass man noch Feinstrukturen abzubilden vermag, die übereinstimmen mit ca. 4 nebeneinander und $1\frac{1}{2}$ mal hintereinander liegenden Eiweissketten eines niedrigen Eiweisses mit einem Molekulargewicht von 17.600. Diese Ergebnisse ermutigen zu weiteren elektronen-mikroskopischen Untersuchungen mit Gewebeschnitten.

8. *Summary.*

1. This communation gives a description of the electron-microscopic technique, used in the investigation of mamma carcinoma in the mouse. This technique can be used also for the electron-microscopic investigation of other tissues.

2. The technique, previously described by us, for the investigation of sections is improved with some supplements.

a. The "Ester wax" of STEEDMAN (1947) was used very advantageously as an embedding-medium.

b. The sectioning was done already with the modern "Cambridge Rocking Microtome for fine sections" at 0,2, 0,4 and 0,6 microns.

3. For orientation in such inhomogeneous systems as tissues are, the use of weak electron-optic magnifications (2000 : 1) is recommended.

4. In connection with the kind of fixation fluid and the thickness of the sections, the influence of the tension on the quality of the electron-optic picture is reviewed with reference to two experimental series. Photographs at 50, 80, 110, 200 and 350 kV emission voltage are reproduced.

5. The effect of two fixation fluids (ZENKER and CHAMPY) on the nucleo- and cyto-plasm was investigated. We came to the conclusion that the fluid containing OsO_4 after CHAMPY gave very satisfactory results. From a stronger electron-optic magnification (at 90.000 : 1) we concluded that one can picture fine structures which correspond to ca. 4 protein-chains side by side and $1\frac{1}{2}$ lengthwise of a lower protein with a molecular weight of 17.600.

This result encourages to further electron-microscopic investigations of sections.

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EXPLANATION

Fig. 1. Moderne Ausführung des "Cambridge Rocking Microtome" Modell 1950. Rechts Messerhalter für Rasierklingen. Natürliche Grösse: 37,5 × 17,5 × 20 cm.

Fig. 2. Mammacarcinom der Maus; Schnittdicke 0,4 μ; Vergrösserung 4.000 : 1; Spannung 110 kV; Fixation: Champy'sche Flüssigkeit; Einbettung: Bienenwachs-Paraffin.

Fig. 3. Mammacarcinom der Maus. Fixation, Einbettung und Schnittdicke wie in Figur 2. Vergrösserung 12.000 : 1.
3*a*. Strahlspannung: 50 kV.
3*b*. Strahlspannung: 80 kV.
3*c*. Strahlspannung: 110 kV.

Fig. 4. Mammacarcinom der Maus. Fixation: nach Champy, Einbettung mit Ester Wax, Schnittdicke 0,6 μ, Vergrösserung: 15.000 : 1.
4*a*. Strahlspannung: 200 kV.
4*b*. Strahlspannung: 350 kV.

Fig. 5. Mammacarcinom der Maus. Fixation nach Zenker, Einbettung: Ester Wax, Schnittdicke 0,4 μ, Vergrösserung 10.000 : 1; Strahlspannung: 110 kV. Links unten Vergrösserung 1700 : 1.

Fig. 6. Kernfragment einer Zelle aus dem Mammacarcinom der Maus. Fixation nach Champy. Einbettung: Ester Wax, Schnittdicke: 0,2 μ, Strahlspannung: 110 kV. Vergrösserung 90.000 : 1.

Fig. 1

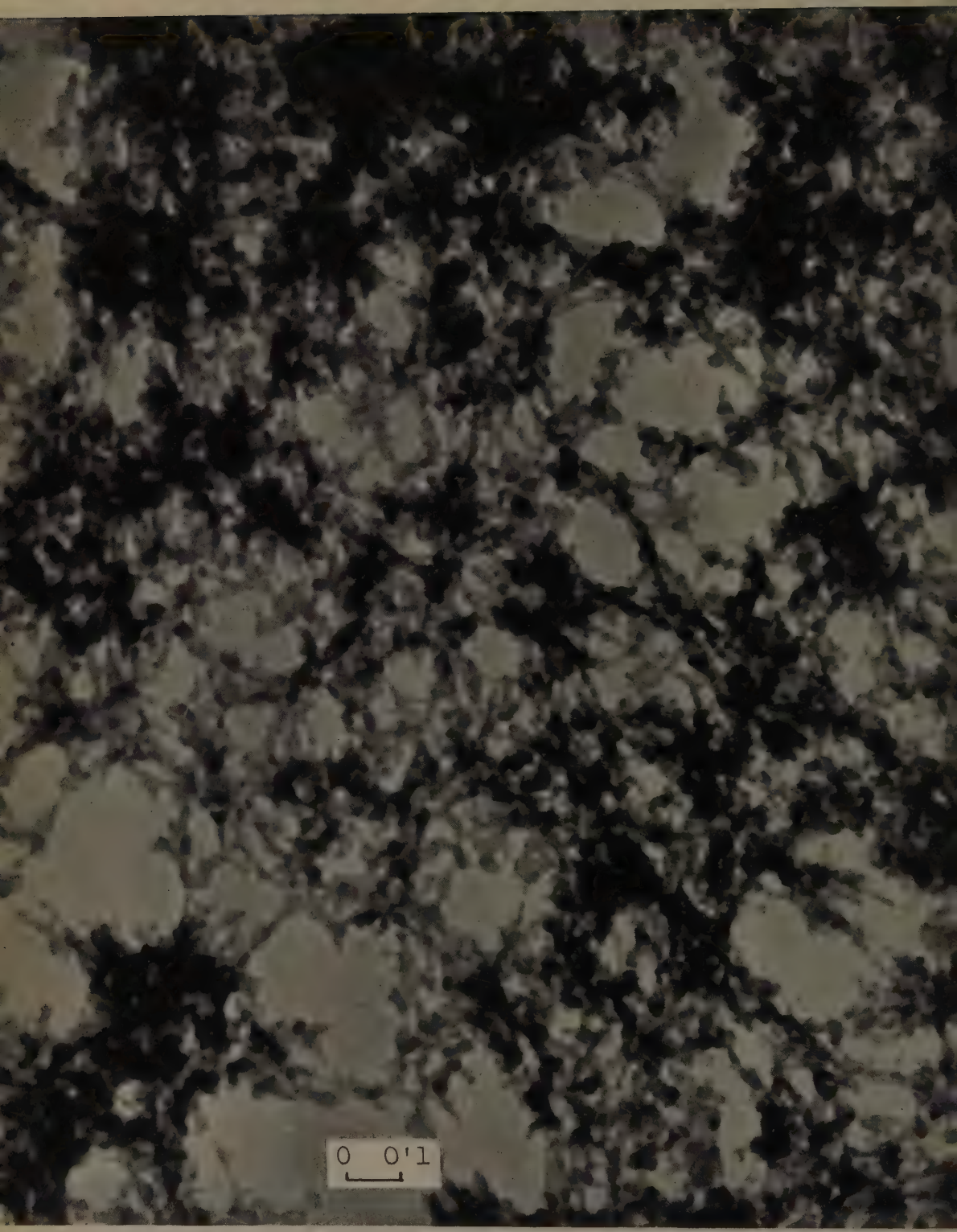

Fig. 6

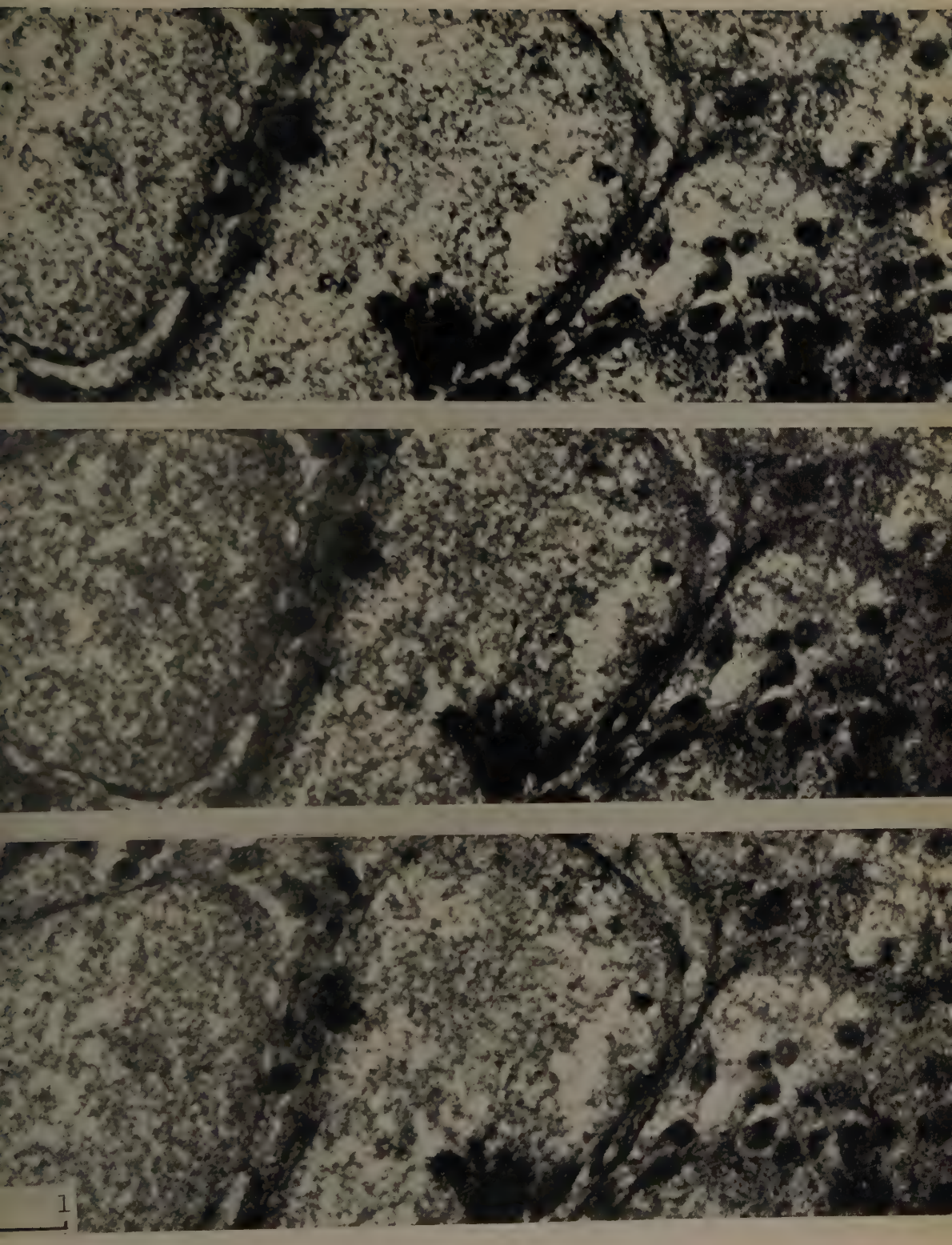

Fig. 3

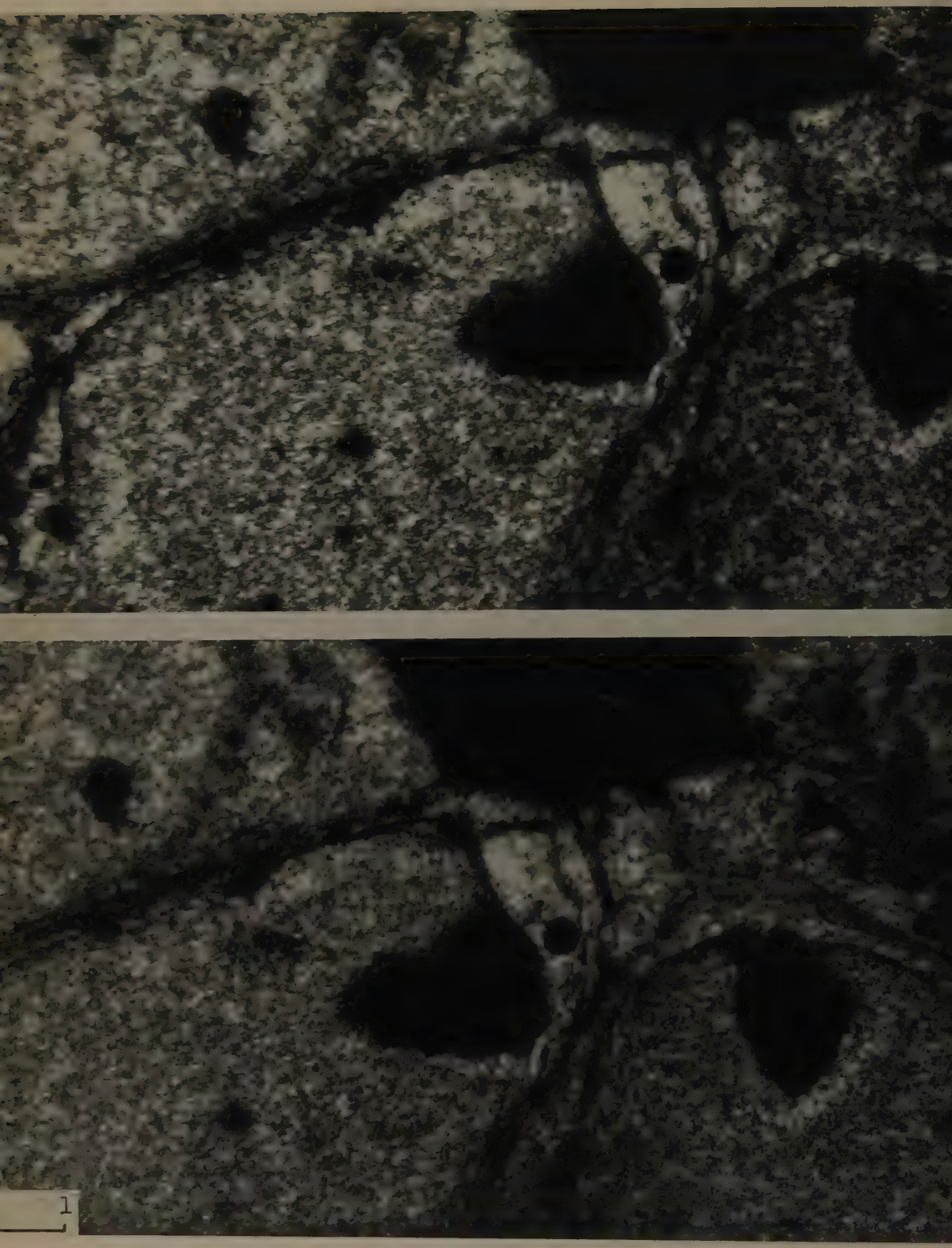

Fig. 4

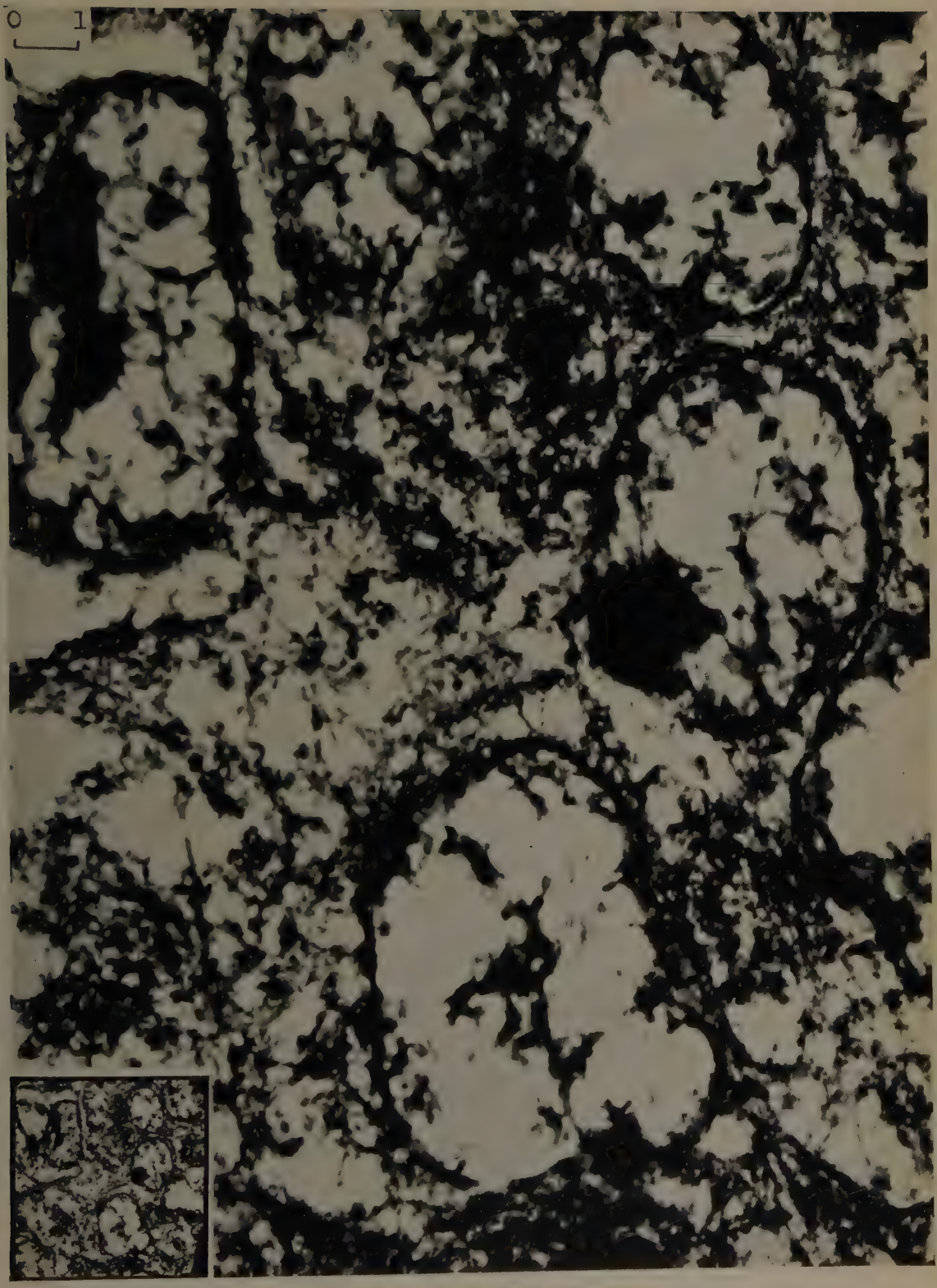

Fig. 5

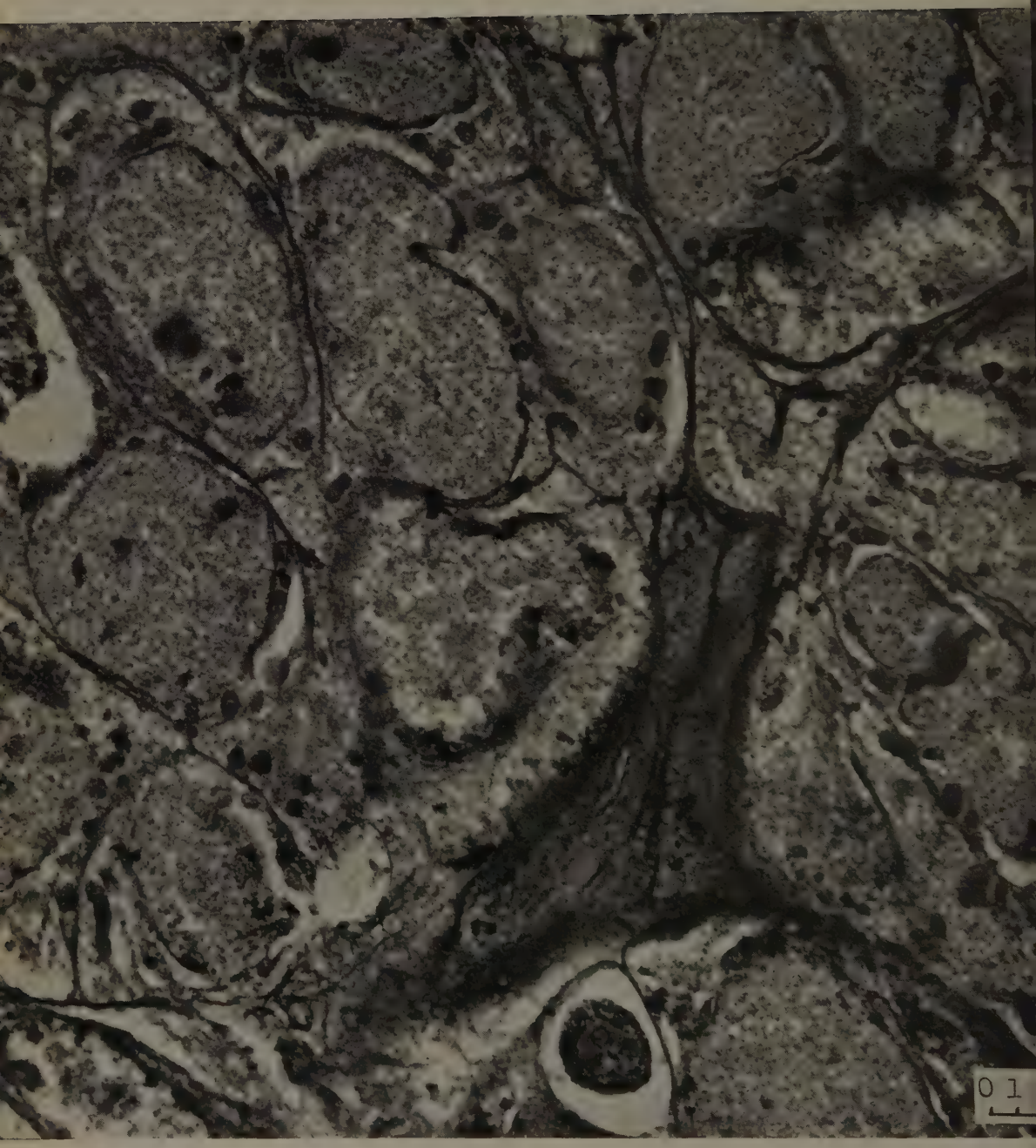

Fig. 2

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